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Local and Global Volumetric strain comparison in sand specimens subjected to drained cyclic and monotonic triaxial compression loading.

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†Deceased June 21, 2017. We dedicate this work to Dr. David Nash, our dearest colleague and friend. who’s many contributions made this work possible.

Abstract

This paper investigates the development of volumetric strain non-uniformities in sand specimens subjected to drained cyclic triaxial compression loading. The assessment is performed by comparing volumetric strain determinations using an external volume gauge and local axial and radial strain measurements mounted on the centre of the specimen. The experimental investigation has been performed for both frictional and enlarged lubricated ends on sand specimens of different densities and fabricated using both moist tamping and dry deposition techniques. It will be shown that considerable discrepancies between the global and local volumetric determination arise even in specimens tested with enlarged lubricated ends, as a result of different volumetric tendencies (contraction or dilation) of the centre and the boundaries of the specimen. These discrepancies are more pronounced for dense specimens cycled at high average stress ratios and amplitudes. The influence of three different assumptions employed to account for the specimens deformed profile (namely right cylinder, parabolic and sinusoidal profile) on the local volumetric determinations will be also assessed. Some recommendations for the need for local volumetric measurements will be attempted.

Keywords

Sand, strains, triaxial testing, end restraint, cyclic loading.

List of notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>β</td>
<td>Cyclic stress amplitude ratio, equal to Δq / p’</td>
</tr>
<tr>
<td>C_u</td>
<td>Coefficient of uniformity</td>
</tr>
<tr>
<td>C_g</td>
<td>Coefficient of gradation</td>
</tr>
<tr>
<td>D_10</td>
<td>Soil particle diameter at which 10% of the soil mass is finer</td>
</tr>
<tr>
<td>D_30</td>
<td>Soil particle diameter at which 30% of the soil mass is finer</td>
</tr>
<tr>
<td>D_60</td>
<td>Soil particle diameter at which 60% of the soil mass is finer</td>
</tr>
<tr>
<td>D_o</td>
<td>Initial diameter of the sample</td>
</tr>
<tr>
<td>D_1</td>
<td>Diameter of the sample measured at mid-height with the radial transducer</td>
</tr>
<tr>
<td>ε_a</td>
<td>Axial strain</td>
</tr>
<tr>
<td>ε_a(G)</td>
<td>Global axial strain</td>
</tr>
<tr>
<td>ε_v</td>
<td>Volumetric strain</td>
</tr>
<tr>
<td>ε_r</td>
<td>Radial strain</td>
</tr>
<tr>
<td>e</td>
<td>Void ratio</td>
</tr>
<tr>
<td>e_{min}</td>
<td>Minimum void ratio</td>
</tr>
<tr>
<td>e_{max}</td>
<td>Maximum void ratio</td>
</tr>
<tr>
<td>H</td>
<td>Height of sample</td>
</tr>
<tr>
<td>L</td>
<td>Gauge length</td>
</tr>
<tr>
<td>LVDT</td>
<td>Linear variable differential transformer</td>
</tr>
<tr>
<td>η</td>
<td>Stress ratio equal to q / p’</td>
</tr>
<tr>
<td>N</td>
<td>Number of cycles</td>
</tr>
<tr>
<td>N_d</td>
<td>Number of cycles when global volumetric strains move from positive to negative values</td>
</tr>
<tr>
<td>M_c</td>
<td>Critical state stress ratio</td>
</tr>
<tr>
<td>p’</td>
<td>Mean effective stress</td>
</tr>
</tbody>
</table>
the friction between the...

s concentrated on the monotonic shearing behaviour of soils...

...the onset of gau... If the e...eiri...d with the entire specimen, and that using measur...}

1 Introduction

One of the major limitations of the triaxial apparatus is the restraint applied at the specimen ends. When conventional end platens are used, the friction between the specimen and the bottom and top caps restricts the specimen from deforming laterally, thereby inducing non-uniform stress distributions and deformations across the specimen, which may assume a “barrel-shape” when undergoing compression. Testing specimens with slenderness ratio $S_r$ (height to diameter ratio) between 1.5 and 3 is common practice to minimise the end restraint effects on the middle third of the specimen (Taylor, 1941; Bishop & Green, 1965), where the deformations are expected to be more uniform and least affected by end restraints. The barrelling can cause a large discrepancy between volume change measurements using global gauges and those deduced from local strain transducers, thus possibly leading to misinterpretation of the soil behaviour (Linton et al., 1988; Klotz & Coop, 2002).

Enlarged lubricated ends are sometimes employed to minimise the friction at the boundaries (Rowe & Barden, 1964; Kirckpatrick & Belshaw, 1968), and even short specimens ($S_r=1$) may experience a more homogeneous deformation pattern than conventional $S_r=2$ specimens tested with frictional end platens (Goto & Tatsuoka, 1988). However, Linton et al (1988) and Klotz and Coop (2002) demonstrated that, despite the use of lubricated ends, specimens may still be subjected to some end restraint and may still develop strain non-uniformities, which in turn affect both local and global determinations of the specimen’s volumetric behaviour. Recent experiments using digital image processing methods (Liu et al., 2013) confirmed that deformations in a triaxial soil specimen were more uniform in the middle region when compared with the entire specimen, and that using measurements from this region was more effective in eliminating the effects of end restraint than the use of lubricated end platens.

The interpretation of the volumetric behaviour from local measurements should also take the deformed shape of the specimens into account. If the specimen assumes a barrel-shape, the radial strain measurements at mid-height of the specimen exceed the average value, and if the barrelling is too great the axial transducers may be pushed out of alignment and give false readings. Thus different deformation assumptions rather than the conventional right cylinder have been employed in the literature, even when lubricated end platens were employed. Germaine & Ladd (1988), and Zhang & Garga (1997), found that the maximum diameter of a triaxial sand specimen occurred at midheight and it changed with height following a parabolic shape. Klotz & Coop (2002) assumed that the sides of the specimen were deforming as an arc and no lateral deformations were developed at the ends of the specimen.

The development of non-uniformities in strain and stress distributions are also influenced by the specimen density and fabrication method. Vardoulakis & Drescher (1985) demonstrated that dense specimens exhibit more pronounced non-uniformities and strain localisation when compared to looser specimens. Before localization develops, the specimen may bulge and present other bifurcation deformation modes, whose onset can be delayed by the presence of lubricated end platens and a uniform method of sample preparation (Desrues, 1990; Desrues, et al. 2007). Depending on the fabrication method of the specimen, non-uniformities of density across the specimen may be present, and thus affect the uniformity of deformation during loading.

While the achievement of precise and accurate measurement of specimen deformations and the proper account of non-uniformities and localisation issues are still ongoing research topics, most of the previous experimental work has concentrated on the monotonic shearing behaviour of soils (Desrues, 1984; Colliat-Dangus et al., 1988). The research reported here aimed to expand the current...
state of the knowledge by exploring the development of non-uniformities in sand specimens subjected
to up to 4000 drained compressive load cycles in the triaxial apparatus, as part of experimental and
constitutive modelling research at the University of Bristol, UK (Corti et al. 2016). The non-uniformity
of deformations across the specimen has been examined by comparing volume changes using both
global and local measurements. Three different deformed shapes for the specimens have been
considered and the sensitivity of the interpretation of experimental results to these assumptions has
been investigated. The influence on the non-uniformities to (i) the end boundary conditions (frictional
and enlarged lubricated ends), (ii) specimen density, and (iii) specimen fabrication method (moist
tamping and dry deposition) will be discussed.

2 Models for calculation of volumetric strains

A schematic representation of a soil specimen instrumented with local axial and radial strain
transducers and deformed during triaxial compression is shown in Fig. 1. The geometry of a specimen
of current height \( H \) is described using an \( x\)-\( y \) coordinate system whose origin is located at the centre
of the specimen. The gauge length of the local axial transducers (LVDT) is defined by \( L \), which varied
between 50 to 52 mm in the present experimental work, and spanned the specimen at mid-height to
monitor the behaviour in its central third. The diameter \( D \) of the specimen varies with height; during a
compression test, the diameter is generally largest at the mid-height \( (D_1) \) and decreases towards the
specimen ends \( (D_0) \).

In order to calculate volumetric strains from the measured values of local axial and radial strains, three
different assumptions for the deformed shape of the specimen have been considered: right cylinder,
parabolic shape, and sinusoidal shape, which are shown in Figures 2a, 2b and 2c respectively. It is
emphasised that the following expressions for estimation of the volumetric strain refer to the
instrumented central portion of the specimen. Compressive strains are assumed positive in this work.

2.1 Right cylinder

The right cylinder assumption considers that lateral deformations are homogeneous through the height
of the specimen (Fig. 2a), with the specimen’s diameter constant and equal to the value measured by
the radial strain transducer located at mid height. For this case, the volumetric strain \( \varepsilon_v \) of the specimen
can be determined from the measured local radial and axial strains, \( \varepsilon_r \) and \( \varepsilon_a \), respectively, using the
following second order expression:

\[
\varepsilon_v = \varepsilon_a + 2\varepsilon_r - 2\varepsilon_a\varepsilon_r - \varepsilon_r^2 + \varepsilon_a^2
\]  

(1)

2.2 Parabolic shape

The assumption of a parabolic specimen profile follows the proposal by Germaine & Ladd (1988) and
Zhang & Garga (1997) from experimental observation of triaxial samples sheared in compression.

With this assumption, the variation of the diameter \( D \) along the sample height can be described with
the following expression:

\[
D = D_1 - (D_1 - D_0) \left( \frac{2y}{H} \right)^2
\]  

(2)

Where \( D_1 \) is the maximum sample diameter located at mid-height \( (y = 0) \), and \( D_0 \) is the diameter at its
ends \( (y = \pm H/2) \) which is equal to the initial sample diameter under the assumption of no lateral
deformation at the sample-cap interface (Klotz & Coop, 2002). By integration of eq. (2) over the
instrumented section of the specimen it is possible to obtain (see Appendix) the following expression for the volumetric strain from the measured local axial and radial strains:

\[ \varepsilon_v = \left[ 2 \left( \frac{L}{H} \right)^2 \varepsilon_r + \left( -1 + \frac{2}{3} \left( \frac{L}{H} \right)^2 - \frac{1}{5} \left( \frac{L}{H} \right)^4 \right) \varepsilon_r^2 \right] \cdot (1 - \varepsilon_a) + \varepsilon_a \]  

(3)

It should be noted that the height (H) and length (L) of the instrumented central portion of the specimen change during the test but their ratio is assumed to stay constant.

2.3 Sinusoidal

An alternative assumption, which assumes that the specimen’s diameter follows a cosinusoidal variation with the sample height, is considered (Fig. 2c). Unlike the parabolic shape, this expression implies that the tangent of the sample profile is vertical at the ends of the sample. Thus, the effect of a different assumption for the deformed shape on the interpretation of volumetric deformations may be examined. The variation of diameter D with the height of the sample is expressed by:

\[ D = D_o - \frac{(D_i - D_o)}{2} \left[ 1 + \cos \left( \pi \cdot \frac{2y}{H} \right) \right] \]  

(4)

In a similar manner (see Appendix), integration of eq. (4) leads to the following expression of the volumetric strains on the instrumented central part of the specimen:

\[ \varepsilon_v = \left[ \left( \frac{H}{\pi \cdot L} \sin \left( \frac{\pi \cdot L}{H} \right) \right) \cdot \varepsilon_r + \left( -\frac{3}{8} - \frac{\sin(\pi \cdot L/H)}{2\pi \cdot L/H} - \frac{\sin(2\pi \cdot L/H)}{16\pi \cdot L/H} \right) \cdot \varepsilon_r^2 \right] \cdot (1 - \varepsilon_a) + \varepsilon_a \]  

(5)

3 Materials

Triaxial experimental tests were conducted on Hostun RF (S28) sand, which is a sub-angular granular siliceous medium material that has been widely used in the past for experimental research and constitutive modelling (Sadek 2006; Doanh & Ibraim, 2000; Diambra et al. 2011). A typical particle size distribution of this sand is given in Fig. 3 and it is characterised by a mean grain size, \( D_{50} = 0.35 \) mm, coefficient of uniformity, \( C_u = D_{50}/D_{0} = 1.59 \), coefficient of curvature \( C_c = (D_{90})^2/(D_{10}D_{90}) = 0.97 \), maximum and minimum void ratio, \( \varepsilon_{smax} = 1.00 \), \( \varepsilon_{smin} = 0.656 \) and specific gravity \( G_s = 2.65 \) (Escribano, 2014) which were determined following BS EN ISO 17892. Further information on other properties of this Hostun RF sand can be found in Flavigny et al. (1990).

4 Specimen preparation

The specimens were tested using two different boundary conditions in order to study their influence on the development of non-uniformities and volumetric deformations during drained cyclic loading: i) fully frictional end platens and ii) enlarged lubricated end platens. In this investigation, two different sample preparation methods were also used: i) moist tamping (MT) with undercompaction (Ladd, 1978) and ii) the dry deposition (DD) method (Ishihara, 1993). The details of each are described in the following paragraphs.

4.1 Specimen boundary conditions

Specimens with fully frictional ends (defined here as restrained ends, RE) had a height to diameter ratio \( Sr = 2 \), a diameter of 75 mm, and height of 150 mm. Specimens with enlarged lubricated ends (defined by LE) had dimensions of 140 mm height and 70 mm diameter with one drainage line at the
centre. The lubricated ends consisted of smooth silicone grease and three layers of latex rubber discs (0.3mm thickness) with several radial cuts to minimise their resistance to radial stretching.

4.2 Specimen fabrication

4.2.1 Moist tamping (MT) technique

In preparing specimens using the moist tamping method with undercompaction (Ladd, 1978), the total amount of dry soil required was mixed with a predetermined amount of water and divided into several equal parts. The value of 10% moisture content, used for the dense specimens, corresponded to the optimum value obtained from compaction tests (Ibraim & Fourmont, 2006), while the use of lower water contents simplified the fabrication of looser specimens. The specimens were then prepared in layers, each tamped to a certain predetermined height with the purpose of controlling its density and avoiding over compaction of the layers underneath. Each specimen was formed from 10 layers, giving a layer thickness of 15 mm for a sample of 150 mm height and 14 mm for samples of 140 mm height.

4.2.2 Dry deposition (DD) technique

In this procedure, a predetermined amount of dry sand was carefully poured inside a mould using a funnel, ensuring a zero height of fall and uniformly spreading the sand across the sample. The target density was then reached by tapping the sides of the mould with a gentle vibrator. Any non-uniformity on the final specimen’s top surface was levelled off with a brush and the top cap was subsequently installed. Particular care was paid in order to minimise the creation of looser zones as a result of the brushing.

5 Experimental equipment and procedures

5.1 Experimental equipment

Drained cyclic triaxial tests were performed using a Bishop-Wesley apparatus equipped with an internal load cell with an accuracy of ±1.5 N, differential pressure transducer with ±0.7 kPa of accuracy, an external LVDT with an accuracy of ±0.02% of strain, and an Imperial College type volume gauge (±0.05ml of accuracy).

5.2 Local Strain Transducers

Three local miniature submersible LVDTs (RDP D5W) similar to the ones described by Cuccovillo & Coop (1997), with a ±5 mm displacement range, output voltage level of ± 10 V and an accuracy of ±0.005 % of strain, were used to measure axial and radial strains in the middle third of the specimens. As shown in Fig. 1, two vertical transducers were mounted opposite each other, each fixed to an upper pad with its armature resting on a lower pad. Average results of the two LVDTs were used to calculate axial strain. A third transducer was mounted horizontally on a radial belt (GDS type) installed at the middle of the specimen. To ensure alignment between upper and lower pads of the axial gauges small aluminium arms were temporarily located between the top and bottom pads. The pads were then glued on the sides of the sample with instant contact adhesive and pins protruded into the sample. A similar connection was used for the radial belt. A fixed connection between the load cell and top cap helped to avoid misalignment errors and therefore minimised the difference in readings between the two opposite axial LVDTs. A photograph of a fully instrumented specimen with enlarged lubricated ends is provided in Fig. 4.

LVDTs can be highly affected by temperature fluctuations of the water in the cell. Even a 1 °C variation of the water temperature produces significant oscillations on the LVDT readings. Room temperature was controlled, and the triaxial cell was covered with bubble wrap, which generally decreased temperature changes to 0.2°C.
5.3 Triaxial test conditions

After its preparation, the initial dimensions of the specimen were taken while applying 25 kPa of vacuum. Fully saturated specimens were then produced by initially circulating carbon dioxide for one hour at a differential pressure of 3 kPa; then de-aired water was circulated from the bottom to the top of the specimen. Finally, the back pressure was raised to between 200 and 400 kPa until a Skempton B value higher than 0.95 was obtained. Changes in specimen’s height and diameter during saturation were monitored through the internal vertical and radial transducers. These measurements have been used to correct the specimen’s initial void ratio.

A schematic representation of the stress path followed during the tests is shown in Fig. 5 in terms of stress ratio \( \eta = q/p' \) (ratio between the deviator stress \( q \) and the mean effective stress \( p' \)) versus the mean isotropic stress \( p' \). Isotropic consolidation to a target value of mean effective stress of \( p'_0 \) was achieved by increasing the cell pressure at a rate of 60 kPa per hour. Any further anisotropic consolidation to reach a desired average stress ratio \( \eta_{\text{avg}} \) was then undertaken maintaining the mean effective stress constant. At the end of consolidation, creep under constant stress conditions was allowed for a maximum of two hours before the start of cyclic loading. In most tests, cyclic loading was applied around the average stress ratio \( \eta_{\text{avg}} \) by varying the axial stress sinusoidally with time, keeping the cell pressure constant. A low frequency of 1 cycle per 5 minutes was applied in order to ensure well defined cycles and homogeneous pore water pressure inside the sample.

The amplitude of the cyclic stress ratio is described using \( \beta = \Delta q/p'_0 \), defined as the ratio of peak to peak cyclic stress amplitude \( \Delta q \) to the average mean effective stress \( p'_0 \) (see Fig. 5). Tests were performed at different stress levels, cyclic stress amplitudes and initial density. Most tests were one of the following types: stresses cycled only in compression i) at low \( \eta_{\text{avg}} \) and ii) at high \( \eta_{\text{avg}} \) iii) stresses cycled from compression to extension and iv) stresses cycled only in extension at low \( \eta_{\text{avg}} \). One test (DDLE_5) utilised several stages of cyclic loading carried out when monotonic loading in compression was interrupted; each stage had a small number of cycles in which the deviatoric stresses were decreased from the current maximum value reached.

Table 1 summarises the tests performed in this campaign providing details on fabrication method and testing conditions. It is possible to divide the test program in three groups characterised by different specimen preparation methods (MT or DD) and employed boundary conditions (RE, LE); hence tests are classified as MTRE, MTLE and DDLE in Table 1 for a total of 16 tests. Table 1 also specifies which samples have been monotonically sheared in drained conditions after the cyclic loading state: some samples sheared at constant mean effective stress \( p' \), while others were sheared at constant cell pressure. The final monotonic shearing was continued until the local strain transducers reached their working limit.
Table 1. List of tests performed (RE: restrained ends; LE: lubricated ends; MT: Moist tamping; DD: dry deposition)

<table>
<thead>
<tr>
<th>Test number</th>
<th>e</th>
<th>$p_0'$ (kPa)</th>
<th>$D_r$ (%)</th>
<th>$\eta_{avg}$</th>
<th>$\beta$</th>
<th>$N$</th>
<th>Monotonic loading</th>
<th>Sample prep. method</th>
<th>End conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>MTRE_1</td>
<td>0.966</td>
<td>100</td>
<td>9.88</td>
<td>0.10</td>
<td>0.32-1.4</td>
<td>3500</td>
<td>* const $\sigma'_3$</td>
<td>MT</td>
<td>RE</td>
</tr>
<tr>
<td>MTRE_2</td>
<td>0.977</td>
<td>100</td>
<td>6.69</td>
<td>0.25</td>
<td>0.31-0.59</td>
<td>1000</td>
<td>* const $\sigma'_3$</td>
<td>MT</td>
<td>RE</td>
</tr>
<tr>
<td>MTRE_3</td>
<td>0.740</td>
<td>100</td>
<td>75.13</td>
<td>0.25</td>
<td>0.33</td>
<td>1500</td>
<td>-</td>
<td>MT</td>
<td>RE</td>
</tr>
<tr>
<td>MTRE_4</td>
<td>0.899</td>
<td>100</td>
<td>29.36</td>
<td>0.50</td>
<td>0.31-0.92</td>
<td>250</td>
<td>const $p'_s$</td>
<td>MT</td>
<td>RE</td>
</tr>
<tr>
<td>MTRE_5</td>
<td>0.727</td>
<td>100</td>
<td>79.36</td>
<td>1.00</td>
<td>0.35</td>
<td>1000</td>
<td>Const $p'_s$</td>
<td>MT</td>
<td>RE</td>
</tr>
<tr>
<td>MTRE_6</td>
<td>0.74</td>
<td>100</td>
<td>75.59</td>
<td>-0.30</td>
<td>0.40</td>
<td>4000</td>
<td>Ext const $\sigma'_s^*$</td>
<td>MT</td>
<td>RE</td>
</tr>
<tr>
<td>MTEL_1</td>
<td>0.937</td>
<td>100</td>
<td>18.31</td>
<td>0.50</td>
<td>0.36</td>
<td>1500</td>
<td>const $\sigma'_3$</td>
<td>MT</td>
<td>LE</td>
</tr>
<tr>
<td>MTEL_2</td>
<td>0.720</td>
<td>100</td>
<td>81.85</td>
<td>0.50</td>
<td>0.34</td>
<td>1500</td>
<td>-</td>
<td>MT</td>
<td>LE</td>
</tr>
<tr>
<td>MTEL_3</td>
<td>0.982</td>
<td>100</td>
<td>5.23</td>
<td>1.00</td>
<td>0.39</td>
<td>350</td>
<td>-</td>
<td>MT</td>
<td>LE</td>
</tr>
<tr>
<td>MTEL_4</td>
<td>0.722</td>
<td>100</td>
<td>80.81</td>
<td>1.00</td>
<td>0.34</td>
<td>1500</td>
<td>const $\sigma'_3$</td>
<td>MT</td>
<td>LE</td>
</tr>
<tr>
<td>MTEL_5</td>
<td>0.717</td>
<td>100</td>
<td>82.27</td>
<td>1.28</td>
<td>0.34</td>
<td>1500</td>
<td>const $\sigma'_3$</td>
<td>MT</td>
<td>LE</td>
</tr>
<tr>
<td>DDLE_1</td>
<td>0.984</td>
<td>50</td>
<td>4.65</td>
<td>0.00</td>
<td>0.35</td>
<td>250</td>
<td>* const $p'_s^*$</td>
<td>DD</td>
<td>LE</td>
</tr>
<tr>
<td>DDLE_2</td>
<td>0.985</td>
<td>100</td>
<td>4.36</td>
<td>0.25</td>
<td>0.34</td>
<td>250</td>
<td>const $p'_s$</td>
<td>DD</td>
<td>LE</td>
</tr>
<tr>
<td>DDLE_3</td>
<td>0.913</td>
<td>75</td>
<td>25.29</td>
<td>0.50</td>
<td>0.47</td>
<td>200</td>
<td>const $p'_s$</td>
<td>DD</td>
<td>LE</td>
</tr>
<tr>
<td>DDLE_4</td>
<td>0.796</td>
<td>100</td>
<td>59.30</td>
<td>0.50</td>
<td>0.47</td>
<td>200</td>
<td>const $\sigma'_3$</td>
<td>DD</td>
<td>LE</td>
</tr>
<tr>
<td>DDLE_5</td>
<td>0.824</td>
<td>100</td>
<td>51.16</td>
<td>0.20-1.40</td>
<td>0.18-1.21</td>
<td>5-50</td>
<td>const $p'_s$</td>
<td>DD</td>
<td>LE</td>
</tr>
</tbody>
</table>

* denotes two-way cycling where $\beta/2 > \eta_{avg}$

6 Comparisons between global and local strain measurements

Differences between global axial strains measured externally and local axial strains measured with transducers mounted on the specimen are commonly observed in monotonic tests, and are due to bedding errors and non-uniformities developed during the test. The external measurement of volume change may be used to determine the average volumetric strain and hence, using the right cylinder assumption, the average radial strain. Such measurements are susceptible to errors arising from leakage and temperature change although in this research attempts were made to minimise these. The direct measurement of radial movements permits the determination of radial strain at mid-height of the specimen, and when combined with the local axial strain measurements and an assumption about the sample shape, the volumetric strains in the instrumented section may be determined.

Examples of the measurements undertaken are shown in Fig. 6 (a) to (d) for the compression test MTRE_5 and the extension test MTRE_6, both on dense samples prepared by moist tamping with restrained ends. Each sample was cycled around an average stress ratio before deviatoric stresses were increased. Fig. 6(a) shows plots of the stress ratio $q/p'$ vs global and local axial, radial and volumetric strains. The local volumetric strains were determined using the sinusoidal shape of the sample deformed profile. Fig. 6(b) reports the trends of global and local volume strains (determined with Eqs (1), (3) and (5) for the right cylinder (RC), parabolic (P) and sinusoidal (S) deformed shape, respectively, and also imposing null radial strains ($\varepsilon_r=0$)) vs local axial strain. Several observations can be made:

1. global axial strains exceed the local axial strains significantly in both cases;
2. during cyclic loading in compression (Fig. 6a), the volume changes measured locally are very small, primarily because the radial strains are extremely small;
3. during cyclic loading at low stress ratios in extension (Fig. 6c), the volume changes measured locally are also small, although the radial strains are slightly larger;
4. at large strains the magnitude and rate of dilation are smaller using the external measurements than those calculated from the local measurements.

Figs. 6(e) and 6(f) show similar plots for test MTRE_1, a loose specimen that was subjected to two-way symmetric cycling before being tested to failure in compression. In this test the strains developed during cycling were larger than those in the previous tests. After cycling the axial strains were small but the sample showed significant volumetric contraction; the volumetric strain measured externally was twice that deduced from the local measurements.

In the following sections, the global and local strains from a number of tests are compared both at large strains and in the small strain range. In some tests, there were indications that temperature fluctuations were unduly influencing the external volume measurements or that the radial displacement transducer was sticking; these tests have been excluded from the comparisons.

6.1 Behaviour at large strains

6.1.1 Axial strain measurements

Fig. 7a, b and c show the comparison between local $\varepsilon_a$ and global $\varepsilon_a(G)$ axial strains for all the compression tests that were loaded monotonically to large strains after experiencing cyclic loading. At large strains there are some slight differences between the axial strains observed in tests with restrained ends (MTRE) and some of those with lubricated ends (MTLE, DDLE). MTRE tests continue through the 1:1 line, instead MTLE tests present larger global axial strains. On the other hand, DDLE tests for 25.3% and 59.3% relative density follow the 1:1 line, but the loose specimen, at 4.4% relative density presents larger global axial strains, as in the case of MTLE tests. This is probably related to inevitable bedding errors related to soft inclusions. However, despite some initial differences, the local and global strain rates at large strain are similar (i.e. the trends are rather parallel to the 1:1 line) in most of the tests.

Only one extension test is reported here (MTRE6) but Fig. 7d shows that the differences are particularly marked. In the small strain range (<0.5%) the externally measured axial strains were more than 10 times those observed in the central part of the sample, possibly produced due to initial friction in the LVDTs armature due to misalignment. At large strains, the local axial strain rates were initially rather similar to the global strain rate but then the global strain rate seems to progressively exceed that measured locally.

6.1.2 Volumetric strains

For each set of compression tests shown in Table 1 (MTRE, MTLE and DDLE groups), two specimens (one in a loose initial state and another in a medium dense to dense initial state) have been selected to illustrate the typical differences between the volumetric strains measured using the external volume gauge, and that calculated using the internal LVDTs. The three assumptions for sample deformation previously introduced in Section 2 (right cylinder, parabolic and sinusoidal deformation shapes) have been considered for the volumetric strains determination using locally installed transducers.

The axial strain – volumetric strain trends for the six selected specimens are given in Fig. 8. The dotted red line included in the graphs corresponds to a 1:1 relationship between strains, to differentiate between expansion at mid-height (data above the 1:1 line), and contraction (data below the 1:1 line). For these tests loaded in compression, the entire data lie above the 1:1 line, meaning that the soil is being deformed as shown in Fig. 2. The initial section of the curves at small strains corresponds to the cyclic loading stage, and all the local strains lie on or very close to the 1:1 line indicating that the radial strains are very small or negligible; as a result, the volumetric strains are initially of similar magnitude to the axial strains.
As shearing progresses and the stress ratio \(q/p'\) exceeds about 1.0 (see Fig. 8), the specimens start to expand at mid-height (radial strains are negative), producing differences among the volumetric strains calculated with the three different deformation models shown in Fig.2. The divergence between the three local strain determinations seems to occur with the onset of dilation, especially for the dense and medium dense specimens (Fig. 8d, e, f), and become significant above an axial strain level of 1%, for both loose and dense specimens. By inspection of the radial strain measurements shown in Figure 6, it can be determined that these differences arise when the radial strain exceeds approximately -0.25%.

For the loose specimens the average volumetric strain measured with the external volume gauge (global measurement) indicates continuous contraction as they are compressed (Fig. 8a, b, c), while the local measurements show that an almost constant volume state is reached above 5% axial strain. In the medium dense to dense specimens (Fig. 8c, d, e) the global measurements indicate much smaller dilatancy rates than those calculated from the local measurements. In the single extension test reported here (see Fig. 6d), the dilatancy rate measured with local instrumentation was approximately twice that measured externally.

These differences are significant and suggest that local strain measurements are essential to interpret the state of the soil correctly. The selection of the most appropriate method for calculating the volumetric strain from the local measurements depends on the degree of end restraint and should be informed by observations of the actual shape after testing. Of the three methods, the right cylinder assumption will always provide a limit, since the radial strain throughout the sample is assumed constant and equal to that at mid-height. Visual observations of the specimens during and after testing suggested that the appropriate shape of deformation to consider when using local measurements on a triaxial sample with \(S_r =2\) for monotonic loading is the sinusoidal assumption (Fig. 2c). Fig.8 also shows that there is no considerable difference on the predicted volumetric strains between the parabolic and sinusoidal assumptions, especially at the small strain level. Therefore, the sinusoidal shape assumption will be used to compare local and global strain measurements in the small strain range.

### 6.2 Behaviour at small strains under cyclic loading

#### 6.2.1 Accumulated axial strains
Since volumetric changes derive from both radial and axial deformations of the sample, it seems appropriate to start the analysis with the comparison of accumulated axial strains measured by both local and external axial strain transducers (\(\Delta \varepsilon_a\) and \(\Delta \varepsilon_{a0}(G)\), respectively) during the cyclic loading stage, defined as:

\[
\Delta \varepsilon_a = \varepsilon_a - \varepsilon_{a0} \quad \text{and} \quad \Delta \varepsilon_{a0}(G) = \varepsilon_{a0}(G) - \varepsilon_{a0}(G)
\]

Where \(\varepsilon_{a0}\) and \(\varepsilon_{a0}(G)\) are the values of axial strain at the start of the first cycle measured with local and external transducers, respectively. Fig. 9 shows a comparison of the two determinations (local and external axial strains) for both fabrication procedures and boundary conditions employed. Except for tests where loads cycled between compression and extension (MTRE 1 in Fig. 9a and DDLE 2 in Fig. 9c), the axial deformations developed during cyclic loading measured with the external transducer were significantly (up to five times) greater than those measured with the local transducers. The differences indicate that proportionately larger movements developed at the ends of the specimen throughout cyclic loading; a factor of five sounds large but actually represents only about 0.25mm additional movement at each end.
6.2.2 Accumulated Radial strains

Direct measurements of the radial strains are only made locally but may be compared with radial strains calculated from the external volume and axial strain measurements. Fig. 6 showed that for test MTRE_5 cycled wholly in compression, the radial strains were very small. Figure 10a shows the data from test MTRE_5 replotted for clarity showing local and global radial strains plotted against each other and against $q/p'$. It can be seen that strains measured locally were extremely small during cycling and did not increase until the stress ratio $q/p'$ reached about 1.4 at which point the sample started to bulge and dilate; this contrasts with the strains calculated from the global measurements. During subsequent monotonic loading, local and global radial strains became more similar at large strains.

Similar trends were observed in all samples tested wholly in compression. The critical stress ratio at which radial strains started to increase has been plotted against relative density in Fig 10b, its value increasing from about 0.8 for loose specimens to 1.4 for dense specimens. In most tests the radial displacement data during cyclic loading were somewhat noisy and the movements were apparently of the same order as the accuracy of the measurements ($\pm 0.001\%$). In reviewing the data, it was obvious that in a small number of the tests the transducer armature had stuck initially, and these tests have been excluded.

Test MTRE_6 was tested wholly in extension (see Fig. 6c) being subjected to 4000 cycles of deviatoric stress before being loaded monotonically to failure. Like the samples tested wholly in compression the radial strains developed during cycling were very small (0.02% contraction).

In contrast to the above, test MTRE_1 was subjected to two stages of two-way cycling before being loaded monotonically to failure in compression (see Fig. 6e). In the first stage ($\beta=0.32$), the sample contracted by 0.16% during cycling, which is significantly more than the volumetric strain observed in the tests loaded wholly in compression or extension (tests MTRE_5 and MTRE_6, Fig.6). In the second stage ($\beta$ increased to 1.5), the accumulated radial strain reached 1.6%. Subsequent monotonic loading resulted in a reversal of the radial strain as the sample dilated. These differences in the radial strains directly affect the calculated volume changes and they are the cause for the larger discrepancies between local and global determinations of volumetric strains observed in Figure 6(f).

6.2.3 Accumulated Volumetric strains

A comparison between local and global accumulated volumetric strains with number of applied cycles is presented in Fig. 11 for all the performed tests. The volumetric strains were calculated from the measurements of the local LVDTs and using Eq.5, which assumes a sinusoidal shape for sample deformation during loading. The tests in Fig. 11 have been divided in the three groups defined in Table 1 (MTRE, MTLE and DDLE) but they have been further subdivided into loose and dense samples depending on whether their relative density (Dr) was lower or higher than 50%, respectively. Corroborating relevant literature (Tatsuoka & Ishihara, 1974; Groot et al. 2006), larger volumetric deformations were observed for the loose rather than for dense samples. Thus, different vertical scales have been used in the representations of Fig.11 and this must be kept in mind when analysing the observed experimental trends. Analysis of the results in Fig. 11, where local volumetric measurements are shown with white filled symbols and global volumetric measurements with black filled symbols, yielded to the following observations:

1. Common to all the groups of loose samples (MTRE, MTLE or DDLE), local and global volumetric trends have the same sign despite differences in magnitude (Fig. 11a, c, e). In contrast, for dense samples, global and local volumetric strains can yield the opposite sign of the volumetric deformations (Fig.11b, d, f). In many tests of dense samples (e.g. MTRE5 in Fig. 11b, MTLE4 and MTLE5 in Fig.11d and DDLE5 in Fig.11f), when the global volumetric
measurements suggest dilation of the sample, the local volumetric measurements indicate progressive compression.

2. For some tests (e.g. MTRE_2, MTRE_4, MTLE_1, MTLE_3, DDLE_2 all belonging to the category of loose samples), there are no significant disparities between local and global volumetric determination. Other tests (e.g. MTRE_1, MTRE_3, MTLE_2, and DDL5_5) exhibit similar local and global volumetric responses up to a certain value of number of cycles \( N \) (between 50 and 200) and then consistent differences can be observed. A final category of tests can be identified (e.g. MTRE_5, MTRE_6, MTLE_4, MTLE_5, DDLE_1, DDLE_3, DDLE_4) where the global and local volumetric strains diverges from the onset of cycling loading.

3. It is not possible to uniquely state whether global measurements overestimate or underestimate volumetric deformations if compared to the local measurements. Generally, the global volumetric determinations indicate more compression than local determinations for loose specimens cycled at low average stress levels (e.g. MTRE1, DDLE1, MTRE2 cycles at \( \eta_{avg} \leq 0.25 \)), but less compressive or more dilative response for all the dense samples or other loose samples cycled at higher average stress ratios.

The magnitude of the difference between local and global volumetric trends is also influenced by the density and stress level. For the MTRE–loose group (Fig.11a), the differences are more pronounced at low stress levels while for all the dense samples (Fig. 11 b, d, f) the difference increases with increasing average cycling stress ratio. These differences appear to be related to magnitude of radial strains developed in the samples during cycling: it has been observed that larger radial strains are developed for two way cycling at low average stress levels (e.g. MTRE1, DDLE1, MTRE2 cycles at \( \eta_{avg} \leq 0.25 \)), but less compressive or more dilative response for all the dense samples or other loose samples cycled at higher average stress ratios.

The above observations suggest that the relative magnitude of the global and local volumetric deformations is affected by both relative density \( (D_r) \) and average cyclic stress ratio \( (\eta_{avg}) \). Indeed, it is expected that the cyclic amplitude \( (\beta) \) is also an important factor but this has been kept rather constant in this experimental investigation. Fig. 12 shows how the average cyclic stress ratio affects the point during cycling (defined by the number of cycle \( N_d \)) when the dense samples start to deform with a global volumetric strain rate of opposite sign to the locally determined volumetric strain rate. As discussed in point 1 above, this divergence of the sign of the volumetric strain rate is rather obvious for many dense samples such as MTRE3, MTRE5, MTLE2, and MTLE5 (Fig. 11b and d), which have been all used for drawing Fig.12. Higher stress levels accelerate the onset of global and local measurements divergences, which can take place from the very early stage of cycling. It is worth noting that in two tests (MTLE_4 and MTLE_5 with lubricated ends) divergence was observed from the start of cycling \( (N_d = 0) \); these samples were subjected to cycling at high stress ratios \( (\eta_{avg} = 1.0 \) and 1.25, respectively).

In order to demonstrate the differences between local and global accumulated volumetric strains and their dependence with both density and average stress ratio, the three dimensional representations in Figure 13 have been developed. In Fig. 13a values of accumulated global and local volumetric strain after 100 cycles are plotted against initial relative density \( (D_r) \) and average cyclic stress ratio \( (\eta_{avg}) \). Each global determination is represented by a circle while the corresponding local determination is indicated by a triangle. The two points have been linked by a line to form an arrow which gives a visual indication of the direction and magnitude of the difference. A grid corresponding to the plane of zero volumetric strains has been included to help in understanding the figure. This representation confirms the previous observation no.2, suggesting that global volumetric determination overestimate compression at low average stress level and for loose samples but they also overestimate dilation for high stress level and/or dense samples. The difference between local and global determination seems to be limited for medium dense samples cycles at an average stress ratio around 0.5.
representation may be obtained in Fig 13b which used the percentage error of volumetric
determination defined as:

\[ \rho_{err} = \frac{\Delta \varepsilon_G - \Delta \varepsilon_L}{|\Delta \varepsilon_L|} \cdot 100 \]  

(6)

With only one exception related to test MTLE3, all the points in Fig 13 fit quite well a second order
polynomial surface. This surface facilitates the understanding of the trends of volumetric error which
can be recorded if external global measurements are used. A reverse of the sign of the error is
observed for average stress ratio of about 0.5. It is also important to note that volumetric errors larger
than 200% can be obtained in extreme situation (e.g. very loose samples cycled at very low stress
ratios or dense samples cycled at very high stress levels).

7 Discussion

The difference between the volumetric trends determined with global and local transducers highlights
the degree of deformation non-uniformities that may emerge throughout a tested specimen during a
monotonic and cyclic triaxial test. The difference in sign of the global and local volumetric strains
determinations observed for the dense specimens, suggest that while the centre of the specimen
undergoes volumetric compression, as evidenced by local strain measurements, the top and bottom
thirds of the specimen must experience important dilative processes to comply with the overall dilative
specimen’s behaviour monitored by the external volume gauge. Thus, continuous cyclic loading could
induce non-uniform stress states throughout the specimen and the sole use of an external volumetric
measurement system may lead to a misinterpretation of the soil response. While it may be a bit
 speculative, it logically follows that the consequences of these non-uniformities may be even more
concerning for cyclic undrained tests. In fact, the different local tendencies for contraction or dilation
between the centre and the boundaries of the specimen will probably cause a differential excess pore
pressure build up throughout the specimen, triggering water flow within the specimen, which will locally
not respect the constant volume hypothesis of an undrained test. This will likely have important
implications for the observed overall mechanical response.

The main parameters that will trigger the development of non-uniform deformations along the
specimen are its density and the stress level. Disparities between global and local volumetric changes
were observed for both loose and dense samples but while for loose samples only differences in
magnitude were observed, for dense samples cycled at high stress level the sign of the volumetric
strains was often in disagreement. Larger discrepancies between local and global determinations of
volumetric strains are associated to those testing conditions triggering larger development of radial
strains, which are observed for two-way cycling at low stress level or cycling at high stress ratios. It
should be also reminded that the increasing discrepancies at higher stress levels may be explained
by larger shear bands or localisation inside the dense specimens, which as suggested by Desrues et
al. (2007) are susceptible to such phenomena from the early stage of shearing.

8 Summary and conclusions

Results from cyclic triaxial tests on sand specimens fabricated with two different techniques (moist
tamping and dry deposition) and tested under both restrained and enlarged lubricated ends conditions
have been analysed to investigate the occurrence of non-uniformities of volumetric deformations and
to assess the implications for the interpretation of the soil response during triaxial cyclic shearing. The
occurrence of non-uniformities was assessed by comparing volumetric strain determinations from
global strain measurement (using an external volume gauge) with strains determined from local axial
and radial strain measurements (using local LVDTs placed on the middle third of the specimens). The
following conclusions can be drawn from the test results:
- It is essential to use local strain measurements during both monotonic and cyclic loading when
accurate measurements of soil deformation are needed. This should be important in the small to
medium strain domain where accurate measurements are necessary for material stiffness
determination, which is essential for constitutive modelling purposes. Divergence between global
and local determinations can be observed from the very beginning of the tests reported here.
- When using local instrumentation, the conventional right cylinder assumption to determine
volumetric deformation gives reliable results only for small radial deformation measured at the
middle third of the specimen. In this investigation, the differences between three different
assumptions for the deformed shape of the specimen (right cylinder, parabolic, sinusoidal)
became important only when the radial strain exceeded approximately -0.25%.
- Under conditions of one-way cyclic loading, the accumulated radial strains were negligible until
the stress ratio reached a critical value which varied with density. Only when cycling was two-way
did significant radial strains develop.
- Comparisons between local and global volumetric determinations during cyclic triaxial loading
revealed considerable discrepancies for both loose and dense specimens at different average
cyclic stress levels. Larger compressive volumetric strains were indicated by global
measurements in very loose specimens at low stress levels. For dense specimens cycled at high
stress levels, global measurements can instead lead to a reversed sign of volumetric
measurements indicating overall dilation while the middle third of the sample is contracting. These
discrepancies between local and global volumetric determination seem to be governed by the
magnitude of radial strain developed during cycling, which was recorded to be larger for two-way
cycling at low average stress ratios or one way cycling at high stress ratios.
- For dense samples, the divergence of local and global volumetric strains appeared to develop
only after a certain number of cycles has been applied. The onset of this divergence appears to
depend on the application of higher average cyclic stress ratios and may occur at a very early
stage of cycling for very high stress ratios.
- While the use of enlarged lubricated ends, opposed to frictional restrained ends, may improve the
overall uniformities of deformations, discrepancies between global and local determination of
volumetric strains were still observed for all the testing conditions and fabrication methods,
especially for dense specimens cycled at high average stress ratios.
- Since the discrepancy between local and global volumetric determination was observed for a wide
range of densities and average cyclic stress levels, the use of local instrumentation is
recommended when accurate measurement of the soil response during cyclic loading are
required.
- Given that non-uniform strains developed during these drained cyclic triaxial tests, even in
specimens tested with enlarged lubricated ends and height to diameter ratio of 2, it appears logical
to point out that the same tendency would significantly affect the behaviour during undrained
cyclic triaxial tests. Differential excess pore pressures could build up within specimens, resulting
in water flows within the specimen which would then locally not respect the constant volume
hypothesis of an undrained test. This would have important implications for interpreting the
observed overall mechanical response of such undrained tests.

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10 References
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Figure 1. Assumed conditions of the triaxial sample for barrelling correction.
Figure 2. Assumed deformation models for triaxial samples subjected to compression.

Figure 3. Grain size distribution of Hostun sand.

Figure 4. Triaxial sample instrumented by vertical and radial LVDTs.
Figure 5. Schematic representation of the stress paths imposed during experimental tests listed in Table 1.
Fig. 6 a) and b) Compression test MTRE_5, c) and d) Extension test MTRE_6, and e) and f) two-way cycled test MTRE_1 showing variation of $q/p'$ versus global and local strains, and global and local volumetric strains versus local axial strain, respectively.
Fig. 7. Comparison of local $\varepsilon_a$ and global axial strain $\varepsilon_a(G)$ measurements for tests loaded monotonically to large strains: (a) MTRE tests, (b) MTLE tests, (c) DDLE tests, (d) Extension test.

Fig. 8. Volumetric strain-axial strain curves for tests sheared in compression.
Fig. 9. Comparison between global and local accumulated axial strains (%) during cyclic loading.

Figure 10: a) \( q/p' \) vs. radial strains for test MTRE_5 and b) critical \( q/p' \) at which radial strains change significantly plotted against relative density.
Fig. 11 Accumulated volumetric strains with number of cycles in logarithmic scale.

Fig. 12. Cyclic number at which dense specimens diverge to negative global $\Delta \varepsilon_v$ during drained cyclic loading, against $\eta$. 
Fig. 13. (a) Plot of local (triangle end of arrow) and global (circle) volumetric strains versus relative density and average cyclic stress ratio, (b) Plot of volumetric percentage error versus relative density and average cyclic stress ratio.
11 APPENDIX – Derivation of formulas for volumetric strains

11.1 Parabolic expression for local volumetric strain

Assuming a parabolic variation of diameter: 
\[ D = D_1 - (D_1 - D_0) \left( \frac{2y}{H} \right)^2 \]

Where \( D_1 \) is the diameter at mid-height \( y/H = 0 \), \( D_0 \) is the diameter at the ends \( 2y/H = 1 \) (Fig. 1), and \( 2y \) corresponds to the total gauge length \( L \).

The average diameter over length +\( y \) to -\( y \) can be obtained from:

Volume \( V = \int_{-y}^{y} \pi \frac{D^2}{4} dy \)

\[ D^2 = D_1^2 - 2D_1(D_1 - D_0) \left( \frac{2y}{H} \right)^2 + (D_1 - D_0)^2 \left( \frac{2y}{H} \right)^4 \]

So for a given value of \( H \):

\[ V = \frac{\pi}{2} \left[ D_1^2 - D_0 \left( D_1 - D_0 \right) \left( \frac{2y}{H} \right)^2 + \frac{1}{5} (D_1 - D_0)^2 \left( \frac{2y}{H} \right)^4 \right] (y_0 + \Delta y) \]

During a test \( L \) and \( H \) vary but we assume \( L/H \) remains constant:

\[ V = \frac{\pi}{2} \left[ D_1^2 - D_0 \left( D_1 - D_0 \right) \left( \frac{L}{H} \right)^2 + \frac{1}{5} (D_1 - D_0)^2 \left( \frac{L}{H} \right)^4 \right] (y_0 + \Delta y) \]

The changing diameter at mid-height \( D_1 \) is measured by the radial strain transducer, and the changing gauge length \( L \) is measured by the axial local strain transducers.

Writing:

\[ \Delta y = y - y_0 ; \Delta D = D_1 - D_0 \]

Hence

\[ V = \frac{\pi}{2} \left[ (D_0 + \Delta D)^2 - \frac{2}{3} D_0 (D_0 + \Delta D) (\Delta D) \left( \frac{L}{H} \right)^2 + \frac{1}{5} (\Delta D)^2 \left( \frac{L}{H} \right)^4 \right] (y_0 + \Delta y) \]

The average area \( A = V/L \) and the average diameter \( \bar{D} = \sqrt{\frac{4A}{\pi}} \)

As

\[ V_0 = \frac{\pi}{2} D_0^2 y_0 \] and \( \Delta V = V - V_0 \)

\[ \Delta V = \frac{\pi}{2} \left[ \left( 2D_0 \Delta D \left( 1 - \frac{1}{3} \left( \frac{L}{H} \right)^2 \right) + \Delta D^2 \left( 1 - \frac{2}{3} \left( \frac{L}{H} \right)^2 + \frac{1}{5} \left( \frac{L}{H} \right)^4 \right) \right) (y_0 - \Delta y) + \Delta D^2 \Delta y \right] \] (1)

Eq. (1) can now be divide by \( V_0 \) to give:

\[ \frac{\Delta V}{V_0} = \left[ 2 \left( 1 - \frac{1}{3} \left( \frac{L}{H} \right)^2 \right) \frac{\Delta D}{D_0} + \left( -1 + \frac{2}{3} \left( \frac{L}{H} \right)^2 - \frac{1}{5} \left( \frac{L}{H} \right)^4 \right) \frac{\Delta D^2}{D_0^2} \right] \left( 1 - \frac{\Delta y}{y_0} \right) + \frac{\Delta y}{y_0} \] (2)

and introducing the expression for volumetric strains \( (\varepsilon_v = -\Delta V/V_0) \), axial strains \( (\varepsilon_a = -\Delta y/y_0) \) and radial strains \( (\varepsilon_r = -\Delta D/D_0) \), it is possible to obtain:
\[ \varepsilon_v = \left[ 2 \left( 1 - \frac{1}{3} \left( \frac{L}{H} \right)^2 \right) \right] \varepsilon_r + \left( -1 + \frac{2}{3} \left( \frac{L}{H} \right)^2 - \frac{1}{5} \left( \frac{L}{H} \right)^4 \right) \epsilon_r^2 (1 - \varepsilon_a) + \varepsilon_a \quad (3) \]

11.2 Sinusoidal expression for local volumetric strain

Assuming a sinusoidal variation of diameter:

\[ D = D_0 - \frac{\Delta D}{2} \left( 1 + \cos \left( \frac{2\pi y}{H} \right) \right) \]

Where \( D_0 + \Delta D \) is the diameter at mid-height \( 2y/H = 0 \), and \( D_0 \) is the diameter at the ends, \( 2y/H = 1 \) (Fig. 1).

The average diameter over length \( +y \) to \(-y \) can be obtained from:

Volume \( V = \int_{-y}^{y} \frac{\pi}{4} D^2 \, dy \)

\[ D^2 = D_0^2 - D_0 \cdot \Delta D \cdot \left( 1 + \cos \left( \frac{2\pi y}{H} \right) \right) + \frac{\Delta D^2}{4} \left( 1 + 2\cos \left( \frac{2\pi y}{H} \right) + \cos^2 \left( \frac{2\pi y}{H} \right) \right) \]

So for a given value of \( H \):

\[ V = \frac{\pi}{2} \left[ D_0^2 - D_0 \Delta D + \frac{3}{8} \Delta D^2 + \frac{\Delta D}{\pi(\pi^2)} \left( -D_0 + \frac{\Delta D}{2} \right) \sin \left( \frac{2\pi y}{H} \right) + \frac{\Delta D^2}{16\pi^2} \sin \left( \frac{4\pi y}{H} \right) \right] \]

During a test \( L \) and \( H \) vary but we assume \( L/H = \) remains constant

So \( V = \frac{\pi}{2} \left[ D_0^2 - D_0 \Delta D + \frac{3}{8} \Delta D^2 + \frac{\Delta D}{\pi(\pi^2)} \left( -D_0 + \frac{\Delta D}{2} \right) \sin \left( \frac{\pi y}{H} \right) + \frac{\Delta D^2}{16\pi^2} \sin \left( \frac{2\pi y}{H} \right) \right] (y_0 - \Delta y) \]

The changing diameter at mid-height \( D_l \) is measured by the radial strain transducer, and the changing gauge length \( y \) is measured by the axial local strain transducers.

Writing \( \Delta y = y - y_0 ; \Delta D = D_l - D_0 \)

\[ V = \frac{\pi}{2} \left[ D_0^2 + D_0 \Delta D + \frac{3}{8} \Delta D^2 + \frac{\Delta D}{\pi(\pi^2)} \left( -D_0 + \frac{\Delta D}{2} \right) \sin \left( \frac{\pi y}{H} \right) + \frac{\Delta D^2}{16\pi^2} \sin \left( \frac{2\pi y}{H} \right) \right] (y_0 - \Delta y) + D_0^2 \Delta y \quad (1) \]

As \( V_0 = \frac{\pi}{2} D_0^2 y_0 \) and \( \Delta V = V_v - V \)

\[ \Delta V = \frac{\pi}{2} \left[ D_0 \cdot \Delta D - \frac{3}{8} \Delta D^2 - \frac{\Delta D}{\pi(\pi^2)} \left( -D_0 + \frac{\Delta D}{2} \right) \sin \left( \frac{\pi y}{H} \right) - \frac{\Delta D^2}{16\pi^2} \sin \left( \frac{2\pi y}{H} \right) \right] (y_0 - \Delta y) + D_0^2 \Delta y \]

Expressing this as strains by dividing by \( V_0 \), it is possible to obtain a final relationship

\[ \frac{\Delta V}{V_0} = \left[ 1 + \frac{1}{\pi L/H} \sin \left( \frac{\pi y}{H} \right) \right] \left( \frac{\Delta D}{D_0} + \left( -\frac{3}{8} - \frac{\sin \left( \frac{\pi y}{H} \right)}{2\pi^2(\pi^2)} \right) \frac{\Delta D^2}{D_0^2} \right) \left( 1 - \frac{\Delta y}{y_0} \right) + \frac{\Delta y}{y_0} \quad (2) \]

\[ \varepsilon_{voi} = \left[ 1 + \frac{H}{\pi L} \sin \left( \frac{\pi y}{H} \right) \right] \varepsilon_r + \left( -\frac{3}{8} - \frac{\sin \left( \frac{\pi y}{H} \right)}{2\pi^2(\pi^2)} \right) \epsilon_r^2 \left( 1 - \varepsilon_a \right) + \varepsilon_a \quad (3) \]