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Consistent estimation of high-dimensional factor models when the factor number is over-estimated

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Abstract

A high-dimensional \( r \)-factor model for an \( n \)-dimensional vector time series is characterised by the presence of a large eigengap (increasing with \( n \)) between the \( r \)-th and the \( (r + 1) \)-th largest eigenvalues of the covariance matrix. Consequently, Principal Component Analysis (PCA) is a popular estimation method for factor models and its consistency, when \( r \) is correctly estimated, is well-established in the literature. However, various factor number estimators often suffer from the lack of an obvious eigengap in finite samples. We empirically show that they tend to over-estimate the factor number in the presence of moderate correlations in the idiosyncratic (not factor-driven) components, and further prove that over-estimation of \( r \) can result in non-negligible errors in the PCA estimators. To remedy this problem, we propose two new estimators based on capping or scaling the entries of the sample eigenvectors, which are less sensitive than the PCA estimator to the over-estimation of \( r \) \textit{without} knowing the true factor number. We show both theoretically and empirically that the two estimators successfully controls for the over-estimation error, and demonstrate their good performance on macroeconomics and financial time series datasets.

\textbf{Key words}: Factor models, principal component analysis, factor number, sample eigenvectors.

1 Introduction

Factor modelling is a popular approach to dimension reduction in high-dimensional time series analysis. It has been successfully applied to large panels of time series, e.g., for building low-dimensional indicators of the whole economic activities and forecasting models \cite{Stock2002a, Stock2002b}, and for analysing dynamic brain connectivity using high-dimensional fMRI data. \textsuperscript{1}Department of Statistics, London School of Economics, Houghton Street, London WC2A 2AE, UK. Email: m.barigozzi@lse.ac.uk.

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data (Ting et al., 2017). In this paper, we consider one of the most general factor models in the literature, the approximate dynamic factor model, which permits serial dependence in the factors and both serial and cross-sectional dependence among the idiosyncratic components (see Bai, 2003, and Fan et al., 2013).

More specifically, we investigate the problem of factor modelling an \( n \)-dimensional vector time series, \( \{x_t = (x_{1t}, \ldots, x_{nt})^\top : t = 1, \ldots, T\} \), as

\[
x_{it} = \lambda_i^\top f_t + \varepsilon_{it},
\]

where \( \lambda_i \) and \( f_t \) are \( r \)-dimensional vectors of loading and factors, respectively (we adopt the notations from the time series factor model literature where \( n \) denotes the dimensionality and \( T \) the sample size). We refer to \( \chi_{it} = \lambda_i^\top f_t \) as the common component and \( \varepsilon_{it} \) the idiosyncratic component, and assume the number of factors, \( r \), to be fixed independent of \( n \) and \( T \).

The main identifying assumption of (1) is the existence of a large (increasing with \( n \)) eigengap between the \( r \) leading eigenvalues of the covariance matrix and the remaining ones. As a consequence, a natural way of estimating (1) is via Principal Component Analysis (PCA), through which the common components are estimated as the projection of the data onto the space spanned by the \( r \) leading eigenvectors of the sample covariance matrix. Intuitively, since the eigengap is assumed to increase with \( n \), the more series are pooled together by means of PCA, the more the contribution of the factors to the total co-variation in the data is likely to emerge over the idiosyncratic components. As a consequence, PCA allows for consistent estimation of an approximate dynamic factor model when both \( n, T \to \infty \).

However, the theoretical properties of PCA estimators have always been investigated under the assumption that the number of factors \( r \) is known, and the problem of determining \( r \) has typically been treated separately. Many methods exist for factor number estimation: Bai and Ng (2002), Alessi et al. (2010), Onatski (2010), Ahn and Horenstein (2013), Bai and Ng (2017), Yu et al. (2018) and Trapani (2018), to name a few, all of which exploit the postulated existence of the eigengap. On the other hand, it is often difficult to identify the large eigengap in finite samples. In particular, the presence of moderate cross-sectional correlations in the idiosyncratic components may shrink the empirical eigengap by introducing ‘weak’ factors (Onatski, 2012), and lead to the over-estimation of \( r \). Moreover, as noted in Barigozzi et al. (2018), instabilities in the factor structure tend to enlarge the factor space and introduce further difficulties to determining the number of factors. Finally, different estimators frequently return different results, thus making it ambiguous for the user to choose a single value to rely on.

The question is, what to do if we have a range of possible candidate estimators of \( r \), or if we believe that none of the estimators is reliable? One solution may be to use the largest number of factors returned by available methods, or set it to be even larger, with the expectation of avoiding the hazard of under-estimating the factor-driven variation, which is a problem
without any clear solution. Indeed, Onatski (2015) noted the negligibility of the cross-sectional average of the estimation error in the common components estimated via PCA with $k > r$ as the number of factors. However, as we show later, over-estimation of $r$ can incur non-negligible estimation error when considering the worst case scenarios for individual common components which, to the best of our knowledge, has not been investigated in the literature.

1.1 Our contribution

To mitigate this problem, we propose modified PCA estimators which are less sensitive than the PCA estimator to the over-estimation of the factor numbers and, therefore, make our estimation procedure less reliant on a precise estimate of $r$.

The factor model (1) is characterised by the following eigengap conditions:

(C1) there exist some fixed $c_j, \bar{c}_j$ such that for $j = 1, \ldots, r$,

$$0 < c_j < \liminf_{n \to \infty} \frac{\mu_{\chi,j}}{n} \leq \limsup_{n \to \infty} \frac{\mu_{\chi,j}}{n} < \bar{c}_j < \infty$$

and $\bar{c}_{j+1} < c_j$ for $j \in [r - 1],$

(C2) $\mu_{\varepsilon,1} < C_\varepsilon$, for any $n$

(see e.g. Fan et al. (2013), where $\mu_{\chi,j}$ and $\mu_{\varepsilon,j}$ denote the $j$-th largest eigenvalues of the covariance matrices of the common and idiosyncratic components. From (C1) it follows that $w_{\chi,j} = (w_{\chi,1j}, \ldots, w_{\chi,nj})^T$, the standardised eigenvector corresponding to $\mu_{\chi,j}$, has its coordinates bounded as $O(1/\sqrt{n})$ for all $j \leq r$ (see Section 2.2). Also, thanks to the eigengap and the Davis-Kahan theorem (Yu et al. 2015), we have the coordinates of $\hat{w}_{x,j}$, $j \leq r$, the $r$ leading eigenvectors of the sample covariance matrix of the data, bounded asymptotically as $\max_{1 \leq i \leq n} |\hat{w}_{x,ij}| = O_p(1/\sqrt{n})$. On the other hand, precisely due to the lack of this eigengap, meaningful control of the behaviour of $\hat{w}_{x,ij}, j \geq r + 1$ is not obvious under the time series factor model in (1).

Motivated by these observations, we propose the following modifications to $\hat{w}_{x,j}$, which ensure that the entries of the modified eigenvectors are bounded by $c_w/\sqrt{n}$ for some fixed $c_w > 0$.

1. **Capping:** returns $\hat{w}_{x,j}^{\text{cap}}$ with its entries

$$\hat{w}_{x,ij}^{\text{cap}} = \hat{w}_{x,ij} I\left( |\hat{w}_{x,ij}| \leq \frac{c_w}{\sqrt{n}} \right) + \text{sign}(\hat{w}_{x,ij}) \cdot \frac{c_w}{\sqrt{n}} I\left( |\hat{w}_{x,ij}| > \frac{c_w}{\sqrt{n}} \right).$$

2. **Scaling:** returns

$$\hat{w}_{x,j}^{\text{sc}} = \nu_j^{-1/2} \hat{w}_{x,j} \quad \text{with} \quad \nu_j = \max \left\{ 1, \frac{\sqrt{n}}{c_w} \max_{i \in [n]} |\hat{w}_{x,ij}| \right\}.$$
While both conceptually and computationally simple, the modified PCA estimators obtained with $\hat{w}_{x,j}^{\text{cap}}$ and $\hat{w}_{x,j}^{\text{sc}}$, show good theoretical and empirical performance in curtailing the error introduced by spurious factors when $\hat{r} \geq r + 1$, compared to the PCA estimator. Asymptotically, both capping and scaling do not alter the $r$ leading sample eigenvectors without the knowledge of the true number of $r$. In particular, we show that the scaled PCA estimator obtained with $\hat{w}_{x,j}^{\text{sc}}$, performs as well as the ‘oracle’ estimator obtained as the PCA estimator with the known $r$, provided that $\hat{r} \geq r$.

1.2 Relationship to the existing literature

The modified PCA estimators are related yet distinguished from the literature on the recovery of low-rank matrices often referred to as the spiked covariance matrices. Under the assumption of serial independence and normality, [Gavish and Donoho, 2017], [Donoho et al., 2018] and [Donoho and Ghorbani, 2018] proposed optimal non-linear singular value shrinkage methods for different loss functions. Further assuming a sparse structure, [Cai et al., 2013] and [Cai et al., 2015] studied estimation of the covariance matrix and the principal subspace. Although we consider the estimation of the common components driven by pervasive factors, rather than the covariance matrix itself, we show the link between our proposed modifications of the PCA estimator and eigenvalue-based shrinkage in Section 3.3. [Bai and Ng, 2017] adopted the singular value shrinkage for minimum-rank factor analysis under time series factor models.

Our approach is distinguished from theirs in the sense that our aim is to bypass the dependence on the factor number estimation itself, in estimating the common components.

We mention two other approaches to factor analysis in time series. First, assuming that all serial dependence in the data is captured by the factors, thus imposing the idiosyncratic component to be a white noise, [Lam et al., 2011] and [Lam and Yao, 2012] proposed an alternative approach to factor model estimation. Since their method is also based on eigenanalysis of a suitable covariance matrix, our methodology can be easily adapted to this case as well. Second, [Forni et al., 2000] considered a richer factor structure where factors have also lagged effects on the data. Estimation of such model is in general based on spectral PCA, but other approaches exist that require standard PCA as the initial or the final steps (e.g., [Forni et al., 2005], [Bai and Ng, 2007], [Forni et al., 2009], and [Doz et al., 2011]), to which our proposed modifications can be adapted.

1.3 Structure of the paper

The rest of the paper is organised as follows. We introduce the approximate factor model and its assumptions in Section 2, where we also discuss its estimation via PCA and investigate the impact of over-estimating the factor number. In Section 3, we motivate and introduce the modified PCA estimators and study their theoretical properties. Comparative simulation study of various PCA-based estimators is conducted in Section 4, and we apply the proposed
estimators to macroeconomics and financial data analysis in Section 5. All the proofs of the theoretical results and additional simulation results are provided in Appendix.

Notation

We write \([m]\) in place of \(\{1, \ldots, m\}\). For a given \(m \times n\) matrix \(B\) with \(b_{ij}\) denoting its \((i, j)\) element, its spectral norm is defined as \(\|B\| = \sqrt{\mu_1(BB^\top)}\), where \(\mu_k(C)\) denotes the \(k\)-th largest eigenvalue of \(C\), and its Frobenius norm as \(\|B\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n b_{ij}^2}\). The sub-exponential norm of a random variable \(X\) is defined as \(\|X\|_{\psi_1} = \inf_k \{k : \mathbb{E}[\exp(|X|/k)] \leq 2\}\).

For a given set \(\Pi\), we denote its cardinality by \(|\Pi|\). Also, we use the notations \(a \vee b = \max(a, b)\) and \(a \wedge b = \min(a, b)\). Besides, \(a \asymp b\) indicates that \(a\) is of the order of \(b\), and \(a \gg b\) indicates that \(a^{-1}b \to 0\). We denote an \(n \times n\)-identity matrix by \(I_n\), and use \(\varphi_i\) to denote an \(n\)-vector of zeros except for its \(i\)-th element being one.

2 Approximate dynamic factor model and its estimation

2.1 Model and assumptions

Recall the factor model in (1), where an \(n\)-dimensional vector time series \(x_t = (x_{1t}, \ldots, x_{nt})^\top\) is divided into the common component \(\chi_t = (\chi_{1t}, \ldots, \chi_{nt})^\top = \Lambda f_t\) driven by the vector of \(r\) latent factors \(f_t = (f_{1t}, \ldots, f_{rt})^\top\), with \(\Lambda = [\lambda_1, \ldots, \lambda_n]^\top = [\lambda_{ij}, i \in [n], j \in [r]]\) as the \(n \times r\)-matrix of loadings, and the idiosyncratic component \(\varepsilon_t = (\varepsilon_{1t}, \ldots, \varepsilon_{nt})^\top\). Without loss of generality, we assume \(\mathbb{E}(f_{jt}) = \mathbb{E}(\varepsilon_{it}) = 0\) for all \(i, j, t\).

We now list and motivate the assumptions imposed on the factor model (1); see e.g., Fan et al. (2013) and Barigozzi et al. (2018) for similar conditions.

Assumption 1 (Identification).

(i) \(T^{-1} \sum_{t=1}^T \mathbb{E}(f_tf_t^\top) = I_r\).

(ii) There exists a positive definite \(r \times r\) matrix \(H\) with distinct eigenvalues and such that \(n^{-1}\Lambda^\top \Lambda \to H\) as \(n \to \infty\).

(iii) There exists \(\bar{\lambda} \in (0, \infty)\) such that \(|\lambda_{ij}| < \bar{\lambda}\) for all \(i, j\).

(iv) There exists \(C_\varepsilon \in (0, \infty)\) such that, for any \(t \geq 1\),

\[\sum_{i, i' = 1}^n a_ia_{i'} \mathbb{E}(\varepsilon_{it}\varepsilon_{i't}) < C_\varepsilon,\]

for any sequence of coefficients \(\{a_i\}^n_{i=1}\) satisfying \(\sum_{i=1}^n a_i^2 = 1\).
We adopt the normalisation given in Assumption 1 (i)–(ii) for the purpose of identification; in general, factors and loadings are recoverable up to a linear invertible transformation only. Assumption 1 (iv) allows for mild cross-sectional dependence across idiosyncratic components. In other words, we are considering an approximate factor structure, as opposed to the classical exact factor model where \( \varepsilon_t \) is assumed to be uncorrelated cross-sectionally. We note that Assumption 1 is sufficient in guaranteeing the commonness of \( \chi_t \) and the idiosyncracy of \( \varepsilon_t \) according to Definitions 2.1 and 2.2 of Hallin and Lippi (2013).

To further motivate and understand the assumptions, we introduce the notations
\[
\Gamma_f = \frac{1}{T} \sum_{t=1}^{T} E(f_t f_t^\top), \quad \Gamma_\varepsilon = \frac{1}{T} \sum_{t=1}^{T} E(\varepsilon_t \varepsilon_t^\top), \quad \Gamma_\chi = \Lambda \Gamma_f \Lambda^\top, \quad \text{and} \quad \Gamma_x = \Gamma_\chi + \Gamma_\varepsilon.
\]

If \( f_t \) and \( \varepsilon_t \) are covariance stationary, then these matrices are the corresponding population covariance matrices. Also, we denote the eigenvalues (in non-increasing order) of \( \Gamma_\chi \), \( \Gamma_\varepsilon \) and \( \Gamma_x \) by \( \mu_{\chi,j} \), \( \mu_{\varepsilon,j} \) and \( \mu_{x,j} \), respectively. Then, Assumption 1 lead to (C1)–(C2) in Section 1.1, i.e., \( \mu_{\chi,j} \), \( j \in [r] \) are diverging as \( n \to \infty \) and, in particular, are of order \( n \), whereas \( \mu_{\varepsilon,1} \) is bounded for any \( n \). The latter coincides with Definition 2 in Chamberlain and Rothschild (1983), Assumption 4 in Forni et al. (2009) and Assumption 2 in Fan et al. (2013), and is also comparable to Assumption C.4 in Bai (2003).

Moreover, (C1)–(C2) imply that, due to Weyl’s inequality, the eigenvalues of \( \Gamma_x \) satisfy the following eigengap conditions:

(C3) the \( r \) largest eigenvalues, \( \mu_{x,1}, \ldots, \mu_{x,r} \), are diverging linearly in \( n \) as \( n \to \infty \);

(C4) the \( (r + 1) \)-th largest eigenvalue, \( \mu_{x,r+1} \), stays bounded for any \( n \).

From (C1)–(C4) above, it is clear that for consistent estimation of the common components, approximate factor models need to be studied in the asymptotic limit where \( n \to \infty \), i.e., these models enjoy what is sometimes referred to as the blessing of dimensionality. In particular, we require:

**Assumption 2.** \( n \to \infty \) as \( T \to \infty \), with \( n = O(T^\kappa) \) for some \( \kappa \in (0, \infty) \).

Under Assumption 2, we operate in high-dimensional settings that permit \( n \gg T \), unlike in the random matrix theory literature where it is typically assumed that \( n/T \to \gamma \in (0, \infty) \). Furthermore we assume:

**Assumption 3** (Tail behaviour).

(i) \( \max_{j \in [r]} \max_{t \in [T]} \| f_{jt} \|_{\psi_1} \leq B_f \) for some \( B_f \in (0, \infty) \).

(ii) \( \max_{t \in [T]} \| \varepsilon_t \|_{\psi_1} \leq B_\varepsilon \) for some \( B_\varepsilon \in (0, \infty) \), where \( \| \varepsilon_t \|_{\psi_1} = \sup_{v \in \mathbb{R}^n : \| v \|_{\psi_1} = 1} \| v^\top \varepsilon_t \|_{\psi_1} \).

**Assumption 4** (Dependence).

\[ \text{(i) max}_{j \in [r]} \text{max}_{t \in [T]} \| f_{jt} \|_{\psi_1} \leq B_f \text{ for some } B_f \in (0, \infty). \]

\[ \text{(ii) max}_{t \in [T]} \| \varepsilon_t \|_{\psi_1} \leq B_\varepsilon \text{ for some } B_\varepsilon \in (0, \infty), \text{ where } \| \varepsilon_t \|_{\psi_1} = \sup_{v \in \mathbb{R}^n : \| v \|_{\psi_1} = 1} \| v^\top \varepsilon_t \|_{\psi_1}. \]

\[ \text{Assumption 4 (Dependence).} \]
(i) $E(f_{jt} \varepsilon_{it'}) = 0$ for all $i \in [n], j \in [r]$ and $t, t' \in [T]$.

(ii) Denoting the $\sigma$-algebra generated by $\{(f_t, \varepsilon_t), s \leq t \leq e\}$ by $\mathcal{F}_e^s$, let

$$\alpha(k) = \max_{1 \leq t \leq T} \sup_{A \in \mathcal{F}_t^{\infty}} |P(A)P(B) - P(A \cap B)|,$$

Then, there exist some fixed $c_\alpha, \beta \in (0, \infty)$, such that we have $\alpha(k) \leq \exp(-c_\alpha k^\beta)$ for all $k \in \mathbb{Z}^+$. The exponential-type tail conditions in Assumption 3, along Assumption 4 (ii), allow us to control the deviation of sample covariance estimates from their population counterparts via Bernstein-type inequality (see e.g., Theorem 1.4 of Bosq, 1998, and Theorem 1 of Merlevède et al., 2011) under the time series factor model. We stress that either strict or weak stationarity of $f_{jt}$ and $\varepsilon_{it}$ is not required in performing the PCA-based estimation, provided that the loadings are time-invariant, see Barigozzi et al. (2018) for further discussion under a piecewise stationary factor model. Finally, it is possible to relax Assumption 4 (i) and allow for weak dependence between the factors and the idiosyncratic components by requiring that $E(T^{-1} \sum_{t=1}^T f_{jt} \varepsilon_{it'}) \leq M/\sqrt{T}$ for some fixed $M$ (c.f. Assumption D of Bai and Ng, 2002).

2.2 Estimation via Principal Component Analysis

The most common way to estimate the approximate factor model (1) is by means of PCA. The asymptotic properties of the PCA estimator have been studied in many contributions: in particular, we refer to Fan et al. (2013) where a set-up very similar to ours is considered. Let us summarise the main features of the PCA estimator. Let $\hat{w}_{x,j}$ denote the $j$-th normalised leading eigenvector of the sample covariance $\hat{\Gamma}_x = T^{-1} \sum_{t=1}^T x_t x_t^\top$. Then, for some estimator of the number of factors $\hat{r} \geq 1$, we have the loadings estimator

$$\hat{\Lambda} = [\hat{\lambda}_1, \ldots, \hat{\lambda}_n]^\top = \sqrt{n} \hat{W}_{x,1:\hat{r}} = \sqrt{n}[\hat{w}_{x,1}, \ldots, \hat{w}_{x,\hat{r}}]^\top,$$

while the factors are estimated by projecting $x_t$ onto the space spanned by the estimated loadings:

$$\hat{f}_t = (\hat{\Lambda}^\top \hat{\Lambda})^{-1} \hat{\Lambda}^\top x_t = \frac{1}{\sqrt{n}} \hat{W}_{x,1:\hat{r}}^\top x_t.$$

Although both loadings and factors are not identifiable unless further conditions are assumed, the common components are always identified and estimated as

$$\hat{\chi}_{it}^{\text{pca}} = \hat{\lambda}_i^\top \hat{f}_t = \sum_{j=1}^{\hat{r}} \hat{w}_{x,ij} \hat{w}_{x,j}^\top x_t. \quad (2)$$

Theorem 1 of Barigozzi et al. (2018) establishes a uniform bound on the estimation error over
Proposition 1 (Theorem 1 of Barigozzi et al., 2018). Under Assumptions 1–4, the PCA estimator \( \hat{\chi}_{it} \) in (2) with \( \hat{r} = r \) satisfies

\[
\max_{i \in [n]} \max_{t \in [T]} |\hat{\chi}_{it}^{\text{pca}} - \chi_{it}| = O_p \left( \left( \sqrt{\frac{\log n}{T}} \vee \frac{1}{\sqrt{n}} \right) \log T \right).
\]

Two key results are required for proving Proposition 1. First, we make use of the eigengap between \( \mu_{x,r} \) and \( \mu_{x,r+1} \) that increases as \( n \) increases (see (C3)–(C4)), which ensures that the eigenspace of \( \Gamma_{\chi} \) are consistently estimated by the \( r \) leading eigenvectors of \( \hat{\Gamma}_{x} \). More specifically, there exists a diagonal \( r \times r \) matrix \( S \) with entries \( \pm 1 \), such that

\[
\| \hat{W}_{x} - W_{\chi} S \| = O_p \left( \sqrt{\frac{\log n}{T}} \vee \frac{1}{\sqrt{n}} \right),
\]

(3)

for \( \hat{W}_{x} = \hat{W}_{x,1:r} \), which follows from the modified Davis-Kahan theorem of Yu et al. (2015), see Lemma 3 in Barigozzi et al. (2018) for its proof.

Second, denoting the eigendecomposition of the covariance matrix of the common components by \( \Gamma_{\chi} = W_{\chi} M_{\chi} W_{\chi}^\top \) with \( M_{\chi} = \text{diag}(\mu_{x,1}, \ldots, \mu_{x,r}) \), we have

\[
\max_{i \in [n]} \sqrt{\sum_{j=1}^{r} w_{\chi,ij}^2} = \max_{i \in [n]} \| \varphi_{\chi}^i \| \leq \max_{i \in [n]} \| \varphi_{\chi}^i \Gamma_{\chi} \| \| W_{\chi} \| \| M_{\chi}^{-1} \| = O \left( \frac{1}{\sqrt{n}} \right)
\]

(4)

under (C1), i.e., asymptotically, each element of \( W_{\chi} \) is \( O(1/\sqrt{n}) \). As mentioned in the Introduction, this result serves as the main motivation behind introducing the modified PCA estimators in Section 3.

Remark 1 (Optimality of PCA). Under the factor model (1) satisfying (C1), PCA is minimax optimal in recovering the principal subspace spanned by the \( r \) factors, provided that \( \varepsilon_t \sim_{\text{iid}} N_0(0, \sigma^2 I_n) \) for some \( \sigma^2 > 0 \), see Theorem 5 of Cai et al. (2013). There, the minimax rate of convergence for \( \mathbb{E} \| \hat{W}_{x} \hat{W}_{x}^\top - W_{\chi} W_{\chi}^\top \|_F^2 \) is given as \( O\left( rn/\left(\mu_{x,r} T\right) \right) \) which, combined with (C1) is comparable to the convergence rate reported in (3).

Remark 2 (De-localisation of factor loadings). Dobriban and Owen (2018) studied the problem of selecting the number of ‘perceptible’ factors using the generalised Marčenko-Pastur distribution under an exact factor model. There, it is assumed that factor loadings after scaling are ‘de-localised’ which, in our framework, amounts to \( \max_{i \in [n]} \sum_{j=1}^{r} |w_{\chi,ij}| \to 0 \) as \( n, T \to \infty \). This is closely related to the properties of the leading eigenvectors of \( \Gamma_{x} \) as noted in (4), which is a direct consequence of Assumption 1 requiring that the factors are pervasive cross-sectionally.
2.3 (Over-)estimation of $r$

In practice, the true number of factors $r$ is unknown and needs to be estimated, and the problem of estimating the number of factors has been one of the most researched problems in the factor model literature; see the references in the Introduction. Based on the conditions (C3)-(C4) the prevailing approach to the estimation of $r$ is to identify a ‘large’ gap between the successive estimated eigenvalues $\hat{\mu}_{x,j}$, $j = 1, \ldots, r_{\text{max}}$ of the sample covariance matrix $\hat{\Gamma}_x$, where $r_{\text{max}}$ denotes the maximum allowable number of factors often required as an input parameter to the estimation procedure. Here we focus on the two most popular methods.

The information criterion-based method proposed in Bai and Ng (2002) estimates $r$ as

$$\hat{r} = \arg \min_{1 \leq q \leq r_{\text{max}}} \text{IC}(q), \text{ where } \text{IC}(q) = \log \left( \frac{1}{n} \sum_{j=q+1}^{n} \hat{\mu}_{x,j} \right) + q \cdot g(n, T), \quad (5)$$

with $g(n, T)$ being a penalty function satisfying $g(n, T) \to 0$ and $\{(n \wedge T) \cdot g(n, T)\} \to \infty$ as $n, T \to \infty$. The eigenvalue ratio-based estimator by Ahn and Horenstein (2013), returns

$$\hat{r} = \arg \max_{1 \leq q \leq r_{\text{max}}} \text{GR}(q), \text{ where } \text{GR}(q) = \log \left( \frac{1 + \hat{\mu}_{x,q}}{1 + \hat{\mu}_{x,q+1}} \right) \text{ with } \hat{\mu}_{x,q} = \frac{\hat{\mu}_{x,q}}{\sum_{j=q+1}^{n} \hat{\mu}_{x,j}}. \quad (6)$$

Implicitly, the information criterion in (5) performs thresholding on the scaled sample eigenvalues $\hat{\mu}_{x,q}$ with respect to $g(n, T)$, and selects an index $q$ among those that correspond to $\hat{\mu}_{x,q}$ surviving the thresholding. On the other hand, the eigenvalue ratio approach in (6) considers the ratio of the successive scaled eigenvalues without taking into account the size of the eigenvalues. This difference frequently leads to distinct estimators from the different approaches, not to mention that, as shown in Alessi et al. (2010), the various choices of $g(n, T)$ often result in different factor number estimators.

Moreover, in finite samples, the lack of an obvious eigengap in the empirical eigenvalues poses a challenge in the estimation of $r$. The estimated number of factors is highly variable as the following quantities vary: the dimensions $n$ and $T$, the degree of cross-sectional correlations in the idiosyncratic components, and the signal-to-noise ratio between the common and idiosyncratic components (represented by the ratio between $\text{Var}(\chi_{it})$ and $\text{Var}(\varepsilon_{it})$). Moreover, Barigozzi et al. (2018) and the references therein showed that large instabilities in the loadings enter the factor model as additional factors, which brings in further difficulties to the problem of factor number estimation. We empirically illustrate the sensitivity of the factor number estimators in Section 4 (Figure 1), and we refer also to the numerical studies in Ahn and Horenstein (2013) for further evidence.

Obviously, when $\hat{r} < r$, the PCA estimator (2) or indeed, any estimator of the common component, does not capture the contribution from more than one factors, which inevitably incurs non-negligible error. To circumvent this problem, the user may then be tempted to
increase the number of factors based on the intuition that the contribution of factors beyond the \( r \)-th one is negligible, and we focus our discussion to the case when \( \hat{r} \geq r \). While Onatski (2015) shows in his Proposition 1 that over-estimation errors, once aggregated over cross-sections and time, is negligible, a formal analysis of the impact of the over-estimated factor number on individual common component estimators has not yet been conducted to the best of our knowledge.

Recalling the construction of the PCA estimator, we have the following decomposition of the estimation error:

\[
\hat{\chi}_{it}^{\text{pca}} - \chi_{it} = \left\{ \sum_{j=1}^{r} \hat{w}_{x,i} \hat{w}_{x,j}^\top x_t - \chi_{it} \right\} + \sum_{j=r+1}^{\hat{r}} \hat{w}_{x,i} \hat{w}_{x,j}^\top x_t. \tag{7}
\]

The rate of convergence for the first term on the right hand side of (7) is given in Proposition 1. Our interest lies in the theoretical treatment of the second term representing the over-estimation error, which faces at least two challenges.

(a) \( \hat{w}_{x,j}, j \in [\hat{r}] \) are obtained from performing PCA on the full sample covariance matrix and thus are dependent on \( x_t, t \in [T] \), which makes it difficult to analyse the stochastic properties of \( \hat{w}_{x,j}^\top x_t \).

(b) The large eigengap between \( \mu_{\chi,r} \) and \( \mu_{\chi,r+1} = 0 \) as noted in (C1) plays a key role in controlling the distance between the empirical principal subspace spanned by the \( r \) leading eigenvectors of \( \hat{\Gamma}_x \) and its population counterpart, as reported in (3). On the other hand, due to the lack of eigengap between successive \( \mu_{x,j}, j \geq r + 1 \) (see (C4)), the behaviour of \( \hat{w}_{x,j} \) for \( j \geq r + 1 \) is not controlled in a meaningful way as \( n \) grows large.

Without any further assumption, a uniform bound for the over-estimation error is derived as

\[
\max_{i \in [n]} \max_{t \in [T]} \left| \sum_{j=r+1}^{\hat{r}} \hat{w}_{x,i} \hat{w}_{x,j}^\top x_t \right| \leq \sum_{j=r+1}^{\hat{r}} \max_{i \in [n]} \| \hat{w}_{x,i} \| \| \hat{w}_{x,j} \| \cdot \max_{t \in [T]} \| x_t \| = O_p(\sqrt{n \log T}) \tag{8}
\]

under Assumptions 3, which is not very informative.

To remedy (a), we adopt the idea of ‘sample splitting’ in a simpler setting where \( x_t \) is serially independent. Letting \( T = 2T_1 \) be even, we define \( x_t^{(1)} = x_{2t-1} \) and \( x_t^{(2)} = x_{2t} \) for all \( t \in [T] \); \( \chi_{t}^{(\ell)} \) and \( \epsilon_{t}^{(\ell)} \) are defined analogously for \( \ell = 1, 2 \). Denoting by \( \hat{w}_{x,j}^{(\ell)} \) the \( j \)-th leading eigenvector of \( \hat{\Gamma}_x^{(\ell)} = T_1^{-1} \sum_{t=1}^{T_1} x_t^{(\ell)} (x_t^{(\ell)})^\top \), we obtain

\[
\hat{\chi}_{it}^{(m)} = \sum_{j=1}^{\hat{r}} \hat{w}_{x,i}^{(\ell)} (\hat{w}_{x,j}^{(\ell)})^\top x_t^{(m)}, \tag{9}
\]
for \((\ell, m) \in \{(1, 2), (2, 1)\}\) and a given \(\hat{r}\) and finally, set \(\hat{\chi}^{\text{pca}}_{2t-1} = \hat{\chi}_t^{(1)}\) and \(\hat{\chi}^{\text{pca}}_{2t} = \hat{\chi}_t^{(2)}\) for all \(t \in [T]\). In other words, the common components are estimated for each split sample as projections onto the principal subspace of the other sample. We emphasise that the sample splitting is adopted solely for the theoretical analysis in order to quantify the over-estimation error in the simplest scenario. In practice, the use of full sample is recommended as (i) after all, serial independence may not be a valid assumption, and (ii) finite sample estimation accuracy may suffer due to sample splitting when \(T\) is small, see the discussion in [Wang and Samworth (2018)] and also Section 4.2 for simulation results comparing the full sample and split sample estimators.

As noted in (b) above, we cannot comment on the ‘consistency’ of \(\hat{w}_{x,j}^{(\ell)}\) as \(n\) grows, whether they estimate the leading eigenvectors of \(\Gamma_{\varepsilon}\) or not, nor such is desired in factor analysis. Therefore, we study the PCA estimator \(\hat{\chi}_{it}^{\text{pca}}\) when \(\hat{r} > r\) in two different cases defined according to the behaviour of \(\hat{w}_{x,j}^{(\ell)}\), \(j \geq r + 1\), rather than that of any population quantities.

**Proposition 2.** Let Assumptions 1–3 and 4 (i) hold and in addition, let \(x_t\) be serially independent over \(t \in [T]\). Suppose \(\hat{r} \geq r + 1\).

**Case 1:** When \(\hat{w}_{x,ij}^{(\ell)} = O_p(1/\sqrt{n})\) for all \(i \in [n]\), \(j = r + 1, \ldots, \hat{r}\) and \(\ell = 1, 2\),

\[
\max_{i \in [n]} \max_{t \in [T]} |\hat{\chi}_{it}^{\text{pca}} - \chi_{it}| = O_p \left( \left( \sqrt{\log n} \lor \frac{1}{\sqrt{n}} \right) \log T \right).
\]

**Case 2:** When there exists \(j^* \in [r + 1, \hat{r}]\) and \(i^* \in [n]\) for which \(|\hat{w}_{x,i^*j^*}^{(\ell)}| \geq c_0 > 0\) for both \(\ell = 1, 2\),

\[
\hat{\chi}_{i^*t}^{\text{pca}} - \chi_{i^*t} = O_p \left( \sqrt{n \log n} \lor \frac{1}{\sqrt{n}} \right) + Z_{i^*t},
\]

for all \(t \in [T]\), where \(Z_{i^*t}\) is a non-degenerate sub-exponential random variable.

The proof of Proposition 2 is provided in Appendix A.2. Although Cases 1–2 do not cover all possible contingencies, they serve as examples when the over-estimation of \(r\) does and does not incur non-negligible errors in the PCA estimator. In particular, Case 2 may naturally arise when \(\Gamma_{\varepsilon}\) is a spiked covariance matrix with sparse support, a setting extensively investigated in [Cai et al. (2013)] and [Cai et al. (2015)], where the sparse support can be well-estimated via PCA when \(n = o(T)\), see Remark 4. In Section 4, we present simulation results that confirm the plausibility of Case 2 empirically when \(\Gamma_{\varepsilon}\) has spiked eigenvalues albeit bounded for all \(n \geq 1\), with and without serial dependence in \(x_t\).
3 Modified principal component estimators

Throughout this section, we propose modified PCA estimators for the common components, and investigate their properties conditional on that the factor number estimator $\hat{r} \geq r$.

3.1 Capped PCA estimator

Recall that under factor models, due to the eigengap properties in (C1)–(C4), we have the elements of the leading $r$ eigenvectors of $\Gamma_\chi$ bounded as $|w_{\chi,ij}| = O(1/\sqrt{n})$, see (4). This, combined with the consistency of the leading $r$ eigenvectors of the sample covariance matrix $\hat{\Gamma}_x$ as noted in (3), leads to

$$\max_{i \in [n]} \max_{j \in [r]} |\hat{w}_{x,ij}| = O_p(1/\sqrt{n}) \quad (10)$$

(see Lemma 3 of Barigozzi et al. (2018)), i.e., with large probability, there exists some fixed $c_w > 0$ such that $|\hat{w}_{x,ij}|, \ j \in [r]$ is uniformly bounded by $c_w/\sqrt{n}$.

Motivated by the above observations, Barigozzi et al. (2018) proposed the capped PCA estimator of $\chi_{it}$ as

$$\hat{\chi}_{it}^{\text{cap}} = \sum_{j=1}^{\hat{r}} \hat{w}_{x,j}^{\text{cap}} (\hat{w}_{x,j}^{\text{cap}})^\top x_t, \quad (11)$$

where each element of $\hat{w}_{x,j}^{\text{cap}}$ is obtained by capping $\hat{w}_{x,ij}$ as

$$\hat{w}_{x,j}^{\text{cap}} = \hat{w}_{x,ij} I\left( |\hat{w}_{x,ij}| \leq \frac{c_w}{\sqrt{n}} \right) + \text{sign}(\hat{w}_{x,ij}) \cdot \frac{c_w}{\sqrt{n}} I\left( |\hat{w}_{x,ij}| > \frac{c_w}{\sqrt{n}} \right),$$

for some fixed $c_w > 0$ (see Remark 3 for its choice). Capping can be viewed as the projection of each $\hat{w}_{x,j}$ onto the $\ell_\infty$-sphere of radius $c_w/\sqrt{n}$. Asymptotically, capping does not alter the contribution from the leading $r$ eigenvectors of $\hat{\Gamma}_x$, while it truncates any large contribution from spurious factors when $\hat{r} \geq r + 1$, all without the knowledge of the true $r$.

Its theoretical property has been studied in Theorem 2 of Barigozzi et al. (2018).

Proposition 3 (Modified Theorem 2 of Barigozzi et al., 2018). Under Assumptions 1–4, the capped estimator $\hat{\chi}_{it}^{\text{cap}}$ with all finite $\hat{r} \geq r + 1$ as the number of factors, satisfies

$$\max_{i \in [n]} \max_{t \in [T]} |\hat{\chi}_{it}^{\text{cap}} - \chi_{it}| = O_p(\log T). \quad (12)$$

Refinement of the upper bound given in (12) is a difficult task, as reasoned in (a)–(b) of Section 2.3. Nonetheless, Proposition 3 shows that the capped estimator improves upon the worst case performance of the PCA estimator reported in (8) in the general time series setting.
3.2 Scaled PCA estimator

Similarly motivated by the boundedness of $|w_{x,ij}|$ and $|\hat{w}_{x,ij}|$, $j \in [r]$ (see (1) and (10)), we propose the scaled PCA estimator

$$\hat{\chi}_{kt}^{sc} = \sum_{j=1}^{\hat{r}} \hat{w}_{x,ij}^{sc}(\hat{w}_{x,j}^{sc})^\top x_t, \quad \text{where}$$

$$\hat{w}_{x,j}^{sc} = \nu_j^{-1/2} \hat{w}_{x,j} \quad \text{with} \quad \nu_j = \max \left\{ 1, \frac{\sqrt{n}}{c_w \max_i |\hat{w}_{x,ij}|} \right\}. \quad (14)$$

As with the capping, the scaling factor satisfies $\nu_j = 1$ for $j \in [r]$ with probability tending to one, and thus the proposed scaling does not alter the contribution from $\hat{w}_{x,j}$, $j \in [r]$ to $\hat{\chi}_{kt}^{sc}$ even though it is applied without the knowledge of $r$. On the contrary, for $\hat{w}_{x,j}$, $j \geq r + 1$, any large contribution from the spurious factors is scaled down.

The scaled PCA estimator is closely related to the capped PCA estimator in (11) in that both are motivated by the asymptotic property of $\hat{w}_{x,j}$, $j \in [r]$. However, scaling shrinks down $||\hat{w}_{x,j}^{sc}||^2$, $j \geq r + 1$ to $\nu_j^{-1}$ from $||\hat{w}_{x,j}||^2 = 1$, which can further help controlling the spurious contribution from $\hat{w}_{x,j}$, $j \geq r + 1$ as demonstrated in the following example.

**Example 1.** Suppose that $\hat{w}_{x,j^*}$ for some $j^* \geq r + 1$ is approximately sparse: there exist $C \subset [n]$ with $|C| = O(1)$ and a fixed $c_0 > 0$ such that $|\hat{w}_{x,ij^*}| \geq c_0$, $i \in C$, while $\max_{i \in C} |\hat{w}_{x,ij^*}| = O(1/\sqrt{n})$ (here we ignore the stochastic nature of $\hat{w}_{x,j}$). Then, we have $||\hat{w}_{x,j^*}^{sc}|| \leq c_w (c_0 \sqrt{n})^{-1}$, which shrinks the overall contribution of the $j^*$-th estimated factor to $\hat{\chi}_{kt}^{sc}$ by the factor of $n$, compared to that to the PCA estimator. In the same scenario, however, capping does not always lead to $||\hat{w}_{x,j^*}^{cap}|| = o(1)$. Consider e.g., $\hat{w}_{x,j^*} = (1/\sqrt{2}, 1/\sqrt{2(n-1)}, \ldots, 1/\sqrt{2(n-1)})^\top$ and $c_w/\sqrt{n} \geq 1/\sqrt{2(n-1)}$, in which case $||\hat{w}_{x,j^*}^{cap}|| = 1/\sqrt{2}$.

Furthermore, scaling preserves the orthogonality among $\hat{w}_{x,j}^{sc}$, $j \in [\hat{r}]$, which facilitates the theoretical treatment of the scaled PCA estimator. As argued in (a)–(b) of Section 2.3 we continue the discussion on the consistency of the scaled PCA estimator with that obtained from sample splitting. Recall the notations from Section 2.3 and let

$$\hat{\chi}_{kt}^{sc,(l)} = \sum_{j=1}^{\hat{r}} \hat{w}_{x,ij}^{sc,(m)}(\hat{w}_{x,j}^{sc,(m)})^\top x_t^{(l)} \quad (15)$$

where $\hat{w}_{x,ij}^{sc,(m)} = (\nu_j^{(m)})^{-1/2} \hat{w}_{x,ij}^{(m)}$ with $\nu_j^{(m)} = \max_1, (\sqrt{n}/c_w \max_i |\hat{w}_{x,ij}^{(m)}|)$. Finally, set $\hat{\chi}_{kt}^{sc,(1)} = \hat{\chi}_{kt}^{sc,1}$ and $\hat{\chi}_{kt}^{sc,2} = \hat{\chi}_{kt}^{sc,2}$ for $t \in [T_1]$.

**Proposition 4.** Let Assumptions 1–3 and 4(i) hold and in addition, let $x_t$ be serially inde-
\[ \max_{i \in [n]} \max_{t \in [T]} |\hat{\chi}_{it} - \chi_{it}| = O_p \left( \sqrt{\frac{\log n}{T}} \lor \frac{1}{\sqrt{n}} \log T \right). \]  

(16)

The proof of Proposition 4 can be found in Appendix A.3. Compared to Proposition 2, Proposition 4 establishes that under the same conditions, the scaled PCA estimator attains the same rate of convergence as the PCA estimator obtained with the true number of factors regardless of the behaviour of \( \hat{w}_{x,j} \), \( j \geq r + 1 \).

**Remark 3** (Choice of \( c_w \)). In our numerical analysis, we observed that the performance of the proposed modified PCA estimators did not vary much with respect to reasonably chosen \( c_w \). It may be explained since, unlike e.g., methods based on singular value thresholding, the modified PCA estimators do not completely ‘kill’ any factors including the spurious ones, and thus avoid the hazard of under-estimating the contribution of the factors provided that \( \hat{r} \geq r \). We recommend the choice of \( c_w = \sqrt{n} \max_{i \in [n]} |\hat{w}_{x,i}| \), which ensures that the leading eigenvector \( \hat{w}_{x,1} \) is not capped or scaled. This choice is shown to work well in controlling for the over-estimation error in our simulation studies (Section 4).

**Remark 4** (Alternative factor models). Lam and Yao (2012) investigated the estimation of the factor space through performing PCA on \( \hat{M} = \sum_{k=1}^{k_0} \hat{\Gamma}(k) \hat{\Gamma}(k)^\top \) with fixed \( k_0 \), where \( \hat{\Gamma}(k) = (T-k)^{-1} \sum_{t=1}^{T-k} x_{t+k} x_t^\top \) denotes the autocovariance matrix of \( x_t \) at lag \( k \). Assuming that the factor structure absorbs the serial correlations of \( x_t \) (and hence \( \epsilon_t \) is serially uncorrelated), they showed the consistency of the estimated factor space spanned by the leading eigenvectors of \( \hat{M} \). Similarly as in (4) and (10), we can show that the entries of the \( r \) leading eigenvectors of \( \hat{M} \) and its population counterpart are bounded as \( O_p(n^{-1/2} \lor T^{-1/2}) \) and \( O(n^{-1/2}) \), respectively. Therefore, the modified PCA estimators can be obtained analogously using the estimation approach proposed in Lam and Yao (2012).

### 3.3 Capped and shrunk PC estimator

From the decomposition \( x_{it} = \sum_{j=1}^{n_T} \hat{w}_{x,ij} \hat{\mu}_{x,j} x_t \), the sample variance of \( x_{it} \), denoted by \( \hat{\text{Var}}(x_{it}) \), satisfies \( \hat{\text{Var}}(x_{it}) = \sum_{j=1}^{n_T} \hat{w}_{x,ij}^2 \hat{\mu}_{x,j} \), and therefore

\[ \max_{i \in [n]} \hat{\text{Var}}(x_{it}) = \sum_{j=1}^{n_T} (\max_{i \in [n]} \hat{w}_{x,ij}^2) \hat{\mu}_{x,j} < \infty \quad \text{a.s.} \]  

(17)

This observation, together with (C3) and Lemma 2 of Barigozzi et al. (2018) showing

\[ \frac{1}{n} |\hat{\mu}_{x,j} - \mu_{x,j}| = O_p \left( \sqrt{\frac{\log n}{T}} \lor \frac{1}{\sqrt{n}} \right), \]
implies that $\max_{i \in [n]} |\hat{w}_{x,ij}| = O_p(1/\sqrt{\hat{\mu}_{x,j}})$ for all $j \in [n]$. Then, with the choice $c_w = \sqrt{n} \max_{i \in [n]} |\hat{w}_{x,i1}|$ as suggested in Remark 3, we may re-write the scaling factor $\nu_j$ in (14) as

$$\nu_j = \max \left\{ 1, \frac{\max_{i \in [n]} |\hat{w}_{x,ij}|}{\max_{i \in [n]} |\hat{w}_{x,i1}|} \right\} = \max \left\{ 1, \sqrt{\frac{C_x}{\hat{\mu}_{x,1}}} \right\},$$

such that

$$\hat{\chi}_{sc}^{it} = \sum_{j=1}^{r} \min \left\{ 1, \sqrt{\frac{\hat{\mu}_{x,j}}{C_x}} \right\} \hat{w}_{x,j} \hat{w}_{x,j}^\top x_t$$

with some fixed $C > 0$. In other words, for some choice of $c_w$, the scaled PCA estimator admits a representation as the PCA estimator combined with shrinkage based on the empirical eigenvalues. In light of Remark 1, the ideal choice of $C$ is $C = \hat{\mu}_{x,r}/\hat{\mu}_{x,1}$ which, however, is unattainable in practice. Alternatively, we may select $C = 1$ which ensures that the contribution from the leading eigenvector of $\hat{\Gamma}_x$ is not curtailed by the shrinkage.

Based on the above observations, we propose a modified PCA estimator that combines the element-wise perturbation of the eigenvectors $\hat{w}_{x,j}$ and the eigenvalue-based shrinkage as

$$\hat{\chi}_{cs}^{it} = \sum_{j=1}^{r} \sqrt{\frac{\hat{\mu}_{x,j}}{\hat{\mu}_{x,1}}} \hat{w}_{x,j} \hat{w}_{x,j}^\top x_t.$$  \hspace{1cm} (18)

As with the capped estimator in (11), theoretical analysis of $\hat{\chi}_{cs}^{it}$ is not straightforward due to that the orthogonality among $\hat{w}_{x,j}$, $j \geq 1$ is not preserved after capping.

**Remark 5** (Eigenvalue shrinkage). The modified PCA estimator in (18) introduces bias, since it typically shrinks the contribution from $\hat{w}_{x,j}$, $j = 2, \ldots, r$ to the estimator. However, simulation studies in Section 4 suggest that any bias incurred by over-shrinkage is richly compensated by its effectiveness in shrinking down the spuriously large over-estimation error. We may connect this phenomenon with the literature on the optimal eigenvalue shrinkage for spiked covariance matrix estimation such as Donoho et al. (2018) and Donoho and Ghorbani (2018), although their problem requires a different asymptotic setting as well as serial independence and normality on $x_{it}$, with a different objective. There, it has been shown that the optimal eigenvalue shrinker under the condition number loss or Stein’s loss, has its asymptotic slope strictly less than 1 even for very large eigenvalues.

**Remark 6** (Weak factors). In the presence of so-called ‘weak factors’ (Lam and Yao, 2012; Onatski, 2012), typically characterised by $\mu_{x,j} \asymp n^a$ with some $a \in (0, 1)$, we can still derive the modified PCA estimators by capping or scaling with $c_w n^{-a/2}$ in place of $c_w n^{-1/2}$ in obtaining $\hat{w}_{x,j}^{cap}$ and $\hat{w}_{x,j}^{sc}$, respectively; see (17) from which it follows that $\hat{w}_{x,ij} = O_p(\mu_{x,j}^{-1/2})$. However, the weakness of the factors, parameterised by $a$, is typically not known and its estimation is a difficult problem in finite samples, and we do not pursue this topic further in the current paper.
We explore the in-sample estimation accuracy of the PCA estimator \( \hat{\chi}_{it} \) and consider \( V \) and \( \Delta \). The diagonal matrix \( \Delta \) has \( n, \Delta \)-variances and \( \hat{V} \) is chosen as the \( r \) leading left singular vectors of \( M \in \mathbb{R}^{n \times r} \), whose first \( s = [0.2n] \) rows are drawn i.i.d from \( N(0, 1) \), and the rest are set to zero. This results in the covariance matrix of the idiosyncratic components to be moderately spiked. We control the ‘signal-to-nose’ ratio by setting \( \vartheta^2 = \phi^2 \cdot n^{-1} \sum_{i=1}^{n} \text{Var}(\chi_{it}) / \text{Var}(\varepsilon_{it}) \) with \( \phi \in \{0.5, 1, 2\} \). Throughout, we set \( \vartheta = 0.05 \).

We consider two scenarios, with and without serial correlations in the factors and idiosyncratic components. When serial correlations are permitted, we choose the autoregressive parameters as \( \rho_{f,j} = \rho_{f} - 0.05(j - 1) \) with \( \rho_{f} = 0.5 \), and \( \rho_{\varepsilon,i} \in \{0.2, -0.2\} \); if not, we set \( \rho_{f,j} = \rho_{\varepsilon,i} = 0 \).

4 Simulation studies

We consider the following data generating model for simulation studies, which allows for serial correlations in \( f_{jt} \) and both serial and cross-sectional correlations in \( \varepsilon_{it} \).

\[
x_{it} = \sum_{j=1}^{r} \lambda_{ij} f_{jt} + \sqrt{\vartheta} \varepsilon_{it}, \quad 1 \leq i \leq n; \quad 1 \leq t \leq T,
\]

where

\[
f_{jt} = \rho_{f,j} f_{j,t-1} + u_{jt}, \quad u_{jt} \sim_{i.i.d} \mathcal{N}(0, 1),
\]

\[
\varepsilon_{it} = \Gamma_{\varepsilon}^{1/2} e_{it}, \quad \varepsilon_{it} \sim_{i.i.d} \mathcal{N}(0, 1),
\]

with \( \lambda_{ij} \sim_{i.i.d} \mathcal{N}(0, 1) \) and \( \Gamma_{\varepsilon} = \mathbf{V} \Delta \mathbf{V}^\top + \mathbf{I}_n \). We consider two scenarios, with and without serial correlations in the factors and idiosyncratic components. When serial correlations are permitted, we choose the autoregressive parameters as \( \rho_{f,j} = \rho_{f} - 0.05(j - 1) \) with \( \rho_{f} = 0.5 \), and \( \rho_{\varepsilon,i} \in \{0.2, -0.2\} \); if not, we set \( \rho_{f,j} = \rho_{\varepsilon,i} = 0 \).

The diagonal matrix \( \Delta \) has \( r \) non-zero eigenvalues taking equidistant values from 20 to 10, and \( \mathbf{V} \) is chosen as the \( r \) leading left singular vectors of \( \mathbf{M} \in \mathbb{R}^{n \times r} \), whose first \( s = [0.2n] \) rows are drawn i.i.d from \( N(0, 1) \), and the rest are set to zero. This results in the covariance matrix of the idiosyncratic components to be moderately spiked. We control the ‘signal-to-nose’ ratio by setting \( \vartheta^2 = \phi^2 \cdot n^{-1} \sum_{i=1}^{n} \text{Var}(\chi_{it}) / \text{Var}(\varepsilon_{it}) \) with \( \phi \in \{0.5, 1, 2\} \). Throughout, we set \( r = 5 \), and consider \( T \in \{500, 1000, 2000\} \) and \( n \in \{200, 500, 1000\} \).

We explore the in-sample estimation accuracy of the PCA estimator \( \hat{\chi}_{it}^{\text{pca}} \) in [2], the capped estimator \( \hat{\chi}_{it}^{\text{cap}} \) in [11], the scaled estimator \( \hat{\chi}_{it}^{\text{sc}} \) in [13] and the capped and shrunk estimator \( \hat{\chi}_{it}^{\text{pca}} \) in [18]. For an estimator of \( r \), we consider the estimators \( \hat{r}_{\text{oracle}} \) (Bai and Ng 2002) and \( \hat{r}_{\text{Ahn and Horenstein}} \) (Ahn and Horenstein 2013), with the maximum number of factors set at \( r_{\text{max}} = \lfloor \sqrt{n} \wedge T \rfloor \). As a benchmark, we also investigate the performance of the oracle estimator \( \hat{\chi}_{it}^{\text{oracle}} = \sum_{j=1}^{r} \hat{w}_{x,j} \hat{w}_{x,j}^\top \chi_{it} \), the PCA estimator obtained with the knowledge of the true \( r \).

We evaluate the accuracy of the various estimators of \( \chi_{it} \) relative to that of the oracle estimator, using the following error measures

\[
\text{err}_{\text{avg}}(\hat{\chi}_{it}) = \frac{\hat{E}\{n^{-1} \sum_{t=1}^{n} \sum_{j=1}^{T} (\hat{\chi}_{it}^{\text{pca}} - \chi_{it})^2\}}{\hat{E}\{n^{-1} \sum_{t=1}^{n} \sum_{j=1}^{T} (\hat{\chi}_{it}^{\text{oracle}} - \chi_{it})^2\}},
\]

\[
\text{err}_{\text{max}}(\hat{\chi}_{it}) = \frac{\hat{E}\{\max_{1 \leq i \leq n} \sum_{j=1}^{T} (\hat{\chi}_{it}^{\text{pca}} - \chi_{it})^2\}}{\hat{E}\{\max_{1 \leq i \leq n} \sum_{j=1}^{T} (\hat{\chi}_{it}^{\text{oracle}} - \chi_{it})^2\}},
\]

\[
\text{err}_{\text{avg},s}(\hat{\chi}_{it}) = \frac{\hat{E}\{s^{-1} \sum_{t=1}^{s} \sum_{j=1}^{T} (\hat{\chi}_{it}^{\text{pca}} - \chi_{it})^2\}}{\hat{E}\{s^{-1} \sum_{t=1}^{s} \sum_{j=1}^{T} (\hat{\chi}_{it}^{\text{oracle}} - \chi_{it})^2\}},
\]

\[
\text{err}_{\text{avg},s}(\hat{\chi}_{it}) = \frac{\hat{E}\{(n-s)^{-1} \sum_{t=s+1}^{n} \sum_{j=1}^{T} (\hat{\chi}_{it}^{\text{pca}} - \chi_{it})^2\}}{\hat{E}\{(n-s)^{-1} \sum_{t=s+1}^{n} \sum_{j=1}^{T} (\hat{\chi}_{it}^{\text{oracle}} - \chi_{it})^2\}}.
\]
where \( \hat{E} \) denotes the empirical expectation over the Monte Carlo repetitions, and \( \circ \) denotes the use of PCA, capped, scaled or capped and shrunk estimator. We note that \( \text{err}_{\text{max}} \) and \( \text{err}_{\text{avg},s} \) are specifically designed to capture the possible deterioration in the performance of the estimators due to the over-estimation resulting from the spiked eigenvalues of \( \Gamma_\varepsilon \).

4.1 When \( x_{it} \) is serially correlated

![Box plots of \( \hat{r} \) returned by (BN) and (AH) over 1000 realisations when \( T = 500 \) (left panel) and \( T = 2000 \) (right panel) with \( n \in \{200, 500, 1000\} \) (top to bottom) and \( \phi \in \{0.5, 1, 2\} \) (left to right); horizontal broken lines indicate \( r = 5 \).](image)

We first investigate \( \hat{r} \) returned by the two different factor number estimators (Figure 1). From the construction of \( \Gamma_\varepsilon \) in [19], it is apparent that the first \( s \) elements of \( \varepsilon_t \) are strongly correlated with one another, despite \( \mu_{e,1} \) still being bounded for all \( n \). This leads to the factor number estimators in [5]–[8] fail in accurately estimating the true \( r \), the former more frequently than the latter with growing \( T \) and \( \phi \) (which decreases the proportion of \( \text{Var}(\chi_{it}) \) in \( \text{Var}(x_{it}) \)), particularly when \( n \) is small. In all cases, the estimated factor number satisfies \( \hat{r} \geq r \).

In order to focus on the performance of various estimators when \( r \) is over-estimated, we report the results when the factor number estimator [5] is used in the main text, see Table 1. Results from using the estimator [6] can be found in Table 5 of Appendix B.

When \( \hat{r} > r \), the PCA-based estimator performs poorly due to the presence of spurious factors. The PCA estimator may perform worse than the oracle estimator by hundredfold in some settings (large \( T \) and \( \phi \)) and, as expected, this phenomenon is most strikingly manifested by the measures \( \text{err}_{\text{max}} \) and \( \text{err}_{\text{avg},s} \) due to the sparse support of \( \mathbf{V} \). Indeed, Figure 2 confirms that the Case 2 of Proposition 2 is a plausible scenario under the spiked covariance structure of \( \varepsilon_t \), as \( \hat{w}_{x,r+1} \) is often closely aligned with the sparse leading eigenvector of \( \Gamma_\varepsilon \).
The errors returned by $\hat{\lambda}^{\text{CAP}}_{it}$, $\hat{\lambda}^{\text{SC}}_{it}$ and $\hat{\lambda}^{\text{CS}}_{it}$ demonstrate that the proposed modified PCA estimators are effective in curtailting the effect of the spurious factors. Occasionally, the three estimators are observed to even improve upon the performance of the oracle estimator when $\text{Var}(\varepsilon_{it})$ is large. Among the three estimators, $\hat{\lambda}^{\text{CS}}_{it}$, the estimator that combines capping and eigenvalue-based shrinkage, often performs the best. The exception is when the average squared error over $i = s + 1, \ldots, n$ is considered; however, we also need to take into account that most severe estimation errors occur for $\hat{\lambda}_{it}$, $i = 1, \ldots, s$.

4.2 When $x_{it}$ is serially independent

We further add to the consideration various estimators combined with the sample splitting as outlined in (9) and (15) for $\hat{\lambda}^{\text{PCA}}_{it}$ and $\hat{\lambda}^{\text{SC}}_{it}$, respectively, and analogously derived for $\hat{\lambda}^{\text{CAP}}_{it}$ and $\hat{\lambda}^{\text{CS}}_{it}$. In this scenario, $x_{it}$ is serially independent and thus the independence condition in Propositions 2 and 4 is met.

Since often the factor number estimators from the full sample and the split samples do not agree, we use $\hat{r}$ obtained from the full sample using (5) for all the estimators. Tables 2-4 compare the performance of the (modified) PCA estimators computed on the full sample and split samples using the four error measures.

Compared to Table 1 it appears that the lack of serial correlations marginally influences the performance of various estimators. While $\hat{\lambda}^{\text{SC}}_{it}$ and $\hat{\lambda}^{\text{CS}}_{it}$ are leading in terms of efficiency as observed in Table 1, more pronounced change in $\text{err}_{\text{max}}$ and $\text{err}_{\text{avg},s}$, is due to the use of sample splitting. Improvement of the split sample estimator upon the full sample one is most noticeable with $\hat{\lambda}^{\text{sc}}_{it}$, which is in accordance with Proposition 4. Generally, all the estimators
Table 1: Summary of estimation error from various estimators for $T \in \{500, 1000, 2000\}$, $n \in \{200, 500, 1000\}$ and $\phi \in \{0.5, 1, 2\}$ when $x_{it}$ is serially correlated; $[5]$ is used for the estimation of $r = 5$.

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<th>$T$</th>
<th>$n$</th>
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<th>$\hat{\epsilon}_{\text{err}}$</th>
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except for $\hat{\chi}_{it}^{cs}$ tend to perform better when combined with sample splitting, apart from when $n$ is large relative to $T$. Then, the poorer efficiency of the PCA performed on the split sample itself (see Proposition [1]) may affect the estimators of the common components adversely, thus supporting our suggestion that sample splitting may be adopted for theoretical consideration only, even when $x_t$ is serially independent (which is often not known and difficult to test for).

## 5 Real data analysis

### 5.1 US macroeconomic data

We analyse the US representative macroeconomic dataset of 101 time series, collected quarterly between 1960:Q2 and 2012:Q3 ($T = 209$). All series are available from the St. Louis Federal Reserve Bank website [https://fred.stlouisfed.org/](https://fred.stlouisfed.org/).

Applying the information criterion of Bai and Ng (2002) with different choices of $g(n, T)$ suggested by the authors returns $\hat{r} = 6, 8$ and $10$ when the maximum allowable number of factors is set at $r_{\text{max}} = 20$. The eigenvalue ratio-based estimator of Ahn and Horenstein (2013) returns $\hat{r} = 1$, and the singular value shrinkage estimator of Bai and Ng (2017) returns $\hat{r} = 3$.

Discordance among different estimators demonstrates the difficulty in determining the number of factors for this dataset, and advocates an estimator of the common components that is less sensitive to factor number estimators. We also note that the same dataset was analysed for multiple change-points in Barigozzi et al. (2018), which yielded 5 structural breaks that may have enlarged the factor space, which brings additional difficulties to the problem of factor number estimation.

Unlike in the simulation studies, the true common components are not accessible, nor is the true number of factors known. Therefore, we evaluate the ‘closeness’ between various estimators with $r_1$ factors, denoted by $\hat{\chi}^o_{it}(r_1)$, and the PCA estimator with $r_2$ factors, $\hat{\chi}^{\text{pca}}_{it}(r_2)$, with $r_1 > r_2$, by calculating the $R^2$ coefficient from regressing $\hat{\chi}^o_{it}(r_1)$ onto $\hat{\chi}^{\text{pca}}_{it}(r_2)$. Figure 3 shows the box plots of the resulting $R^2$ coefficients for different estimators when $(r_1, r_2) = (6, 3)$ and $(r_1, r_2) = (10, 6)$, respectively.

We observe that the ‘height’ of the box for the PCA estimator in the left panel of Figure 3 is much greater than that in the right panel. In light of the discussion in Section 2.3 if $r_1 > r_2 \geq r$, we expect $\hat{\chi}^{\text{pca}}_{it}(r_1)$ and $\hat{\chi}^{\text{pca}}_{it}(r_2)$ to be close (and hence the corresponding $R^2$ large) for the majority of $i \in [n]$ with a few exceptions. Therefore, we may conclude that $r_2 = 3$, returned from the factor number estimator proposed in Bai and Ng (2017), underestimates the true number of factors for the dataset, thus resulting in biased estimators for some common components. With a larger choice of $r_2$, it is observed in the right panel of Figure 3 that compared to the PCA estimator, $\hat{\chi}^o_{it}(r_1)$ and $\hat{\chi}^{\text{pca}}_{it}(r_1)$ attain $R^2$ coefficients much more skewed towards one, which confirms the effectiveness of the modified PCA estimators.
Table 2: Summary of estimation error from various estimators for $T = 500$, $n \in \{200, 500, 1000\}$ and $\phi \in \{0.5, 1, 2\}$ when $\rho_{f,j} = \rho_{\epsilon,i} = 0$; (5) is used for the estimation of $r = 5$.

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<th>$err_{c^{\text{cap}}}$</th>
<th>$err_{c^{\text{sc}}}$</th>
<th>$err_{c^{\text{cs}}}$</th>
<th>$err_{c^{\text{cap}}}$</th>
<th>$err_{c^{\text{sc}}}$</th>
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Table 3: Summary of estimation error from various estimators for $T = 1000$, $n \in \{200, 500, 1000\}$ and $\phi \in \{0.5, 1, 2\}$ when $\rho_{f,j} = \rho_{\epsilon,i} = 0$; (5) is used for the estimation of $r = 5$.

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Table 4: Summary of estimation error from various estimators for $T = 2000$, $n \in \{200, 500, 1000\}$ and $\phi \in \{0.5, 1, 2\}$ when $\rho_{f,j} = \rho_{\varepsilon,i} = 0$; (5) is used for the estimation of $r = 5$.

| $n$ | $\phi$ | sample | $\hat{\chi}_{\text{pca}}^{\text{avg}}$ | $\hat{\chi}_{\text{cap}}^{\text{avg}}$ | $\hat{\chi}_{\text{sc}}^{\text{avg}}$ | $\hat{\chi}_{\text{cs}}^{\text{avg}}$ | $\hat{\chi}_{\text{pca}}^{\text{max}}$ | $\hat{\chi}_{\text{cap}}^{\text{max}}$ | $\hat{\chi}_{\text{sc}}^{\text{max}}$ | $\hat{\chi}_{\text{cs}}^{\text{max}}$ | $\hat{\chi}_{\text{pca}}^{\text{avg},s}$ | $\hat{\chi}_{\text{cap}}^{\text{avg},s}$ | $\hat{\chi}_{\text{sc}}^{\text{avg},s}$ | $\hat{\chi}_{\text{cs}}^{\text{avg},s}$ |
|-----|-----|-------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
5.2 S&P100 stock returns

Next, we analyse the log returns of the daily closing values of the stocks composing the Standard and Poor’s 100 (S&P100) index, observed between 4 January 2000 and 30 September 2013 (n = 90 and T = 3456). The dataset is available from Yahoo Finance.

The information criterion of Bai and Ng (2002) returns \( \hat{\lambda} = 4 \) regardless of the choice of the penalty function \( g(n, T) \), while the estimators proposed in Ahn and Horenstein (2013) and Bai and Ng (2017) return \( \hat{\lambda} = 1 \). Setting \( r_1 = 4 \) and \( r_2 = 1 \), we repeat the analysis conducted in Section 5.1, see Figure 4. For this dataset, \( \hat{\mu}_{x,1} \) markedly dominates \( \hat{\mu}_{x,j}, j \geq 2 \) such that the eigenvalue-based shrinkage adopted by \( \hat{\chi}_{it}^{\text{PCA}}(r_1) \) leads to the estimators that are strikingly close to \( \hat{\chi}_{it}^{\text{PCA}}(r_2) \). Although \( \hat{\chi}_{it}^{\text{CS}}(r_1) \) does not explicitly adopt such shrinkage, it efficiently controls for the effect of possibly spurious factors. The capped estimator \( \hat{\chi}_{it}^{\text{CAP}}(r_1) \) also performs reasonably well, while the PCA estimator \( \hat{\chi}_{it}^{\text{PCA}}(r_1) \) may account for only about 50% of the variability in \( \hat{\chi}_{it}^{\text{PCA}}(r_2) \) for some \( i \).

A Proofs

A.1 Preliminaries

Lemma 1. Under Assumptions 1, 3 and 4,

\[
\max_{i \in [n]} \max_{t \in [T]} |\chi_{it}| = O_p(\log T), \quad \max_{i \in [n]} \max_{t \in [T]} |\varepsilon_{it}| = O_p(\log T) \quad \text{and} \quad \max_{i \in [n]} \max_{t \in [T]} |x_{it}| = O_p(\log T).
\]
Figure 4: S&P100 DATA: The box plots of $R^2$ from regressing $\hat{\chi}_it^{CAP}(r_1)$, $\hat{\chi}_it^{CS}(r_1)$, $\hat{\chi}_it^{PCA}(r_1)$ and $\hat{\chi}_it^{SC}(r_1)$ onto $\hat{\chi}_it^{PCA}(r_2)$, respectively. Where $(r_1, r_2) = (4, 1)$.

Proof. Assumptions [1](iii) and [2] and Proposition 2.7.1 in Vershynin (2018) lead to

$$\max_{i \in [n]} \max_{t \in [T]} |\chi_{it}| \leq r \bar{\lambda} \max_{j \in [r]} \max_{t \in [T]} |\hat{w}_{x,ij}(m)_{x,j}^\top \chi_t^{(l)}| = O_p(\log T).$$

Similarly, the second and the third statements follow. □

A.2 Proof of Proposition 2

Case 1: Note that

$$\max_{t \in [T]} |\hat{\chi}_it^{(l)} - \chi_{it}^{(l)}| \leq \max_{t \in [T]} \left| \sum_{j=1}^{r} \hat{w}_{x,ij}^{(m)} (\hat{w}_{x,j}^{(m)})^\top x_t^{(l)} - \chi_{it}^{(l)} \right| + \max_{t \in [T]} \left| \sum_{j=r+1}^{\tilde{r}} \hat{w}_{x,ij}^{(m)} (\hat{w}_{x,j}^{(m)})^\top x_t^{(l)} \right| + \max_{t \in [T]} \left| \sum_{j=r+1}^{\tilde{r}} \hat{w}_{x,ij}^{(m)} (\hat{w}_{x,j}^{(m)})^\top \epsilon_t^{(l)} \right| = I + II + III. \quad (20)$$

Split sample size is still of order $T$ and thus adopting the arguments used in the proof of Theorem 1 in Barigozzi et al. (2018), we can easily show that $I = O_p\{\sqrt{\log n/T} \lor 1/\sqrt{n}\log T\}$. Lemma 3 of Barigozzi et al. (2018) implies that, for any fixed $k \geq r + 1$,

$$\| (\hat{W}_{x,(r+1):k}^{(l)})^\top A \| = O_p \left( \sqrt{\frac{n \log n}{T}} \lor \frac{1}{\sqrt{n}} \right), \quad l = 1, 2. \quad (21)$$

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Besides, $\hat{w}_{x,j}^{(m)}$ and $\epsilon_t^{(\ell)}$ are independent under the serial independence of $x_t$. Therefore

$$II = O_p\left\{ \frac{1}{\sqrt{n}} \cdot \left( \sqrt{\frac{n \log n}{T}} \vee \frac{1}{\sqrt{n}} \right) \cdot \log T \right\} = O_p\left\{ \left( \sqrt{\frac{\log n}{T}} \vee \frac{1}{n} \right) \log T \right\},$$

$$III = O_p\left( \frac{\log T}{\sqrt{n}} \right)$$

from the boundedness of $\hat{w}_{x,ij}^{(m)}$, Assumption 1(iv) and Lemma 1, and the conclusion follows after taking the maximum over $i \in [n]$.

**Case 2:** As in Case 1, we have

$$\hat{\chi}^{(\ell)}_{it} - \chi_{it} = \sum_{j=1}^{r} \hat{w}_{x,ij}^{(m)} (\hat{w}_{x,j}^{(m)})^\top x_t^{(\ell)} - \chi_{it}^{(\ell)} + \sum_{j=r+1}^{\tilde{r}} \hat{w}_{x,ij}^{(m)} (\hat{w}_{x,j}^{(m)})^\top \chi_{it}^{(\ell)}$$

$$+ \sum_{j=r+1}^{\tilde{r}} \hat{w}_{x,ij}^{(m)} (\hat{w}_{x,j}^{(m)})^\top \epsilon_t^{(\ell)} = IV + V + Z_{it}.$$ For $i = i^*$, we have $IV + V = O_p(\sqrt{n \log n / T} \vee 1/\sqrt{n})$ from the proof of Theorem 1 of Barigozzi et al. (2018) and (21). Also, $(\hat{w}_{x,ij}^{(m)})^\top \epsilon_t^{(\ell)}$ behaves like a non-degenerate, sub-exponential random variable due to the normalisation $\|\hat{w}_{x,j}^{(m)}\| = 1$ and Assumption 3(ii), and it carries over to $Z_{i^*t}$ as $|\hat{w}_{x,i^*j^*}^{(m)}|$ is bounded away from zero.

### A.3 Proof of Proposition 4

Note that

$$\max_{i \in [n]} \max_{t \in [T]} |\hat{\chi}^{sc,(\ell)}_{it} - \chi_{it}^{(\ell)}| \leq \max_{i \in [n]} \max_{t \in [T]} \left| \sum_{j=1}^{r} \hat{w}_{x,ij}^{sc,(m)} (\hat{w}_{x,j}^{sc,(m)})^\top x_t^{(\ell)} - \chi_{it}^{(\ell)} \right|$$

$$+ \max_{i \in [n]} \max_{t \in [T]} \left| \sum_{j=r+1}^{\tilde{r}} \hat{w}_{x,ij}^{sc,(m)} (\hat{w}_{x,j}^{sc,(m)})^\top \chi_{it}^{(\ell)} \right|$$

$$+ \max_{i \in [n]} \max_{t \in [T]} \left| \sum_{j=r+1}^{\tilde{r}} \hat{w}_{x,ij}^{sc,(m)} (\hat{w}_{x,j}^{sc,(m)})^\top \epsilon_t^{(\ell)} \right| = I + II + III.$$
Since scaling does not alter the $r$ leading eigenvectors with probability converging to one, $I = O_p\{(\sqrt{\log n/T} \vee 1/\sqrt{n}) \log T\}$. Next, due to the orthogonality of $\hat{w}_{x,j}^{sc}$, $j \in [\tilde{r}]$,

$$
II = \max_{i \in [n]} \max_{t \in [T]} \left| \sum_{j=r+1}^{\tilde{r}} \hat{w}_{x,i,j}^{sc,(m)} (\hat{w}_{x,j}^{sc,(m)})^\top \{\chi_t^{(l)} - \bar{W}_{x,1:r}(\bar{W}_{x,1:r})^\top x_i^{(l)}\} \right|
$$

$$
= O_p \left\{ \left( \sqrt{\frac{\log n}{T}} \vee \frac{1}{\sqrt{n}} \right) \log T \right\}
$$

from the bound on $I$, the boundedness of $|\hat{w}_{x,i,j}^{sc}|$ and Lemma [1]. Finally, thanks to sample splitting, $\hat{w}_{x,j}^{sc,(m)}$ and $\epsilon_t^{(l)}$ are independent and thus $III = O_p(\log T/\sqrt{n})$ from Assumptions [1](iv) and [3](ii), which concludes the proof.
### Table 5: Summary of estimation error from various estimators for $T \in \{500, 1000, 2000\}$, $n \in \{200, 500, 1000\}$ and $\phi \in \{0.5, 1, 2\}$ when $x_{it}$ is serially correlated; $[6]$ is used for the estimation of $r = 5$.

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