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Small Infinitary Epistemic Logics

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Abstract

We develop a series of small infinitary epistemic logics to study deductive inference involving intra/inter-personal beliefs/knowledge such as common knowledge, common beliefs, and infinite regress of beliefs. Specifically, propositional epistemic logics $GL(L_{\alpha})$ are presented for ordinal $\alpha$ up to a given $\alpha^\omega$ ($\alpha^\omega \geq \omega$) so that $GL(L_0)$ is finitary $KD^n$ with $n$ agents and $GL(L_\alpha)$ ($\alpha \geq 1$) allows conjunctions of certain countably infinite formulae. $GL(L_\alpha)$ is small in that the language is countable and can be constructive. The set of formulae $L_\alpha$ is increasing up to $\alpha = \omega$ but stops at $\omega$. We present Kripke-completeness for $GL(L_\alpha)$ for each $\alpha \leq \omega$, which is proved using the Rasiowa-Sikorski lemma and Tanaka-Ono lemma. $GL(L_\alpha)$ has a sufficient expressive power to discuss intra/inter-personal beliefs with infinite lengths. As applications, we discuss the explicit definability of Axioms T (truthfulness), 4 (positive introspection), 5 (negative introspection), and of common knowledge in $GL(L_\alpha)$. Also, we discuss the rationalizability concept in game theory in our framework. We evaluate where these discussions are done in the series $GL(L_\alpha)$, $\alpha \leq \omega$.

1 Introduction

We develop a series of infinitary epistemic logics to study deductive inference involving intra/inter-personal beliefs/knowledge in social situations. In these situations, people’s beliefs may include infinitary components such as common knowledge, common beliefs, and infinite regress of beliefs. To approach such situations, we extend the finitary epistemic logic $KD^n$ with $n$ agents to infinitary logics, illustrated as

$$KD^n = GL(L_0) \implies GL(L_1) \implies \cdots \implies GL(L_\omega).$$

Each logic $GL(L_\alpha)$ is “small” in that the set of formulae is countable and can be constructive. These logics are formulated in a Hilbert-style, and each is complete with respect to Kripke
semantics. This implies that the logics in (1) are connected by the conservative extension relation \( \vdash \), and the series can be used in various manners to evaluate infinitary concepts. Our approach offers a new framework, alternative to the existing literatures on related issues on infinitary epistemic concepts, with applications to evaluations of epistemic axioms and of decision-making processes in game theory.

First, we compare our approach with two literatures on infinitary epistemic concepts: the infinitary logic literature since Karp [19] (for epistemic logics, Kaneko-Nagashima [20], Tanaka-Ono [33], Tanaka [32], Heifetz [12]), and the fixed-point logic literature (for epistemic logics, Fagin et al. [8], Meyer-van der Hoek [25], and for \( \mu \)-calculus, Enqvist, et al. [7], Jäger, et al. [16], and Jäger-Studer [17]). Both approaches have some merits and demerits; to discuss such merits and demerits, we note that the infinitary epistemic concepts we consider in applications are typically constructed by iterated substitution of the belief operators.

The infinitary logic approach is capable of discussing various infinitary concepts in an explicit and unified manner. However, the languages are very large (at least continuum) in terms of sets of formulae. A large language is not only unnecessary but also sometimes imposes an obstacle for a precise study of targeted infinitary concepts. The fixed-point logic approach has a merit to be specific to targeted infinitary concepts, but has the inconvenience that targeted concepts are indirectly expressed by a fixed-point argument. In contrast to these approaches, ours allows for explicit and unified treatments of targeted concepts and enables us to evaluate, as in (1), how large a given targeted concept requires. The key to our approach is a syntactical concept of germinal forms, upon which we build a series of languages, as explained below.

Our base logic is a finitary KD\( ^n \) with language \( L_0 \) (the set of formula); the agents have classical logical abilities and contradiction-free beliefs, described by the belief operators \( B_i(\cdot) \) for agents \( 1, ..., n \). We extend the finitary language \( L_0 \) by adding conjunctions of certain infinite sequences of formulae in \( L_0 \). Specifically, we consider a countable number of infinite sequences \( \langle C^\nu(p) : \nu \geq 0 \rangle = \langle C^0(p), C^1(p), ... \rangle \) from \( L_0 \), which we call germinal forms. A typical example is common knowledge. The germinal form for it is given as \( \langle C^\nu(p) : \nu \geq 0 \rangle = \langle B_N^0(p) : \nu \geq 0 \rangle \);

\[
C^0(p) = p, \quad C^1(p) = \land_{i \in N} B_i(p), ..., \quad C^{\nu+1}(p) = \land_{i \in N} B_i C^\nu(p), ...
\]  

The conjunction \( B_N^\nu(p) := \land \langle B_N^\nu(p) : \nu \geq 0 \rangle \) is the common knowledge of \( p \), meaning that \( p \) holds, all agents believe \( p \), all agents believe all believe \( p \), and so on. This is not in \( L_0 \), and we extend \( L_0 \) to \( L_1 \) to have \( B_N^\nu(A) \) as a targeted formula.

The next layer \( L_2 \) is obtained from \( L_1 \) by adding the infinite conjunctions \( \land \langle C^\nu(A) : \nu \geq 0 \rangle \) for \( A \in L_1 \), e.g., \( B_N^\nu(B_N^\nu(A)) \); roughly speaking, each formula in \( L_2 \) includes infinitary conjunctions nested at most twice. Assuming that the set of germinal forms are unchanged, we define \( L_0, L_1, ..., L_\alpha, ... \) up to some ordinal \( \alpha_0 \geq \omega := \{0, 1, ...\} \). We show that this extension stops at \( L_\omega = \cup_{\alpha < \omega} L_\alpha = L_{\omega+1} = ... = L_\omega^\omega \). The language \( L_\alpha \) is kept countable for all \( \alpha \leq \omega \). Also, we show that the ordinal depth of each formula in \( L_\omega \) is less than \( \omega^2 \).

Infinitary concepts such as common knowledge are typically constructed by iteration of substitutions. Our formulation of a germinal form is rich enough to capture these infinitary concepts. In our approach, however, germinal forms are more generally defined even to allow nonconstructive sequences \( \langle C^\nu(p) : \nu \geq 0 \rangle \). This implies that our theory is quite flexible and could go beyond our current applications.

The proof systems in the series (1) are uniform; they share the same logical axiom schemata.
and inference rules only with the restriction to each $L_\alpha$.\footnote{Below $KD^n$, a hierarchy of logics of shallow epistemic depths is developed in Kaneko-Suzuki [22]. Each system is a fragment of $KD^n$ with a finite epistemic structure, and continues to $KD^n$.} The Kripke semantics is defined also in a uniform manner over $\alpha$. Each $GL(L_\alpha)$ is proved to be sound and complete with respect to Kripke semantics. It follows from this result that $GL(L_{\alpha+1})$ is a conservative extension of $GL(L_\alpha)$, i.e., for any formula $A \in L_\alpha$, $A$ is provable in $GL(L_\alpha)$ if and only if it is provable in $GL(L_{\alpha+1})$. In (1), the double arrow $\equiv$ describes the conservative extension relation.

To prove Kripke-completeness, we adopt the $Q$-filter method developed in Tanaka-Ono [33]. $Q$-filters play the corresponding role to that of maximal consistent sets of formulae in the standard construction of a canonical model. The $Q$-filter method is crucial, since $GL(L_\alpha)$ deals with both particular infinitary conjunctions and modality. To treat these aspects, our proof relies upon two lemmas, the Rasiowa-Sikorski lemma and Tanaka-Ono lemma; the countability of the language $L_\alpha$ is crucial in applications of these lemmas. Although we use various algebraic concepts, our model theory is Kripke semantics, but not algebraic semantics. Our completeness theorem can be modified to systems including additional epistemic axioms, Axioms $T$ (truthfulness – $B_i(A) \supset A$), 4 (positive introspection – $B_i(A) \supset B_iB_i(A)$), and/or 5 (negative introspection – $\neg B_i(A) \supset B_i(\neg B_i(A))$).

We deliberately choose the base logic $KD^n = GL(L_0)$. In the literature of epistemic logic, all, some, or none of Axioms $T$, 4, and 5 for $B_i(\cdot)$ are adopted depending upon purposes/environments. Axioms 4 and 5 include infinitary aspects, though they are expressed in a finitary way. In our approach, we can study these axioms in terms of explicit definability in $GL(L_\alpha)$ in the series in (1), that is, we ask whether there is a formula in $GL(L_\alpha)$ such that it is an extension of $B_i(\cdot)$ and satisfies each of $T$, 4, and 5. For $T$, it is affirmatively answered in all $\alpha$, for 4, we need $\alpha = \omega$, and for 5, the answer is entirely negative. Also, we consider faithful embedding of the logics added $T$ and/or 4 in $GL(L_\alpha)$. Axiom $D$ is included as a basic axiom in our framework, since it is crucial in proving (20) for playability in Section 5.

We also consider the faithful embedding of the common knowledge logic, denoted $CK(L^C)$, which is the fixed-point extension of $KD^n$, to $GL(L_\alpha)$. As a whole, $CK(L^C)$ is faithfully embedded into $GL(L_\omega)$. Logic $CK(L^C)$ is also a fragment of modal $\mu$-calculus (Alberucci [1]). In this context, we show that a comparison between the rank function given in Alberucci, et al. [2] and our ordinal depth for $L_\omega$ coincide.

Although $CK(L^C)$ can be regarded as being in the intersection of our approach and modal $\mu$-calculus, these two approaches differ from each other not only in that the former is infinitary while the latter is finitary, but also in that the differences are substantive. We make a small summary of comparisons between our approach and modal $\mu$-calculus in the end of Section 4.3.

Using our framework, we study a decision making process in game theory, called “rationalizability” (cf., Osborne-Rubinstein [27]). In this theory, an agent “rationalizes” his possible decision by looking for a prediction about his opponent’s decision, assuming that the opponent uses the same criterion. This leads to an infinite regress of such rationalization. We show that the full discourse from a consideration of decision-making to the stage of playing a final decision can be given in logic $GL(L_2)$. Thus, our framework allows for explication of game theoretic decision making with a clear-cut notion of depths of infinitary reasoning.

The paper format is as follows: Section 2 gives the definition of the sets of formulae. Section 3 formulates the system $GL(L_\alpha)$ and the Kripke semantics, and states the completeness result. In Sections 4 and 5, we give discussions on applications of our framework and the completeness
result to the definability problems of various epistemic concepts, and also on an application to the rationalizability concept in game theory. A proof of Kripke-completeness is given in Section 6. Section 7 concludes the paper.

2 Small Infinitary Languages $L_\alpha$

We fix an ordinal $\alpha^o$ with $\alpha^o \geq \omega = \{0, 1, \ldots \}$. We define the class of infinitary languages $\{L_\alpha : \alpha \leq \alpha^o\}$. For each $\alpha$, $L_\alpha$ is constructed from $\bigcup_{\beta<\alpha}L_\beta$ in an inductive manner, and we will show that $L_\alpha$ becomes constant after $\alpha = \omega$. We also evaluate the depths of formulae in $L_\alpha$, and show that the depth of the entire set $L_\omega$ is $\omega^2$. In the end of this section, we make brief comparisons with the set of formulae in the literatures of infinitary logics. We stipulate that Greek letters $\alpha, \beta, \gamma$ are ordinals up to $\alpha^o$, but Greek $\nu$ runs over the natural numbers $0, 1, \ldots$

We adopt the following list of primitive symbols:

- propositional variables: $p_0, p_1, \ldots$;
- logical connectives: $\neg$ (not), $\supset$ (implies), $\land$ (and);
- unary belief operators: $B_1(\cdot), \ldots, B_n(\cdot)$ ($1 \leq n < \omega$);
- parentheses: $\langle, \rangle$;
- brackets: $\{, \}$.

The conjunction symbol $\land$ is applied to a finite set of formulae and some infinite sequences of formulae. An infinitary conjunction is written as $\land(C^\nu : \nu \geq 0)$ and will be specified below. We denote $P_0 = \{p_0, p_1, \ldots\}$, and the set of agents (the subscripts for the beliefs operators) by $N = \{1, \ldots, n\}$. We may abbreviate the parentheses $\langle, \rangle$ and use different brackets when they cause no confusions.

Let $\alpha$ be an ordinal with $\alpha \leq \alpha^o$. Let $\mathcal{F}_\alpha$ be a given set of formulae with $\mathcal{F}_0 = \emptyset$, which is the source of infinitary conjunctions and is specified below. We define the set $L_\alpha$ for $\alpha \geq 0$ by a double induction. Specifically, when $\alpha = 0$, $P_0 = \{p_0, p_1, \ldots\}$, and when $\alpha > 0$, $P_\alpha = \bigcup_{\beta<\alpha}L_\beta$, provided that the set of formulae $L_\beta$ is already defined for all $\beta < \alpha$. We define the set $L_\alpha$ for each $\alpha \geq 0$ by the following three steps:

**Io0:** all formulae in $P_\alpha \cup \mathcal{F}_\alpha$ belong to $L_\alpha$;

**Io1** (finitary extension): if $A, B$ are formulae in $L_\alpha$, so are $(A \supset B), (\neg A), B_i(A)$ ($i \in N$); and if $\Phi$ is a nonempty finite set of formulae in $L_\alpha$, then $\langle \land \Phi \rangle$ is a formula in $L_\alpha$;

**Io2** (infinitary extension): if $\langle \land(C^\nu : \nu \geq 0) \rangle, \langle \land(D^\nu : \nu \geq 0) \rangle \in L_\alpha$ and $A \in L_\alpha$, then

(i) $\langle \land(A \supset C^\nu : \nu \geq 0) \rangle \in L_\alpha$;

(ii) $\langle \land(B_i(C^\nu) : \nu \geq 0) \rangle \in L_\alpha$ for all $i \in N$;

(iii) $\langle \land\{C^\nu, D^\nu\} : \nu \geq 0 \rangle \in L_\alpha$.

When $\alpha = 0$, step Io2 is vacuous since $\mathcal{F}_0 = \emptyset$; thus, $L_0$ is the set of all finitary formulae.

In Io1, the conjunction symbol $\land$ is applied to finite sets of formulae. We write $A \land B, A \land B \land C$ for $\langle \land \{A, B\} \rangle$ and $\langle \land \{A, B, C\} \rangle$, etc., and $A \equiv B$ for $(A \supset B) \land (B \supset A)$. Io1 and Io2 are interactive since formulae generated by Io2 may be used in Io1, and vice versa.

The set $\mathcal{F}_\alpha$ is determined by a given set of germinial forms specified as follows. A sequence $\langle C^\nu : \nu \geq 0 \rangle$ is called a germinial form iff $C^\nu \in L_0$ for all $\nu \geq 0$ and a finite number of propositional variables occur in $\langle C^\nu : \nu \geq 0 \rangle$. Let $p_1, \ldots, p_m$ be the propositional variables occurring in $\langle C^\nu : \nu \geq 0 \rangle$. We often denote each $C^\nu$ in $\langle C^\nu : \nu \geq 0 \rangle$ by $C^\nu(p_1, \ldots, p_m)$, though some of them may not be included in $C^\nu$. Let $A_1, \ldots, A_m$ be formulae in $P_\alpha = \bigcup_{\beta<\alpha}L_\beta$, which are
called germs. By substituting $A_t$ for each occurrence of $p_t$ in $(C''(p_1, ..., p_m) : \nu \geq 0)$, we obtain the sequence $(C''(A_1, ..., A_m) : \nu \geq 0)$. We say that $\Phi = (C''(A_1, ..., A_m) : \nu \geq 0)$ is generated by a germinal form $(C''(p_1, ..., p_m) : \nu \geq 0)$ and germs $A_1, ..., A_m$ in $P_\alpha$. This generation is illustrated as follows:

\[
(C''(p_1, ..., p_m) : \nu \geq 0) \quad \rightarrow \quad \Phi = (C''(A_1, ..., A_m) : \nu \geq 0)
\]

For example, $(C''(p) : \nu \geq 0) = (B''(p) : \nu \geq 0)$ is the germinal form for common knowledge. We remark that germinal forms do not require $(p_1, ..., p_m)$ to enter $C''(p_1, ..., p_m)$ positively, e.g., $(C''(p) : \nu \geq 0) = (\neg p, \neg \neg p, ...)$. A germinal form, and a less trivial one will be given later.

Let $G$ be a nonempty countable (possibly finite) set of germinal forms. We define:

\[
F_\alpha = \{\wedge \Phi : \Phi \text{ is generated some germinal form in } G \text{ and germs in } P_\alpha\}. \quad (4)
\]

Since $G$ is at most countable and used uniformly for all $\alpha \leq \alpha_0$, we can see that the sets $F_\alpha$ and $L_\alpha$ remain countable for each $\alpha \leq \alpha_0$.

In addition, Ito to Ito generate the other infinite conjunctions. We call $\wedge (C'' : \nu \geq 0) \in L_\alpha$ an $\alpha$-infinite conjunction, and $(C'' : \nu \geq 0)$ an $\alpha$-permissible sequence. Sometimes, we simply call $\wedge (C'' : \nu \geq 0)$ an infinite conduction. We stipulate that $A \in (C'' : \nu \geq 0)$ iff $A \in (C'' : \nu \geq 0)$. We use the same expression, $\wedge \Phi$, for a finite conjunction or an infinite conjunction. We write $B_\alpha(\Phi)$ for $\langle B_i(C) : C \in \Phi \rangle$ if $\Phi$ is an $\alpha$-permissible sequence or $\langle B_i(C) : C \in \Phi \rangle$ if $\Phi$ is a finite set of formulæ in $L_\alpha$.

A series of languages $\{L_\alpha : \alpha \leq \alpha^o\}$ is determined by a set of germinal forms $G$; we may write $L_\alpha = L_\alpha(G)$ to emphasize the choice of $G$ for $L_\alpha$. Each $L_\alpha$ serves a language for an epistemic logic GL($L_\alpha$) to be given in Section 3. Thus, $\{L_\alpha : \alpha \leq \alpha^o\} = \{L_\alpha(G) : \alpha \leq \alpha^o\}$ is not only a series of languages but also determines a series of epistemic logics. When $G$ is changing with fixed $\alpha$, we have another series of languages and logics. Using these series, we discuss the required depth $\alpha$ and germinal forms $G$ for a discourse involving infinitary concepts.

In Section 1, we gave the germinal form $(C''(p) : \nu \geq 0) = (B''(p) : \nu \geq 0)$ for common knowledge, which is defined by (2). As emphasized in Section 1, this is generated by iterations of substitutions. Here, we give a few more examples; the last one is not based on iterations of substitutions.

**Example 2.1 (1) Positive introspection:** Let $i \in N$ be fixed. We define

\[
B_0^i(p) = B_i(p) \text{ and } B_{\nu+1}^i(p) = B_i(B_j^\nu(p)) \text{ for } \nu \geq 0. \quad (5)
\]

The sequence $(B_\nu^i(p) : \nu \geq 0)$ is a possible germinal form. Then, we denote $B^{\omega^o}(p) := \wedge (B_j^\nu(p) : \nu \geq 0)$. For $A \in P_\alpha$, $B_\alpha^\nu(A)$ belongs to $F_\alpha$ as long as $(B_j^\nu(p) : \nu \geq 0) \in G$. We will see in Section 4 that the formula $B_\alpha^\nu(A)$ is regarded as the infinitary extension of finitary $B_i(A)$ in that $B_\alpha^\nu(A)$ enjoys the positive introspection property (Axiom 4) in GL($L_\omega$).

For both common knowledge and positive introspection, the germinal forms are obtained by substituting for one propositional variable. The next example needs two propositional variables. Game theoretical examples may involve more propositional variables; one example is given in Section 5.\footnote{The common belief of $A$ is defined by plugging germ $\wedge_{i \in N}B_i(A)$ to $p$ in $(B''(p) : \nu \geq 0)$, that is, $B''(\wedge_{i \in N}B_i(A))$.}
(2) Infinite regress: Let \( n = 2 \). We prepare two formulae \( B_i(p_j) \) and \( B_j(p_i) \) with \( \{i,j\} = \{1,2\} \). Then, the germinal forms \( \langle \text{Ir}_i^\nu[p_1,p_2] : \nu \geq 0 \rangle, i = 1,2, \) are generated as follows: for \( i,j = 1,2 \) (\( i \neq j \)),
\[
\text{Ir}_i^0[p_1,p_2] = B_i(p_i); \quad \text{and } \text{Ir}_i^{\nu+1}[p_1,p_2] = B_i(\text{Ir}_j^\nu[p_1,p_2]) \quad \text{for } \nu \geq 0.
\]

We write the conjunction \( \text{Ir}_i[p_1,p_2] := \land \langle \text{Ir}_i^\nu[p_1,p_2] : \nu \geq 0 \rangle \) for \( i = 1,2 \). Let \( A_1,A_2 \in \mathcal{P}_\alpha \). The epistemic infinite regress for agent \( i \) from \( A_i \) and \( A_j \) is given \( \text{Ir}_i[A_1,A_2] = \land (B_i(A_i), B_j(A_j), B_iB_j(A_i), ...) \).

Epistemic infinite regress is a subjective concept in that each formula for \( i \) occurs in the scope of \( B_i(\cdot) \), and is an extension of common belief. When \( A_1 = A_2 = A \), \( \text{Ir}_i[A_1,A_2] \land \text{Ir}_2[A_1,A_2] \) is equivalent to the common belief of \( A \). The epistemic infinitary regress takes subjectivity (and individuality) more seriously than common knowledge and common belief.

(3) More general germinal forms: We do not assume positivity for germinal forms. The example already given is \( \langle \neg p, \neg p, ... \rangle \), which is generated by iterated substitutions with \( \neg p \). This is inconsistent, but still allowed in our theory. A consistent example is the germinal forms
\[
\begin{align*}
&\langle B_1(p_1), \neg B_1B_2(p_2), \neg B_1\neg B_2B_1(p_1), \neg B_1\neg B_2\neg B_1B_2(p_2), \ldots \rangle; \\
&\langle B_2(p_2), \neg B_2B_1(p_1), \neg B_2\neg B_1B_2(p_2), \neg B_2\neg B_1\neg B_2B_1(p_1), \ldots \rangle,
\end{align*}
\]
each of which is obtained by \( C_1^\nu(p_1,p_2) = \neg B_1C_2^{\nu-1}(p_1,p_2) \) and \( C_2^\nu(p_1,p_2) = \neg B_2C_2^{\nu-1}(p_1,p_2) \) for each \( \nu \geq 1 \) with \( C_1^0(p_1,p_2) = B_1(p_1) \) and \( C_2^0(p_1,p_2) = B_2(p_2) \). Their conjunctions are consistent in our logic containing them in the language.

The above examples are constructed by iteration of substitutions. However, our formulation also allows for infinite conjunctions that cannot be obtained by iterated substitutions. For example, let \( \{k_\nu : \nu \geq 0\} \) be the sequence of Fibonacci numbers and define \( C^\nu(p) = B_1^{k_\nu}B_2^{k_{\nu-1}}B_1^{k_{\nu-2}}(p) \), where \( i = 1 \) if \( \nu \) is even and \( i = 2 \) otherwise. This sequence \( \langle C^\nu(p) : \nu \geq 0 \rangle \) is a germinal form but cannot be generated by iteration of substitutions. Moreover, germinal forms defined by uncomputable \( \{k_\nu : \nu \geq 0\} \) are also allowed.

The subformulae of \( A \in L_\alpha = L_\alpha(\mathcal{G}) \) are defined in the standard manner. Then, \( L_\alpha \) is subformula-closed. It is proved by the double induction over ordinals \( \alpha \) and over \( \text{Ia}0 - \text{Ia}2 \).

**Lemma 2.1.** Any subformula of \( A \in L_\alpha \) belongs to \( L_\alpha \).

The set of formulae \( L_\alpha \) is increasing up to \( \alpha = \omega \), but it becomes constant after \( \alpha = \omega \).

**Theorem 2.1.** (Stopping at \( \omega \)) Let \( \mathcal{G} \) be a fixed nonempty set of germinal forms. If \( \alpha < \omega \), then \( L_\alpha \subseteq L_{\alpha+1} \); and if \( \omega \leq \alpha \leq \alpha^\circ \), then \( L_\alpha = L_\omega = \mathcal{P}_\omega (= \cup_{\beta < \omega} L_\beta) \).

**Proof.** Let \( \langle C^\nu(p_1,\ldots,p_m) : \nu \geq 0 \rangle \) be a germinal form in \( \mathcal{G} \). Since \( \land \langle C^\nu(p_1,\ldots,p_m) : \nu \geq 0 \rangle \in L_1 - L_0 \), we have \( L_0 \subseteq L_1 \). Let \( 1 \leq \alpha < \omega \). Suppose \( L_{\alpha-1} \subseteq L_\alpha \). By \( I(\alpha + 1)0 \), \( L_\alpha \subseteq L_{\alpha+1} \). Take \( A_1,\ldots,A_m \in L_\alpha - L_{\alpha-1} \). Then, \( \land \langle C^\nu(A_1,\ldots,A_m) : \nu \geq 0 \rangle \) is in \( \mathcal{F}_{\alpha+1} \) but not in \( \mathcal{F}_{\alpha} \); so, it is not in \( L_\alpha \). Hence, \( L_\alpha \not\subseteq L_{\alpha+1} \).

Consider the latter assertion of the theorem. By \( \text{Iw}0 - \text{Iw}2 \), \( \mathcal{P}_\omega \subseteq L_\omega \). Now, we show \( L_\omega \subseteq \mathcal{P}_\omega \). Take germs \( A_1,\ldots,A_m \in \mathcal{P}_\omega \). These germs belong to \( L_\gamma \) for some \( \gamma < \omega \). Hence, \( \land \langle C^\nu(A_1,\ldots,A_m) : \nu \geq 0 \rangle \) belongs to \( \mathcal{F}_{\gamma+1} \). Thus, any formulae generated by \( \text{Iw}0 - \text{Iw}2 \) belong to \( L_\beta \) for some \( \beta < \omega \). Hence, \( L_\omega \subseteq L_\omega = \cup_{\beta < \omega} L_\beta \). Now, by induction over \( \alpha \) up to \( \alpha^\circ \), we have \( \mathcal{P}_\omega = L_\omega = L_\alpha \) for all \( \alpha \) (\( \omega \leq \alpha \leq \alpha^\circ \)).
The set $L_\alpha = L_\alpha(\mathcal{G})$ ($0 \leq \alpha \leq \omega$, a countable $\mathcal{G}$) is small in the sense that it remains countable. Also, the depths of formulae in $L_\alpha$ are relevant to evaluations of infinitary concepts such as common knowledge. We introduce the depth function $\delta$ over $L_\omega$, which assigns an ordinal number to each formula in $L_\omega$. We define $\delta$ inductively along the definition of formulae as follows:

**d0:** $\delta(p) = 0$ for all propositional variables $p$;

**d1:** $\delta(\neg A) = \delta(A) + 1$, and $\delta(A \supset B) = \max(\delta(A), \delta(B)) + 1$;

**d2:** $\delta(B_i(A)) = \delta(A) + 1$ for all $i \in N$;

**d3:** $\delta(\land \Phi) = \sup\{\delta(C) + 1 : C \in \Phi\}$.

Step d3 have several cases; $\Phi$ may be a finite set of formulae in $\text{Io1}$ and $\Phi$ may be an $\alpha$-permissible sequence in $\mathcal{F}_\alpha$ or generated by $\text{Io2}$. If $\sup\{\delta(C) + 1 : C \in \Phi\}$ is a limit ordinal, then $\delta(\land \Phi) = \sup\{\delta(C) : C \in \Phi\}$, and otherwise, $\delta(\land \Phi) = \sup\{\delta(C) : C \in \Phi\} + 1$. For any set of formulae $\Gamma$, we define $\delta(\Gamma) = \sup\{\delta(A) : A \in \Gamma\}$. Since $L_0$ consists only of finitary formulae, we have $\delta(L_0) = \sup\{\delta(A) : A \in L_0\} = \omega$. It follows from d0-d3 that for any $A \in L_\omega$, $\delta(C) < \delta(A)$ for any proper subformula $C$ of $A$.

Consider the formula $B^\gamma_i(p) = \land\{B^\gamma_i(p) : \nu \geq 0\}$ in Example 2.1.(1). Then, $\delta(B^\gamma_i(A)) = \omega + 1$ and $B^\gamma_i(p) \in L_1 - L_0$, provided $\{B^\gamma_i(p) : \nu \geq 0\} \in \mathcal{G}$. Any formula $D$ in $L_1$ including $B^\gamma_i(A)$ takes the form $\omega + k$ for some finite $k$, and this $k$ may be arbitrary large; thus, $\delta(D) < \omega + \omega = \omega^2$ and $\delta(L_1) = \omega^2$. The following theorem generalizes this observation.

**Theorem 2.2. (Depths of Formulae)** Suppose that $\mathcal{G}$ has a germinal form $\langle C^\nu(p_1, ..., p_m) : \nu \geq 0 \rangle$ such that $\sup\{\delta(C^\nu(p_1, ..., p_m)) : \nu \geq 0\} = \omega$.

(1): If $0 \leq \alpha < \omega$, then $\delta(A) < \omega(\alpha + 1)$ for all $A \in L_\alpha$; and $\delta(L_\alpha) = \omega(\alpha + 1)$.

(2): $\delta(A) < \omega^2$ for all $A \in L_\omega$; and $\delta(L_\omega) = \omega^2$.

**Proof.** (1): As mentioned above, $\delta(A) < \omega$ for all $A \in L_0$ and $\delta(L_0) = \omega$. Let $1 \leq \alpha < \omega$, and suppose the induction hypothesis that $\delta(A) < \omega\alpha$ for all $A \in L_{\alpha-1}$ and $\delta(L_{\alpha-1}) = \omega\alpha$. Then, we prove the assertions for $\alpha$. First, we show that $\delta(A) < \omega(\alpha + 1)$ for all $A \in L_\alpha$.

Let $\land \Phi \in \mathcal{F}_\alpha$. Since $\delta(A) < \omega\alpha$ for all $A \in \Phi$ by the induction hypothesis, we have $\delta(\land \Phi) \leq \omega\alpha$ by d3. Thus, $\delta(A) \leq \omega\alpha$ for any $A \in \mathcal{P}_\alpha \cup \mathcal{F}_\alpha$. Now, consider Io1. Suppose the other induction hypothesis that for any immediate subformula $C$ of $A$ generated by Io1, $\delta(C) \leq \omega\alpha + k$ for some $k < \omega$. Then, by d1-d3, we have $\delta(A) \leq \omega\alpha + k'$ for some $k' < \omega$.

Consider Io2. The induction hypothesis is that $\delta(D) \leq \omega\alpha + k$ and $\delta(\land \Phi) \leq \omega\alpha + k$ for some $k < \omega$. Then, $\delta(D \supset C) \leq (\omega\alpha + k) + 1$ for any $C \in \Phi$; and so $\delta(\land \{D \supset C : C \in \Phi\}) \leq (\omega\alpha + k) + 1$. Also, $\delta(B_i(C)) \leq (\omega\alpha + k) + 1$ for any $C \in \Phi$; and so $\delta(\land \{B_i(C) : C \in \Phi\}) \leq (\omega\alpha + k) + 1$. The case of Io2.(iii) is similar. Thus, for a formula $A$ generated by Io2, it still holds that $\delta(A) \leq \omega\alpha + k'$ for some $k' < \omega$. By these two paragraphs and induction, it holds that $\delta(A) < \omega(\alpha + 1)$ for all $A \in L_\alpha$.

For $\delta(L_\alpha) = \omega(\alpha + 1)$, we show that for any $k < \omega$, there is a formula $C \in L_\alpha$ so that $\delta(C) \geq \omega + k$. Now, since $\delta(L_{\alpha-1}) = \omega\alpha$, there are formulae $A_1, ..., A_m \in L_{\alpha-1}$ such that $\delta(A_t) \geq \omega(\alpha - 1)$ for $t = 1, ..., m$. Let $\langle C^\nu(p_1, ..., p_m) : \nu \geq 0 \rangle$ be a germinal form given in the assumption of the theorem. Consider $\land \{C^\nu(A_1, ..., A_m) : \nu \geq 0\}$. Since $\sup\{\delta(C^\nu) : \nu \geq 0\} = \omega$, there is a $\nu$ for any $k < \omega$ such that $\delta(C^\nu(A_1, ..., A_m)) \geq \omega(\alpha - 1) + k$. Hence, $\delta(\land \{C^\nu(A_1, ..., A_m) : \nu \geq 0\}) = \omega(\alpha - 1) + \omega = \omega\alpha$. Then, using Io1, for any $k < \omega$, we find a formula $F \in L_\alpha$ so
that $\delta(F) \geq \omega \alpha + k$. Thus, $\delta(L_\alpha) \geq \sup_k(\omega \alpha + k) = \omega \alpha + \omega = \omega(\alpha + 1)$, and by the conclusion of the previous paragraph, we have $\delta(L_\alpha) = \omega(\alpha + 1)$.

The first part of (2) follows (1), since $L_\omega = \cup_{\alpha<\omega} L_\alpha$ by Theorem 2.1. The second part follows $L_\omega = \cup_{\alpha<\omega} L_\alpha$ and (1); indeed, $\delta(L_\omega) = \delta(\cup_{\alpha<\omega} L_\alpha) = \sup\{\delta(L_\alpha) : \alpha \geq 0\} = \sup\{\omega(\alpha + 1) : \omega > \alpha \geq 0\} = \omega^2$. □

Theorem 2.2 is summarized in Table 2.1; our infinitary languages $L_\alpha$ ($1 \leq \alpha \leq \omega$) include infinitary conjunctions but are not much larger than the finitary language $L_0$. These extensions are large enough for treatments of infinitary concepts mentioned above.

<table>
<thead>
<tr>
<th>Table 2.1: Depths and cardinalities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$#\quad L_0$</td>
</tr>
<tr>
<td>depth</td>
</tr>
</tbody>
</table>

Let us compare the above theorem with the infinitary logic approach. Following Kaneko-Nagashima [20], we construct $L_\alpha$ ($0 \leq \alpha \leq \alpha^\omega$) as follows. Let $F_0 = \emptyset$, $P_0 = \{P_0, P_1, \ldots\}$. Let $L_0 = L_0$. For any $\alpha$ ($1 \leq \alpha \leq \alpha^\omega$), assuming that $L_\beta$ are defined for any $\beta < \alpha$, we define

$K\text{N}a : F_\alpha = \{\wedge \Phi : \Phi$ is a countable subset of $\cup_{\beta < \alpha} L_\beta\}$,

and then $L_\alpha$ is defined by Io0 with $F_\alpha$ and $\cup_{\beta < \alpha} L_\beta$ and by Io1-Io2. We denote the set of formulae for step $\alpha$ by $L_\alpha$. The set $L_1$ is already uncountable. Also, $L_\alpha$ does not stop at $\alpha = \omega$, e.g., $\cup_{\beta < \alpha} L_\beta \subseteq L_\omega \subseteq L_{\omega+1}$ for all $\alpha \leq \omega$. Then, $\delta(\cup_{\alpha<\omega} L_\alpha) = \omega^2$ but $\delta(L_\omega) = \omega^2 + \omega$. This sequence $L_\alpha$ increases up to the first uncountable ordinal $\omega_1$, where we assume $\alpha^\omega \geq \omega_1$. Tanaka-Ono [33] considered the smallest set, $L_{TO}$, that is closed with respect to finitary operations on $\neg$, $\supset$, $B_i(\cdot)$ and countable conjunctions:

$TO$: for any countable subset $\Phi$ of $L_{TO}$, $\wedge \Phi$ belongs to $L_{TO}$.

Then, it holds that $L^{TO} = \cup_{\beta < \omega_1} L_\beta$. This $L^{TO}$ is the smallest infinitary language in the sense of Karp [19].

### 3 Epistemic Logics $GL(L_\alpha)$ ($0 \leq \alpha \leq \omega$)

We formulate a Hilbert-style proof theory and Kripke-semantics for epistemic logic $GL(L_\alpha) = GL(L_\alpha(G))$ with $0 \leq \alpha \leq \omega$ and a countable set of germinal forms $G$. We state the soundness-completeness theorem (Theorem 3.1), which will be proved in Section 6. We discuss the hierarchy of $GL(L_\alpha(G))$ with respect to both $\alpha$ and $G$, and provide four meta-lemmas to be used in Section 4.

#### 3.1 Hilbert-style proof theory

The base logic for epistemic logic $GL(L_\alpha)$ is an infinitary classical logic defined by the following four axiom schemata and two inference rules: for all formulae $A, B, C, \wedge \Phi$ in $L_\alpha$,

$L_1$: $A \supset (B \supset A)$;

$L_2$: $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$;

$L_3$: $(\Phi \supset \neg \neg \Phi)$;

$L_4$: $(\neg \neg \Phi \supset \Phi)$.
L3: \((\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)\);
L4: \(\land \Phi \supset C\), where \(C \in \Phi\);

**Modus Ponens:** \[
\frac{A \supset B \quad A}{B}
\]

and **\(\land\)-rule:** \[
\frac{A \supset C : C \in \Phi}{A \supset \land \Phi}
\]

We add the following epistemic axiom schemata and inference rule: for any \(A, C, \land \Phi\) in \(L_\alpha\) and \(i \in N\),

**K:** \(B_i(A \supset C) \supset (B_i(A) \supset B_i(C))\);

**D:** \(\neg B_i(\neg A \land A)\);

**\(\land\)-Barcan:** \(\land B_i(\Phi) \supset B_i(\land \Phi)\);

**Necessitation:** \[
\frac{A}{B_i(A)}
\]

The above axiomatization is an infinitary version of epistemic logic KD\(^n\) with the \(\land\)-Barcan axiom (conjunctive analogue of the Barcan axiom \(\forall x(\Box A(x)) \supset \Box(\forall x A(x))\) in the first order modal logic). Infinitary aspects are included in L4, \(\land\)-rule, and \(\land\)-Barcan, while the other axioms and inference rules do not directly operate on infinitary structures. The definition of \(L_\alpha\) guarantees the well-definedness of L4, \(\land\)-rule, and \(\land\)-Barcan. Indeed, an instance \(\land \Phi \supset C\) for L4 is in \(L_\alpha\) for all \(C \in \Phi\) by Lemma 2.1 and Ioa1. The sequence \(\langle A \supset C : C \in \Phi\rangle\) of the upper formulae in \(\land\)-rule is \(\alpha\)-permissible by Ioa2.(i). Since \(B_i(\land \Phi) \in L_\alpha\) by Ioa1 and \(\land B_i(\Phi) \in L_\alpha\) by Ioa2.(ii), the formula \(\land B_i(\Phi) \supset B_i(\land \Phi)\) of the \(\land\)-Barcan axiom is in \(L_\alpha\). An equivalent form of Axiom D is \(B_i(\neg A) \supset \neg B_i(A)\), which is used in (20) in Section 5.

A proof \(P = \langle X, <, f \rangle\) in GL\((L_\alpha)\) consists of a countable tree \(\langle X, < \rangle\) and a function \(f : X \rightarrow L_\alpha\) with the following requirements:

(o): \(\langle X, < \rangle\) has no infinite path from its root;

(i): for each node \(x\) in \(\langle X, < \rangle\), \(f(x)\) is a formula attached to \(x\);

(ii): for each leaf \(x\) in \(\langle X, < \rangle\), \(f(x)\) is an instance of the axiom schemata;

(iii): for each non-leaf \(x\) in \(\langle X, < \rangle\),

\[
\frac{\{f(y) : y \text{ is an immediate successor of } x\}}{f(x)}
\]

is an instance of the inference rules, MP, \(\land\)-rule, and Nec.

Infinite branching is possible in (iii) to allow inferences with \(\land\)-rule. Thus, the width of \(\langle X, < \rangle\) can be countably infinite and also the supremum of the depths can be infinite.

When \(A\) is attached to the root node of \(P = \langle X, <, f \rangle\), we call \(P\) a proof of \(A\). We say that \(A\) is **provable** in GL\((L_\alpha)\), denoted by \(\vdash A\), iff there is a proof of \(A\) in GL\((L_\alpha)\).

Lemma 3.1 states basic properties of the provability relation \(\vdash\) in GL\((L_\alpha)\). Since we adopt a particular axiomatization of classical logic, these should be proved. Since the fragment determined by \(\supset\) and \(\neg\) with L1-L3, MP is a standard formulation of classical proposition logic, a proof of (1) is found in a textbook (e.g., Mendelson [24]). Since our system additionally includes the connective \(\land\), (2) is crucial; a proof is given in Kaneko [18], Lemma 11.1. (3) is the converse of \(\land\)-Barcan, which is proved for any permissible or finite \(\Phi\); indeed, since \(\vdash \land \Phi \supset A\) for \(A \in \Phi\) by L4, we have \(\vdash B_i(\land \Phi) \supset B_i(A)\) by Nec and K. Since this holds for all \(A \in \Phi\), we have, by \(\land\)-rule, \(\vdash B_i(\land \Phi) \supset \land B_i(\Phi)\). Incidentally, when \(\Phi\) is a finite set, the \(\land\)-Barcan axiom is unnecessary, i.e., \(\land B_i(\Phi) \supset B_i(\land \Phi)\) is derived without using \(\land\)-Barcan.
Lemma 3.1. For any $A, B, C, \land \Phi \in L_\alpha$, and $i \in N$,

(1): $\vdash A \supset B$ and $\vdash B \supset C$ imply $\vdash A \supset C$;

(2): $\vdash [A \land B \supset C] \equiv A \supset (B \supset C)$;

(3): $\vdash B_i(\land \Phi) \supset \land B_i(\Phi)$.

Remark 3.1. (1): We can take the standard de Morgan method to define the disjunction formula as $\lor \Phi := \neg \land \{\neg A : A \in \Phi\}$ for a finite set of formulae $\Phi$. For an $\alpha$-permissible sequence $\Phi$, this could work when we extend $I_\alpha 2$ to include $(\neg A : A \in \Phi)$ for any $\land \Phi \in L_\alpha$, which is not included in this paper.

(2): In $GL(L_\alpha)$, the substitution-rule is stated as follows: for any $A[p]$ and $B$ in $L_\alpha$,

$$\text{if } \vdash A[p] \text{ and } A[B] \in L_\alpha, \text{ then } \vdash A[B],$$

where $A[p]$ is a formula in $L_\alpha$ and $A[B]$ is the formula obtained from $A[p]$ by substituting $B$ for all occurrences of $p$. This fact will be used in Lemma 4.2.

3.2 Kripke completeness

A Kripke frame $K = (W; R_1, \ldots, R_n)$ is an $(n + 1)$-tuple of a set of possible worlds and $n$ accessibility relations over $W$, where $W$ is an arbitrary nonempty set and $R_i$ is a serial binary relation over $W$ for each $i \in N$, i.e., for any $w \in W$, $(w, u) \in R_i$ for some $u \in W$. A truth assignment $\tau$ is a function from $W \times P_0$ to $\{\top, \bot\}$. A pair $(K, \tau)$ is a Kripke model.

Let $\mathcal{G}$ be a fixed countable set of germinal forms. The valuation $(K, \tau, w) \models w \in W$ is inductively defined over $L_\alpha = L_\alpha(\mathcal{G})$ as follows: for any $A, C, \land \Phi \in L_\alpha = L_\alpha(\mathcal{G})$, and any $w \in W$,

V0: for any $p \in P_0$, $(K, \tau, w) \models p \iff \tau(w, p) = \top$;

V1: $(K, \tau, w) \models \neg A \iff (K, \tau, w) \not\models A$;

V2: $(K, \tau, w) \models A \supset C \iff (K, \tau, w) \not\models A$ or $(K, \tau, w) \models C$;

V3: $(K, \tau, w) \models \land \Phi \iff (K, \tau, w) \models A$ for all $A \in \Phi$;

V4: $(K, \tau, w) \models B_i(A) \iff (K, \tau, v) \models A$ for all $v$ with $w R_i v$.

Since $L_\alpha \subseteq L_\omega$ ($\alpha \leq \omega$), the valuation $(K, \tau, w) \models$ is uniform over $L_\alpha$ for all $\alpha \leq \omega$; that is, it is defined over $L_\omega$ and it can be restricted to $L_\alpha$. For any $A \in L_\alpha$, we write $\models A$ iff $(K, \tau, w) \models A$ for all $K, w \in W$ and $\tau$.

We have the following soundness-completeness theorem; the proof of soundness is standard and mentioned below, and completeness will be proved in Section 6. In the theorem, let $\mathcal{G}$ be a fixed (at most countable) set of germinal forms.

Theorem 3.1. (Soundness and completeness for $GL(L_\alpha)$) Let $\alpha$ be an ordinal with $0 \leq \alpha \leq \omega$. For any $A \in L_\alpha$, $GL(L_\alpha) \vdash A$ if and only if $\models A$.

Soundness (the only-if part) implies the contradiction-freeness of logic $GL(L_\alpha)$, which will be used in the proof of completeness. Also, by soundness, we can see consistency of the conjunctions of both germinal forms in (7) by the following Kripke model; both are true in the middle world.

$$\begin{align*}
\varnothing p_1, \neg p_2 & \iff 1 \\
\varnothing p_1, p_2 & \iff 2 \\
\neg p_1, p_2 & \iff 2
\end{align*}$$

Figure 3.1
Soundness is proved as follows: Let \( P = (X, <; f) \) be a proof of \( A \) in \( GL(L_\alpha) \). We prove by induction on the tree structure of \( P \) from its leaves that \( \models C \) for each formula \( C = f(x) \) attached to a node \( x \) in \( P \). Each step is verified in the following lemma.

**Lemma 3.2.** (1): Let \( A \) be an instance of \( L 1-L4 \) in \( L_\alpha \). Then \( \models A \).

(2): Let \( A \) be an instance of Axioms \( K, D, \land \)-Barcan in \( L_\alpha \). Then \( \models A \).

(3): \( \models \) satisfies inference rules \( MP, \land \)-rule, and Necessitation.

**Proof.** We see only the truthfulness of \( \land \)-Barcan. Let \( (K, \tau, w) \models \land B_i(\Phi) \). Then, \( (K, \tau, w) \models B_i(C) \) for any \( C \in \Phi \). Then, for any accessible \( v \in W \) from \( w \) by \( R_i \), it holds that for any \( C \in \Phi \), \( (K, \tau, v) \models C \), equivalently, \( (K, \tau, v) \models C \); thus, \( (K, \tau, v) \models \land \Phi \) holds for any accessible \( v \in W \) by \( R_i \). This implies \( (K, \tau, w) \models B_i(\land \Phi) \). Thus, \( (K, \tau, w) \models \land B_i(\land \Phi) \) is true. \( \blacksquare \)

For completeness, a difficulty is to show the existence of a maximal consistent set. For this aim, Karp [19] assumes Axiom of Choice within her axiomatic system. We do not choose this method; instead, we adopt the \( Q \)-filter method due to Rasiowa-Sikorski [29] and the multi-modal extension given by Tanaka-Ono [33]. Here, a \( Q \)-filter plays the role of a maximal consistent set. A sketch of a proof of our proof will be given in Section 6.1.

The above completeness result holds when we add Axioms T, 4, and 5 (or drop D), either in combination or in isolation, and add the corresponding conditions, reflexivity, transitivity, and euclidean (or drop seriality) on accessibility relation \( R_i \) \((i \in N)\). Required modifications of the proof will be stated in Remark 6.1. On the contrary, in our framework, we can evaluate these axioms by studying explicit definability of each axiom, which will be undertaken in Section 4.

### 3.3 Conservativity and four meta-lemmas

We have the conservativity result between two logics with orders over \( \alpha \)'s and \( \mathcal{G} \)'s.

**Theorem 3.2.** (Conservativity) Let \( \alpha \leq \beta \leq \omega \) and \( \mathcal{G}, \mathcal{G}' \) two sets of germinal forms with \( \mathcal{G} \subseteq \mathcal{G}' \). Then, for any \( A \in L_\alpha(\mathcal{G}) \), \( GL(L_\alpha(\mathcal{G})) \vdash A \) if and only if \( GL(L_\beta(\mathcal{G}')) \vdash A \).

**Proof.** The if part is essential. Let \( GL(L_\alpha(\mathcal{G})) \vdash A \). Let \((\mathcal{K}, \tau)\) be any serial Kripke model, and \( w \) any world in \( \mathcal{K} \). By Theorem 3.1, we have \((\mathcal{K}, \tau, w) \models A \). Because of subformula-closedness (Lemma 2.1) and the definition V0-V4 for \( (\mathcal{K}, \tau, w) \models A \), the statement \((\mathcal{K}, \tau, w) \models A \) is determined in \( L_\alpha(\mathcal{G}) \). Since this holds for any \( \mathcal{K}, \tau, w \in W \), we have \( GL(L_\alpha(\mathcal{G})) \vdash A \) by Theorem 3.1.\( \blacksquare \)

By Theorem 3.2, our infinitary logics form the hierarchy with the conservative extension relation \( \Rightarrow \), described as in Table 3.1: each row is a series of logics with the same \( \mathcal{G} \), corresponding to (1), and each column is a series with the same \( \alpha \) with \( \mathcal{G} \subseteq \mathcal{G}' \subseteq \mathcal{G}'' \). The weakest logic is \( GL(L_0) = KD^n \) and the strongest is \( GL(L_\omega(\mathcal{G})) \) in the row with the same \( \mathcal{G} \). It holds that for each fixed \( A \in L_\omega(\mathcal{G}) \), we can find the smallest \( \alpha_A < \omega \) and \( \mathcal{G}_A \subseteq \mathcal{G} \) such that \( A \in L_{\alpha_A}(\mathcal{G}_A) \);
then Theorem 3.2 implies that $\text{GL}(L_{\alpha\beta}(G_A)) \vdash A \iff \text{GL}(L_\omega(G)) \vdash A$.

### Table 3.1. Hierarchy of infinitary epistemic logics

$$
\begin{array}{cccc}
\text{GL}(L_0) & \Rightarrow & \text{GL}(L_1(G)) & \Rightarrow \\
\downarrow & & \downarrow & \\
\text{GL}(L_1(G')) & \Rightarrow & \text{GL}(L_\omega(G')) & \\
\downarrow & & \downarrow & \\
\text{GL}(L_1(G'')) & \Rightarrow & \cdots & \Rightarrow \\
\end{array}
$$

In terms of languages, the arrows $\Rightarrow$ and $\downarrow$ are strict; $L_{\alpha}(G)$ is a proper subset of $L_{\beta}(G')$ whenever $\alpha < \beta$ or $G \subsetneq G'$. In terms of provability, it is more subtle. Consider positive introspection (Example 2.1.(1)) and let $G = \{\langle B_i^n(p) : \nu \geq 0 \rangle\}$, and $G' = \{\langle B_i^n(p) : \nu \geq 0 \rangle, \langle B_j^n(p \lor \neg p) : \nu \geq 0 \rangle\}$. Then, the vertical relation $\downarrow$ between $\text{GL}(L_1(G))$ and $\text{GL}(L_1(G'))$ collapses in the sense that for any $A' \in L_1(G')$, there is a formula $A \in L_1(G)$ such that $\text{GL}(L_1(G')) \vdash A' \equiv A$.\(^\text{3}\) When $G'' = \{\langle B_i^n(p \lor \neg p) : \nu \geq 0 \rangle\}$, we have the entire collapse result from $\text{GL}(L_{\alpha}(G''))$ to $\text{GL}(L_0) = \text{KD}^\alpha$ for any $\alpha \geq 0$, though $L_{\alpha}(G'')$ contains infinite formulae.

Conversely, both arrows can be strict. Here, we give only two examples. The strictness holds between $\text{GL}(L_0(G)) = \text{KD}^\alpha$ and $\text{GL}(L_1(G))$; we show by Lemma 3.3, given below, that for any $A \in L_0(G) = L_0$,

$$
\text{GL}(L_1(G)) \vdash A \supset B_i^n(p) \Rightarrow \text{GL}(L_1(G)) \vdash \neg A. \quad (9)
$$

Thus, there is no formula $A \in L_0(G)$ such that $\text{GL}(L_1(G)) \vdash A \equiv B_i^n(p)$. Now, let $G' = \{\langle B_i^n(p) : \nu \geq 0 \rangle, \langle B_j^n(p) : \nu \geq 0 \rangle\}$ \((i \neq j)\). It holds that for any $A \in L_1(G)$,

$$
\text{GL}(L_1(G')) \vdash A \supset B_j^n(p) \Rightarrow \text{GL}(L_1(G)) \vdash \neg A. \quad (10)
$$

A proof is given in the working paper version of this paper.\(^\text{4}\) Then, there is no formula $A \in L_1(G)$ such that $\text{GL}(L_1(G')) \vdash A \equiv B_j^n(p)$. However, a general study of the hierarchy in Table 2.1 is beyond the scope of the current paper.

Here, we give four meta-results; two are known in a finitary $\text{KD}^\alpha$ (cf., Kaneko-Suzuki [22]) and the other two are new. First, the depth lemma for $\text{GL}(L_0) = \text{KD}^\alpha$ is converted to $\text{GL}(L_\alpha)$ by Theorem 3.2. Recall the depth measure $\delta$ given in Section 2.\(^\text{5}\)

**Lemma 3.3.** (Depth lemma) Let $A$ and $C$ be two formulae in $L_0$. Let $(i_1, \ldots, i_k)$ be a sequence of agents in $N$ and $\delta(A) < k$. In $\text{GL}(L_\alpha)$, if $\vdash A \supset B_{i_1} \ldots B_{i_k}(C)$, then $\vdash \neg A$ or $\vdash C$.

Assertion (9) is proved by this lemma. Let $\text{GL}(L_1(G)) \vdash A \supset B_i^n(p)$ and $k > \delta(A)$. Then, $\text{GL}(L_1(G)) \vdash A \supset B_i^n(p)$, which implies $\vdash \neg A$ by Lemma 3.3.

The second result is an extension of the epistemic disjunction lemma for $\text{KD}^\alpha$. The following lemma is stated in $\text{GL}(L_\alpha)$, but can be proved in the same manner as in [22], i.e., by constructing a counter-model based upon Theorem 3.1. Recall Remark 3.1 about disjunction $\lor$.

**Lemma 3.4.** (Epistemic Disjunction lemma) Let $A, C \in L_\alpha$. In $\text{GL}(L_\alpha)$, $\vdash B_i(A) \lor B_i(C)$ if and only if $\vdash B_i(A)$ or $\vdash B_i(C)$.\(^\text{6}\)

\(^{3}\)A referee gave a similar example to show the collapse of $\downarrow$.


\(^{5}\)In [22], the epistemic depth to count only the nested occurrences of $B_i$, $i \in N$ is used for this lemma.
The third result enables us to move forward/backward from the beliefs and their contents. This will be used in Section 5.

Lemma 3.5. (Scope Lemma) Let \( A, C \in L_\alpha \). In \( GL(L_\alpha), \vdash B_i(A) \supset B_i(C) \) if and only if \( \vdash A \supset C \).

Proof. The if part is straightforward. We show the contrapositive of the only-if part. Suppose \( \not\vdash A \supset C \). By Theorem 3.1, there is a model \((K, \tau)\) such that \((K, \tau, w) \models A\) but \((K, \tau, w) \not\models C\) for some world \( w \in W \). Now, we add a new world \( w^* \) to \( W \) so that \( W^* = W \cup \{w^*\} \), \( R_i^* = R_i \cup \{(w^*, w)\}\) for all \( i \neq j \). We extend \( \tau \) to \( \tau^*: W^* \times P_0 \to \{\top, \bot\} \) so that \( \tau^*(u, p) = \tau(u, p) \) for all \( (u, p) \in W \times P_0 \) and \( \tau^*(w^*, p) \) is arbitrary for all \( p \in P_0 \). We have a new model \((K^*, \tau^*)\). In this new model, all valuations are preserved from \((K, \tau)\). Since agent \( i \) refers only to \( w \) at \( w^* \), we have \((K, \tau, w^*) \models B_i(A)\) but \((K, \tau, w^*) \not\models B_i(C)\). Hence, \((K, \tau, w^*) \not\models B_i(A) \supset B_i(C)\). By Theorem 3.1, \( \not\vdash B_i(A) \supset B_i(C)\). \(\square\)

Using this lemma and Theorem 3.1, we can prove that in \( GL(L_\alpha) \), \( \not\vdash B_i(p) \supset B_i(p) \) and \( \not\vdash B_i B_i(p) \supset B_i(p) \). Thus, Axioms 4 and T are not provable in our logic. Nevertheless, \( \vdash B_i^+(p) \supset B_i B_i^+(p) \) but \( \not\vdash B_i B_i^+(p) \supset B_i^+(p) \). This unprovability is shown by the counter-model:

\[
\begin{align*}
\bigcirc p &\rightarrow_i \bigcirc \neg p \rightarrow_i \bigcirc p \\
\end{align*}
\]

This is a counter-model also for \( B_i^+ B_i^+(p) \supset B_i^+(p) \) in \( GL(L_2(\mathcal{G})) \).

The next lemma, which is the dual of \( \land \)-rule, will be used in Section 5.

Lemma 3.6. (Infinitary conjunctions) Let \( A, \land(C^\nu : \nu \geq 0) \in L_\alpha \). In \( GL(L_\alpha) \), if \( \vdash A \supset \neg C^\nu \) for some \( \nu \geq 0 \), then \( \vdash A \supset \neg \land(C^\nu : \nu \geq 0) \).

Proof. Let \( \vdash A \supset \neg C^\nu \) for some \( \nu \geq 0 \). Let \((K, \tau)\) be any model and \( w \) any world in \( W \) with \((K, \tau, w) \models A\). By Theorem 3.1, \((K, \tau, w) \models \neg C^\nu\), i.e., \((K, \tau, w) \not\models C^\nu\). Thus, \((K, \tau, w) \not\models \land(C^\nu : \nu \geq 0)\), equivalently, \((K, \tau, w) \models \neg \land(C^\nu : \nu \geq 0)\). Thus, \((K, \tau, w) \models A \supset \neg \land(C^\nu : \nu \geq 0)\). Since \((K, \tau)\) and \( w \) are arbitrary, we have, by Theorem 3.1, \( \vdash A \supset \neg \land(C^\nu : \nu \geq 0)\). \(\square\)

4 Application 1: Evaluations of Various Epistemic Concepts

From the viewpoint of epistemic logics, the choices of Axioms T, 4, and 5 are of great importance. Completeness is one criterion but is neutral in the sense that our logics accommodate all these axioms, as stated after Theorem 3.1. Axioms 4 and 5 include infinitary aspects, though they are formulated in a finitary logic. Here, we ask whether each can be explicitly defined in our infinitary logics. The answers differ for T, 4, and 5. Then, we consider the possibility of embedding a logic with such an axiom to \( GL(L_\alpha) \). A similar consideration is given to the concept of common knowledge. In the end of Section 4.3, we give a small summary of differences between our approach and modal \( \mu \)-calculus.
4.1 Explicit definabilities of Axioms T, 4, and 5 in GL($L_\alpha$)

We fix one agent $i$ throughout Sections 4.1 and 4.2. Also, a set of germinal forms $G$ is fixed here. We begin with the following requirements for a target formula $F_i(p)$ in $L_\alpha$: for any $A, C \in L_\alpha$,

\begin{align*}
F_{0i} & : F_i(A) \in L_\alpha; \\
F_{Ei} & : \vdash F_i(A) \supset B_i(A); \\
F_{Ki} & : \vdash F_i(A \supset C) \supset (F_i(A) \supset F_i(C)); \\
F_{Ni} & : \vdash A \text{ implies } \vdash F_i(A),
\end{align*}

where $F_i(p)$ contains only propositional variable $p$ and $\vdash$ is the provability relation in GL($L_\alpha$). $F_{0i}$ means that $F_i(\cdot)$ is applicable to any $A \in L_\alpha$, and $F_{Ei}$ that $F_i(\cdot)$ is an extension of the belief operator $B_i(\cdot)$. $F_{Ki}$ and $F_{Ni}$ correspond to Axiom K and Nec. The corresponding requirement to Axiom D, $\vdash \neg F_i(A \land \neg A)$, is implied by the contrapositive of $F_{Ni} \vdash \neg B_i(A \land \neg A) \supset \neg F_i(A \land \neg A)$ and Axiom D for $B_i(\cdot)$.

The above requirements are conditions not only for $F_i(p)$ but also for $L_\alpha$, since formulae $A, C$ vary in $L_\alpha$. Lemma 4.1 states that when $F_i(p) \in L_\alpha$ satisfies $F_{0i}$, $F_i(p)$ is finitary or $\alpha = \omega$.

**Lemma 4.1.** If $F_{0i}$ holds for $F_i(p) \in L_\alpha$, then $\delta(F_i(p)) < \omega$ or $\alpha = \omega$.

**Proof.** Let $\delta(F_i(p)) \geq \omega$. Then, some infinitary conjunction $\land \Phi$ with $\delta(\land \Phi) \geq \omega$ is included in $F_i(p)$. Since $F_i(p)$ contains only propositional variable $p$, so does $\land \Phi$. Since $F_i(F_i(p)) \in L_\alpha$ by $F_{0i}$ and $\land \Phi(F_i(p))$ is a subformula of $F_i(F_i(p))$, it holds by Lemma 2.1 that $\land \Phi(F_i(p)) \in L_\alpha$. But $\delta(\land \Phi(F_i(p))) \geq \omega + \omega$. This implies $\delta(F_i(F_i(p))) \geq \omega \cdot 2$. In general, we can prove by induction on $\beta \geq 1$ that $\delta(F_i^\beta(p)) > \omega \cdot \beta$ for all $\beta < \omega$. Using $F_{0i}$, $F_i^\beta(p) \in L_\alpha$ for any $\beta < \omega$. Thus, $\omega^2 \leq \sup_{\beta < \omega} \delta(F_i^\beta(p)) \leq \delta(L_\alpha)$. By Theorem 2.2, we have $\alpha = \omega$.

Another lemma is about the consistency of $F_i(p)$. We say that a formula $A$ is consistent in GL($L_\alpha$) iff $\not\vdash A \supset \neg p \land p$ in GL($L_\alpha$). A formula $A$ is not consistent if and only if $\vdash \neg A$.

**Lemma 4.2.** Let $0 \leq \alpha \leq \omega$. Any $F_i(p)$ satisfying $F_{Ni}$ is consistent in GL($L_\alpha$).

**Proof.** Suppose that $F_i(p)$ is not consistent in GL($L_\alpha$), i.e., $\vdash \neg F_i(p)$. By the substitution-rules mentioned in Remark 3.1.2, it holds that $\vdash \neg F_i(p \supset p)$. On the other hand, by $F_{Ni}$, $\vdash F_i(p \supset p)$. This is impossible because GL($L_\alpha$) is contradiction-free, as remarked just after Theorem 3.1.

The conditions corresponding to Axioms T, 4, and 5 are as follows: for any $A \in L_\alpha$,

\begin{align*}
F_{Ti} & : \vdash F_i(A) \supset A; \\
F_{4i} & : \vdash F_i(A) \supset F_i(F_i(A)); \\
F_{5i} & : \vdash \neg F_i(A) \supset F_i(\neg F_i(A)).
\end{align*}

We look for a formula $F_i(p)$ satisfying each of these in addition to $F_{0i}$ to $F_{Ni}$. Whether or not such an $F_i(p)$ exists is explicit definability of Axioms T, 4, and/or 5 in GL($L_\alpha$).

In the case of Axiom T, we observe that $B_i(p) \land p$ satisfies $F_{0i}$, $F_{Ei}$, and $F_{Ti}$, and it is also the deductively weakest among such formulae; we say that $F_i(p)$ is the deductively weakest among the formulae satisfying given conditions iff it satisfies them and for any $F_i'(p)$ among those formulae, $\vdash F_i'(A) \supset F_i(A)$ for any $A \in L_\alpha$. 

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Theorem 4.1. (Explicit definability for Axiom T) Let $0 \leq \alpha \leq \omega$. In $GL(L_\alpha)$, $B_\alpha(p) \land p$ is the deductively weakest among the formulae satisfying $F_0$, $F_E$, and $F_T$.

Proof. We can verify that $B_\alpha(p) \land p$ satisfies $F_0$, $F_E$, and $F_T$ in $GL(L_\alpha)$. Let $F'_\alpha(p)$ satisfy $F_0$, $F_E$, and $F_T$. By $F_E$ and $F_T$, $\vdash F'_\alpha(A) \supset B_\alpha(A)$ and $\vdash F'_\alpha(A) \supset A$. By $\wedge$-rule, $\vdash F'_\alpha(A) \supset B_\alpha(A) \land A$, which holds for any $A \in L_\alpha$. Thus, $B_\alpha(p) \land p$ is deductively weakest among $F_\alpha(p)$ satisfying $F_0$, $F_E$, and $F_T$.

This theorem holds for every $\alpha$ ($0 \leq \alpha \leq \omega$). Also, we can include $FK_t$ and $FN_t$ as required conditions in Theorem 4.1. Note that $G$ is arbitrary up to this theorem.

Now, we go to the evaluation of Axiom 4. We assume that $G$ contains $B^\omega_\nu(p) = \land(B^{\nu+1}_i(p))$ for any $\nu \geq 0$ with $B^\nu_0(p) = B_i(p)$.

Theorem 4.2. (Explicit definability for Axiom 4) (1): Let $F_\alpha(p) \in L_\alpha$ satisfy $F_0$, $F_E$, $FK_t$, $FN_t$, and $F_4$. Then $\alpha = \omega$ and $\vdash F_\alpha(p) \supset B_\omega^\nu(p)$ in $GL(L_\omega)$.

(2): $B_\omega^\nu(p)$ is the deductively weakest among the formulae $F_\alpha(p)$ satisfying $F_0$, $F_E$, $FK_t$, $FN_t$, and $F_4$ in $GL(L_\omega)$.

Proof. (1): We prove $GL(L_\omega) \vdash F_\omega(p) \supset B_\omega^\nu(p)$ for all $\nu < \omega$ by induction over $\nu \geq 0$. For $\nu = 0$, the claim is $F_E$. Suppose the induction hypothesis that $\vdash F_\nu(p) \supset B_\omega^\nu(p)$. Then, by $F_0$, $F_E$, $FK_t$, and $FN_t$, we have $\vdash F_\nu(F_\nu(p)) \supset B_\omega^\nu(B_\omega^\nu(p))$. By this and $\vdash F_\nu(p) \supset F_\nu(F_\nu(p))$ by $F_4$, we have $\vdash F_\nu(p) \supset F_\nu(B_\omega^\nu(p))$. Since $\vdash F_\nu(B_\omega^\nu(p)) \supset B_\omega^\nu(B_\omega^\nu(p))$ by $F_E$, we have $\vdash F_\nu(p) \supset B_\omega^\nu(A)$.

Let $\delta(F_\nu(p)) < \omega$. Take a $\nu > \delta(F_\nu(p))$. By Lemma 3.3, we have $\not\vdash \neg F_\nu(p)$ or $\vdash p$ in $GL(L_\omega)$. The first is impossible since $F_\nu(p)$ is consistent in $GL(L_\omega)$ by Lemma 4.2. The second is also impossible. Hence, $\delta(F_\nu(p)) \geq \omega$. By Lemma 4.1, $\alpha = \omega$. Using $F_0$, $F_E$, $FK_t$, $FN_t$, $F_4$, we have $GL(L_\omega) \vdash F_\omega(p) \supset B_\omega^\nu(p)$ for all $\nu < \omega$. Thus, $GL(L_\omega) \vdash F_\omega(p) \supset B_\omega^\nu(p)$ by $\wedge$-rule.

(2): We can verify that $F_0$, $F_E$, $FK_t$, $FN_t$, $F_4$ hold for $B_\omega^\nu(p)$ in $GL(L_\omega)$. By (1) of this theorem, it is deductively weakest among $F_\alpha(p)$ satisfying these requirements.

In contrast to Theorem 4.1, Theorem 4.2 states that Axiom 4 is explicitly definable only in $GL(L_\omega)$. It has the implication that $\vdash B_\omega^k(p) \supset B_\omega^\nu(B_\omega^k(p))$ for any $k < \omega$. In $GL(L_\omega)$, though $\not\vdash B_\omega^k(p) \supset B_\omega^\nu(p)$ for $\nu < \omega$: i.e., after $\omega$, further introspection carries no additional information. $F_4$ with the closure property $F_0$ directly brings us to infinity.

We showed that both Axiom T and 4 can be explicitly defined in our system, though the depth requirements differs. For Axiom 5, the answer is entirely negative, independent of the choices of $\alpha$ and $G$.

Theorem 4.3. (Explicit indefinability of Axiom 5) There is no consistent formula $F_\alpha(p)$ in $GL(L_\alpha)$ ($0 \leq \alpha \leq \omega$) such that it satisfies $F_E$ and $F_5$.

Proof. Suppose that there is some consistent formula $F_\alpha(p)$ in $GL(L_\alpha)$ satisfying $F_E$ and $F_5$. Then, $F_5$ is equivalent to $\vdash F_\alpha(p) \lor F_\alpha(\neg F_\alpha(p))$, which further implies, by $F_E$, $\vdash B_\alpha(p) \lor B_\alpha(\neg F_\alpha(p))$. By Lemma 3.4, we have $\vdash B_\alpha(p)$ or $\vdash B_\alpha(\neg F_\alpha(p))$. By Lemma 3.5, we have $\vdash p$ or $\vdash \neg F_\alpha(p)$. The former is impossible; and so is the latter because $F_\alpha(p)$ is consistent in $GL(L_\alpha)$.

Thus, Axiom 5 cannot be defined explicitly by a formula in $GL(L_\alpha)$. However, it can still be treated as a logical axiom keeping completeness, as remarked in Section 3.2.
4.2 Faithful embedding

The explicit definability results for Axioms T and 4 may imply that an extension GL($L_\alpha$) with Axiom T or 4 is faithfully embedded into GL($L_\alpha$). For Axiom T, the embedding result is available from $L_\alpha$ to $L_\alpha$ for any $\alpha$ in terms of language, but for Axiom 4, it can be only from $L_0$ to $L_\omega$.

We have no embedding result for Axiom 5.\(^6\) Here, we give a full embedding argument in the case of Axiom 4, and a sketch for the embedding result in the case of Axiom T.

Consider the case of Axiom 4 and recall $F^4_i(p) = B^\psi_i(p)$. Let $(B^\psi_i(p) : \nu \geq 0) \in \mathcal{G}$. We define the $F^4_i$-translator $\psi^4 : L_0 \rightarrow L_\omega(\mathcal{G})$ inductively as follows: for all $A, C \in L_0$ and $\land \Phi \in L_0$,

\begin{align*}
E0: & \quad \psi^4(p) = p \text{ if } p \in \mathcal{P}_0; \\
E1_0: & \quad \psi^4(\neg A) = \neg \psi^4(A); \\
E2_0: & \quad \psi^4(A \supset C) = \psi^4(A) \supset \psi^4(C); \\
E3_0: & \quad \psi^4(\land \Phi) = \land \psi^4(\Phi); \\
E4_0: & \quad \psi^4(B_i(A)) = F^4_i(\psi^4(A)) \text{ and } \psi^4(B_j(A)) = B_j(\psi^4(A)) \text{ for } j \neq i.
\end{align*}

The following theorem states that KD$^n + 4_i$ is faithfully embedded to GL($L_\omega$). The depth of the embedded fragment $\psi^4(L_0)$ is $\delta(\psi^4(L_0)) = \sup_{\nu<\omega} \psi^4(B^\psi_i(A)) = \sup_{\nu<\omega}(\omega \cdot \nu) = \omega^2 = \delta(L_\omega)$.

**Theorem 4.4.** (Faithful embedding of KD$^4$ to GL($L_\omega$)) (1): For any $A \in L_0$, KD$^n + 4_i \vdash A$ in if and only if GL($L_\omega$) $\vdash \psi^4(A)$.

(2): For any $A \in L_0$, there exists an $\alpha < \omega$ such that KD$^n + 4_i \vdash A$ if and only if GL($L_\alpha$) $\vdash \psi^4(A)$.

**Proof.** (1): Take an arbitrary Kripke model $(\mathbb{K}, \tau)$ for KD$^n$, which is also a model for GL($L_\omega$). We replace the accessibility relation $R_i$ in $(\mathbb{K}, \tau)$ by its transitive closure $R^\tau_i$, and we denote the resulting Kripke model by $(\mathbb{K}^\tau, \tau)$. Then, KD$^n + 4_i$ is Kripke complete with respect to those models $(\mathbb{K}^\tau, \tau)$. Then, we prove by induction on the length of $A \in L_0$ that for any world $w \in W$, $(\mathbb{K}^\tau, \tau, w) \models A$ if and only if $(\mathbb{K}, \tau, w) \models \psi^4(A)$. We consider only case of $A = B_i(C)$.

Let $(\mathbb{K}^\tau, \tau, w) \models B_i(C)$. Then, $(\mathbb{K}^\tau, \tau, v) \models C$ for any $v \in R^\tau_i(w)$. By the induction hypothesis, $(\mathbb{K}, \tau, v) \models \psi^4(C)$ for any $v \in R^\tau_i(w)$. Since $R^\tau_i$ is the transitive closure of $R_i$, it is equivalent to that $(\mathbb{K}, \tau, v) \models \psi^4(C)$ for any $v$ reachable from $w$ by $R_i$. This means $(\mathbb{K}, \tau, w) \models B^\psi_i(\psi^4(C))$ for any $\nu \geq 0$, i.e., $(\mathbb{K}, \tau, w) \models B^\psi_i(\psi^4(C))$, implying $(\mathbb{K}, \tau, w) \models \psi^4(B_i(C))$. Tracing this argument back, we have a proof of the converse. For the cases of other connectives, the argument is similar.

(2): For a given $A \in L_0$, we find the maximal iterations, $\alpha$, of $B_i(\cdot)$ inside $A$; then, by Theorem 3.2 (conservativity), KD$^n + 4_i \vdash A \iff$ GL($L_\alpha$) $\vdash \psi^4(A)$.

Now, consider the embedding of Axiom T to GL($L_\alpha$). Now, we do not need $(B^\psi_i(p) : \nu \geq 0) \in \mathcal{G}$. In this case, we use the translator $\psi^T$ based on $F^T_i(p) = B_i(p) \land p$. Then, the formal definition of $\psi^T : L_0 \rightarrow L_0$ is obtained by the same rules E0, E10-E30, but E40 with $F^T_i(p) = B_i(p) \land p$ instead of $F^4_i(p)$. This translator $\psi^T$ is also uniquely defined. Then, we have

\[ \text{KD}^n + T_i \vdash A \iff \text{KD}^n \vdash \psi^T(A). \] 

This embedding result is essentially the same as the result given in Kaneko [18], Section 5.

\(^6\)Halpern et al. [11] consider two modalities, one called belief (KD45) and the other called knowledge (S5), and discuss whether the latter can be reduced to the former via various notions of definability. In contrast, our embedding results are about reducing one logic system (e.g., KD4) to GL($L_\omega$).
However, the result (13) holds, under a minor additional condition, from GL(\(L_\alpha\)) + T_i to GL(\(L_\alpha\)) for all \(\alpha\) (0 ≤ \(\alpha\) ≤ \(\omega\)). When \(\alpha\) ≥ 1, the definition \(\psi_T\) over \(L_\alpha\) needs one requirement on the set of germinal forms \(G\) to be closed under the translation \(\psi_T\):

\[
\langle C^\nu : \nu \geq 0 \rangle \in G \implies \langle \psi_T(C^\nu) : \nu \geq 0 \rangle \in G.
\]

This implies that \(G\) is countably infinite.

We have the following lemma. Proofs of this lemma and the next theorem are found in the working paper version of this paper.\(^7\)

**Lemma 4.3.** \(\psi_T : L_\omega \to L_\omega\) is uniquely defined by \(E\theta, E1_\alpha\) to \(E4_\alpha\) (\(\alpha\) ≤ \(\omega\)).

Now, we have the following theorem, where GL(\(L_\alpha\)) + T_i denotes the logic GL(\(L_\alpha\)) plus Axiom T for \(B_i(\cdot)\). Then, the logic GL(\(L_\alpha\)) + T_i is faithfully embedded into GL(\(L_\alpha\)) with the translator \(\psi_T\). Let 0 ≤ \(\alpha\) ≤ \(\omega\).

**Theorem 4.5.** For any \(A \in L_\alpha\), GL(\(L_\alpha\)) + T_i \(\vdash\) \(A\) if and only if GL(\(L_\alpha\)) \(\vdash\) \(\psi_T(A)\).

Theorem 4.5 compares logic GL(\(L_\alpha\)) + T_i with the fragment \(\psi_T(GL(L_\alpha))\) obtained by the translator \(\psi_T\). It is the main difference from Theorem 4.4 that the translator \(\psi_T\) does not change the layer, i.e., it embeds \(L_\alpha\) to \(L_\alpha\) for each \(\alpha\), while \(\psi^T\) embeds \(L_0\) to \(L_\omega\). We remark here that Io2.(iii) is used in proving Lemma 4.3 and Theorem 4.5, but otherwise, it is not needed for any other results in the present paper.

### 4.3 Evaluation of common knowledge in GL(\(L_\alpha\))

The concept of common knowledge can be formulated in a fixed-point extension of a finitary epistemic logic, often S5-type, (Halpern, et al. [8], Meyer-van der Hoek [25]). Here, we consider its KD\(^n\) variant, and show that this fixed-point logic is embedded to GL(\(L_\alpha\)).

The finitary language \(L_0\) is extended by adding the unary operator symbol \(C_N(\cdot)\) to the basic symbols listed in Section 2.1, and use \(L^{C_N}\) to denote the extended language. A formula \(C_N(A)\) means the common knowledge of \(A\) among the group of agents \(N\). The common knowledge logic CK(\(L^{C_N}\)) is defined to be the extension of KD\(^n\) with the language \(L^{C_N}\) by adding the following axiom scheme and an inference rule: for any \(A, D \in L^{C_N}\),

**Axiom CKA:** \(C_N(A) \supset [A \land \land_{i \in N} B_i C_N(A)]\);

**Rule CKI:** \[
\frac{D \supset [A \land \land_{i \in N} B_i(D)]}{D \supset C_N(A)}.
\]

A (finite) proof is defined in the same way as in Section 3.1. In this logic, it is shown by repeated use of CKA that \(\vdash C_N(A) \supset B_\nu^N(A)\) for all \(\nu \geq 0\), where \(B_\nu^N(A)\) is defined in (2). Thus, \(C_N(A)\) contains the common knowledge of \(A\). Rule CKI means that if any \(D\) has the property described by CKA, then \(D\) contains \(C_N(A)\), i.e., \(C_N(A)\) is the deductively weakest among the formulae having the property.

In CK(\(L^{C_N}\)), the formula \(C_N(A)\) is not explicitly expressed in terms of \(B_1(\cdot),..., B_n(\cdot)\) in CK(\(L^{C_N}\)), but \(C_N(A)\) is *implicitly definable*. To see this, we add another operator symbol

---

We have

\[ \mathcal{C}_N(\cdot) \]

to the language \( L^C_N \) and assume CKA, CKI for \( \mathcal{C}_N(\cdot) \). By CKA for \( \mathcal{C}_N(A) \) and CKI with \( D = C_N(A) \), we have \( \vdash C_N(A) \supset C_N(A) \). We have the converse by a parallel argument. Thus, \( \vdash C_N(A) \equiv C_N(A) \).

In contrast, our infinitary logic \( GL(L_\omega) \) allows us to express the concept of common knowledge explicitly, i.e., \( B_N^*(\nu : \nu \geq 0) \), assuming \( \langle B_N^*(p : \nu \geq 0) \rangle \in G \). In a similar manner to Section 4.1, we look for a formula \( F(p) \in L_\alpha \) in \( GL(L_\alpha) \) having the following properties: for \( A \in L_\alpha \), and \( D \in L_\alpha \), F0 with the replacement of \( F_i(p) \) by \( F(p) \) and

\[
\begin{align*}
\text{FCA}_\alpha & : \vdash F(A) \supset A \land [\land_{i \in N} B_i(F(A))]; \\
\text{FCL}_\alpha & : \text{if } \vdash D \supset A \land [\land_{i \in N} B_i(D)], \text{ then } \vdash D \supset F(A).
\end{align*}
\]

These require \( F(p) \) satisfy the properties corresponding to CKA and CKI in \( CK(L^C_N) \).

The following theorem states that the common knowledge is explicitly definable in \( GL(L_\alpha) \). Since it follows from FCA\(_\alpha\) and Nec, K for \( B_i(\cdot) \)'s that \( F(A) \) is an infinitary formula, Lemma 4.1 is applied to \( F(p) \), the explicit definability holds only for \( \alpha = \omega \).

**Theorem 4.6.** (Explicit definability of common knowledge). In \( GL(L_\omega) \), the common knowledge \( F(p) = \land(B_N^*(p : \nu \geq 0)) \) is a unique, up to the deductive equivalence, formula satisfying FCA\(_\omega\), \( FCL_\omega \), and FCI.

Now, we look at the relation between \( CK(L^C_N) \) and \( GL(L_\alpha) \). The Kripke semantics for \( CK(L^C_N) \) is the same as that for \( GL(L_\alpha) \). Here, \( M = (W; R_1, \ldots, R_n, \tau) \) is a serial model as in Section 3.2 and the valuation of \( \mathcal{C}_N(A) \) is defined in the same way except the following:

\[
(M, w) \models \mathcal{C}_N(A) \text{ iff } (M, v) \models A \text{ for all } \mathcal{C}_N\text{-reachable } v \text{ from } w,
\]

where \( v \) is \( \mathcal{C}_N\text{-reachable from } w \) iff there is a finite sequence \( \langle w_0, \ldots, w_m \rangle (m \geq 0) \) in \( W \) such that \( w_0 = w, w_m = v \), and for all \( k = 0, \ldots, m - 1, (w_k, w_{k+1}) \in R_i \) for some \( i \in N \).

We have the completeness/soundness result for \( CK(L^C_N) \), which is a variant of the well-known result (cf., Fagin et al. [8]): for any \( A \in L^C_N \), \( A \) is valid if and only if \( CK(L^C_N) \vdash A \).

Now we show that \( CK(L^C_N) \) can be faithfully embedded into \( GL(L_\omega) \) with the translator \( \psi^C_N : L^C_N \rightarrow L_\omega \) by E0 and E1_0 - E3_0, and

\[
\begin{align*}
\text{E4}_0 & : \psi^C_N(B_i(A)) = B_i(\psi^C_N(A)) \text{ for all } i \in N; \\
\text{EC} & : \psi^C_N(\mathcal{C}_N(A)) = B_N^*(\psi^C_N(A)).
\end{align*}
\]

Then, we have the following theorem.

**Theorem 4.7.** (Faithful embedding of \( CK(L^C_N) \) to \( GL(L_\omega) \)) (1): For any \( A \in L^C_N \), \( CK(L^C_N) \vdash A \) if and only if \( GL(L_\omega) \vdash \psi^C_N(A) \).

(2): For any \( A \in L^C_N \), there exists an \( \alpha_A < \omega \) such that \( CK(L^C_N) \vdash A \) if and only if \( GL(L_\alpha) \vdash \psi^C_N(A) \).

**Proof.** (1) can be proved by observing that with the translation \( \psi^C_N \), the Kripke semantics for \( CK(L^C_N) \) and for \( GL(L_\omega) \) are the same. For (2), we take the maximum nested depth \( \alpha \) of \( \mathcal{C}_N(\cdot) \) in \( A \in L^C_N \). By Theorem 3.2, we have \( GL(L_\alpha) \vdash \psi^C_N(A) \Leftrightarrow GL(L_\omega) \vdash \psi^C_N(A) \). By part (1) and this, we have (2).
This theorem is similar to Theorem 4.4 with respect to the depths required, that is, the finitary logics are faithfully embedded to GL($L_\omega$).

It may be relevant to see the rank function given by Alberucci et al. [2] in this context; this concept is defined in modal $\mu$-calculus, but Alberucci [1] shows that CK($L^{CN}$) (based on K-type) can be regarded as a fragment of modal $\mu$-calculus. In our context with $\mathcal{G} = \{\langle B^\nu_N(p) : \nu \geq 0 \rangle\}$, their problem is to find a function $f$ over $L^{CN}$ assigning an ordinal to each formula in $L^{CN}$ having the following two properties: for all $A \in L^{CN}$,

(a): if $B$ is a proper subformula of $A$, then $f(B) < f(A)$;

(b): $f(CN(A)) > f(B^\nu_N(A))$ for all $\nu < \omega$.

The second is motivated by the fact that $\vdash CN(A) \supset B^\nu_N(A)$ for all $\nu \geq 0$. In the present context, their rank function $f$ is defined by the inductive definition of our depth function $\delta$ by replacing the second part of d3 by: $f(CN(A)) = f(A) + \omega$ for all $A \in L^{CN}$. This function $f$ satisfies the requirements (a) and (b). Furthermore, we have:

$$f(A) = \delta(\psi^{CN}(A))$$

Thus, their rank function for $L^{CN}$ corresponds to our depth function $\delta$ for $L_\omega$. In the same manner as Theorem 4.7.(2), we can evaluate the depth for each $A \in L^{CN}$. Since each $A \in L^{CN}$ has the maximum nested depth $\alpha < \omega$ of $CN(\cdot)$, it follows from (15) and Theorem 2.2 that for each $A \in L^{CN}$, there is an $\alpha_A < \omega$ such that $\omega \alpha_A \leq f(A) = \delta(\psi^{CN}(A)) < \omega(\alpha_A + 1)$.

We remark that Theorem 4.7 does not hold for generic common knowledge (Sato [30], Artemov [4], Antonakos [3]). In one version of such logics, the language $L^I$ is obtained from $L^{CN}$ by adding $J(\cdot)$. Here, we consider the extension JL$_1(L^I)$ of CK($L^{CN}$) in which the belief operators $B_i(\cdot)$ obey KD$^n$ and $J(\cdot)$ obeys S4 axioms (including Nec), and

**Interaction axiom (IA): $J(A) \supset \land_{i\in N} B_i(A)$ for all $A \in L^I$.**

The expression $J(A)$ is interpreted as meaning that $A$ is “obvious fact” in that it is known to all agents. Interaction Axiom connects $J(A)$ to $\land_{i\in N} B_i(A)$, but the converse is not guaranteed. Also, $JL_1(L^I) \vdash J(A) \supsetl CN(A)$; since $\vdash JJ(A) \supsetl \land_{i\in N} B_i(J(A))$ by plugging $J(A)$ to $A$ in IA and $\vdash J(A) \equiv JJ(A)$ by the S4 axioms for $J(\cdot)$, we have $\vdash J(A) \supsetl \land_{i\in N} B_i(J(A))$, and since this is the upper formula of CKI, we have $\vdash J(A) \supsetl CN(A)$.

In JL$_1(L^I)$, the operator $J(A)$ is not explicitly defined in terms of $B_1(\cdot),...,B_n(\cdot)$ and $CN(\cdot)$. Contrary to this, in GL($L_\omega$), there are multiple formulae satisfying the corresponding properties to the axioms for $J(\cdot)$. The formula $F(p) = B_N^\omega(p)$ enjoys the S4 properties and IA, but for another propositional variable $q \neq p$, the formula $F'(p) = B_N^\omega(p) \land B_N^\omega(q)$ also enjoys all of these properties, but is deductively stronger than $F(p)$.

A more general development in the fixed-point logic literature is given in the study of modal $\mu$-calculus (cf., Enqvist, et al. [7]). Our approach looks similar in that germinal forms can be based on iterated substitutions. However, the two approaches also have significant difference, as summarized below.

(i) The definition of germinal forms in Section 2 allows non-constructive germinal forms, and even when germinal forms are constructive in terms of iterated substitutions, they may include negative occurrences of propositional variables for substitution. See Example 2.1.(3). In contrast, the positivity assumption that the $\mu$-operator (and $\nu$-operator) is applied only to a formula is crucial. See Enqvist, et al. [7], Section 3, and Fountaine [9] for related problems.
(iii) The required depth for the language of GL($L_{\alpha}$) is $\omega(\alpha + 1)$ ($0 \leq \alpha < \omega$) and that of GL($L_{\omega}$) is $\omega^2$. On the other hand, Alberucci et al. [2] showed that their notion of ordinal ranks to evaluate the depths of formulae in modal $\mu$-calculus and it goes up to $\omega^\omega$. Our germinal forms are sequences in GL($L_0$) and are assumed to be uniform in generating the series GL($L_0$), GL($L_1$), ... In modal $\mu$-calculus, this is regarded as corresponding to $A(\mu x.A(x))$, $A^2(\mu x.A(x))$, ..., and the $\mu$-operator is also applied to formulae already including the $\mu$-operator, that is, $\mu y(\mu x.A(x, y))$ as long as the variable condition is satisfied. The difference in the required depths is caused by these facts.

5 Application 2: Rationalizability in Game Theory

We apply our framework to the study of decision making in game theory, called the theory of rationalizability (cf., Bernheim [5], Pearce [28], and Osborne-Rubinstein [27]). This application has two purposes. First, we show that our framework enables us to formalize each agent’s decision-making process in terms of agents’ logical inference. Second, it gives a concrete example of a discourse requiring GL($L_{\alpha}$) exactly with $\alpha = 2$, which differs from the infinitary concepts discussed in Section 4. Also, the theory requires more complex germinal forms involving disjunctions, and we will use the sound/completeness theorem (Theorem 3.1) to prove one step (Lemma 5.4) of the main theorem (Theorem 5.2). We remark that Axiom D is used for (20) in this section.

A 2-person game is given as $G = \{1, 2\}, S_1, S_2, g_1, g_2$, where 1 and 2 are agents, $S_i$ is a finite nonempty set of available actions, and $g_i : S_1 \times S_2 \rightarrow \mathbb{R}$ (reals) is the payoff function of agent $i = 1, 2$. Before the actual play of the game, each agent chooses his action to be played without knowing the other’s choice. The focus is on this ex ante decision making.

A crucial component for rationalizability is the best response property: an action $s_i \in S_i$ for agent $i$ is a best response to an action $t_j \in S_j$ for agent $j$ if $g_i(s_i; t_j) \geq g_i(s_i'; t_j)$ for all $s_i' \in S_i$, where we often write $g_i(s_1, s_2)$ as $g_i(s_i; s_j)$. We stipulate that when agent $i$ is focused, the other agent is denoted by $j$. We say that an action $s_i \in S_i$ for agent $i$ is rationalizable iff $s_i$ is a best response to some action $s_j \in S_j$ for $j$, and $s_j$ is a best response to some $s_i^2$, and $s_i^2$ is a best response to some $s_j^3$, and so on ad infinitum. The referred action $s_i^{t+1}$ for $t$ is interpreted as a prediction inferred in the interpersonal beliefs of depth $t$ in the mind of agent $i$. Here, this interpretation is informal; to make it explicit, we go to our formal system.

To express the above game theoretical concepts, we add the following atomic propositions as propositional variables to the basic symbols listed in the beginning of Section 2: for $i = 1, 2$,

preference symbols $Pr_i(s_1, s_2 : t_1, t_2)$ for $(s_1, s_2), (t_1, t_2) \in S_1 \times S_2$;

decision symbols $I_i(s_i)$ for $s_i \in S_i$.

The atomic proposition $Pr_i(s_1, s_2 : t_1, t_2)$ intends to mean that “agent $i$ weakly prefers $(s_1, s_2)$ to $(t_1, t_2)$”, which is also written as $Pr_i(s_i; s_j : t_i; t_j)$ with $(i, j) = \{1, 2\}$. The expression $I_i(s_i)$ means that “$s_i$ is a possible final decision for agent $i$”. The finitary language $L_0$ is now defined.

---

*In the literature, this is called point-rationalizability, which is the degenerate version of "rationalizability" allowing mixed strategies with the interpretation that they express probabilistic beliefs about the other’s choices (Bernheim [5], Peace [28]). In the recent game-theory literature, rationalizability is studied in a state space with probabilistic (common) beliefs (cf., Tan and Werlang [31] and Hu [13]). However, this approach does not explicitly formulate logical inferences as in proof theory, since it does not have a formal language.
by Io0 and Io1 with \( \alpha = 0 \) based on these additional symbols and the list of primitive symbols in Section 2. In \( L_0 \), the best response property is described as a formula: for \( s_i \in S_i \) and \( t_j \in S_j \),

\[
\text{Bst}_i(s_i; t_j) := \land \{ \text{Pr}_i(s_i; t_j : s'_j; t_j) : s'_j \in S_j \}. \tag{16}
\]

For rationalizability, we use two types of germinal forms. The first is the germinal forms for epistemic infinite regresses \( \langle \text{Ir}_i' [p_1, p_2] : \nu \geq 0 \rangle \) in Example 2.1(2). We denote \( G^{IR} = \{ \langle \text{Ir}_i' [p_1, p_2] : \nu \geq 0 \rangle : i = 1, 2 \} \). The other will be introduced after giving the decision making criterion.

Consider the following criterion for decision making by agent \( i \):

\[
D_i^R := \land_{s_i \in S_i} (I_i(s_i) \lor t_j \in S_j (B_j(I_j(t_j)) \land \text{Bst}_i(s_i; t_j))).
\]

This is used in his mind, i.e., \( D_i^R \) occurs in the scope \( B_i(\cdot) \). It states that agent \( i \) makes some prediction about the other’s decision \( t_j \) and his decision \( s_i \) is a best response to the prediction \( t_j \). The disjunction \( \lor_{t_j \in S_j} \) is specific to the rationalizability theory and to capture the idea of rationalization.

The criterion \( D_i^R \) is self-insufficient in that it lacks the description of how agent \( j \) infers \( t_j \) in agent \( i \)'s mind; that is, agent \( i \) needs to have a certain criterion for it. We assume that agent \( i \) has the same (symmetric) criterion, \( D_j^R \), to predict a possible \( t_j \) for the imaginary agent \( j \) in agent \( i \)'s mind. This is formally expressed as \( B_iB_j(D_j^R) \). However, this formula includes \( B_i(I_j(t_j)) \) in the innermost \( D_j^R \) and by the parallel argument to the above, \( B_iB_j(D_j^R) \) is required. Unless we force this argument to stop at some finite level, this leads to an infinite regress:

\[
B_i(D_i^R) \rightarrow B_iB_j(D_i^R) \rightarrow B_iB_jB_j(D_i^R) \rightarrow ... \tag{17}
\]

The conjunction of this sequence is exactly the infinite regress formula \( \text{Ir}_i[D_i^R] = \text{Ir}_i[D_i^R, D_j^R] \).

We regard the infinite regress \( \text{Ir}_i[D_i^R] \) as a system of equations with unknowns \( I_1(s_i) \) and \( I_2(s_2) \); agent \( i \) may find some formulae so that they could be regarded as solutions for \( \text{Ir}_i[D_i^R] \). To discuss solutions for \( \text{Ir}_i[D_i^R] \), we introduce the germinal forms to express the rationalizability property.

First we choose subsets of propositional variables \( \{ p_i(t_1; t_2) : (t_1, t_2) \in S_1 \times S_2 \} \) for \( i = 1, 2 \) from \( \{ p_0, p_1, \ldots \} \), where \( p_i(t_i; t_j) \)'s are all distinct. We define two sets of sequences \( \{ \langle \text{rat}_{i'}^{s_i} : \nu \geq 0 \rangle : s_i \in S_i \} \), \( i = 1, 2 \), interactively as follows: for \( i = 1, 2 \),

\[
\text{rat}_i^0(s_i) = \land_{t_j \in S_j} p_i(s_i; t_j);
\]

\[
\text{rat}_i^\nu(s_i) = \land_{t_j \in S_j} (B_j(\text{rat}_j^{\nu-1}(t_j)) \land p_i(s_i; t_j)) \quad \text{for } \nu \geq 1.
\]

Recall \( \lor \Phi = \neg \land \{ \neg A : A \in \Phi \} \) for a finite nonempty set \( \Phi \) in \( L_\alpha \). Let \( G^R = \{ \langle \text{rat}_i^\nu(s_i) : \nu \geq 0 \rangle : s_i \in S_i, i = 1, 2 \} \). Hence, \( G^R \) consists of \( |S_1| + |S_2| \) germinal forms, and each \( \langle \text{rat}_i^\nu(s_i) : \nu \geq 0 \rangle \) contains \( 2 \times |S_1| \times |S_2| \) propositional variables for substitution. set of these germinal forms by \( G^R \). We adopt the set of germinal forms \( G^{IR+R} := G^R \cup G^R \). The series of languages \( \{ L_\alpha : \alpha \geq 0 \} \) is defined based on \( G^{IR+R} \).

Let \( s_i \in S_i \) and \( i = 1, 2 \). For each \( \nu \geq 0 \), let \( \text{Rat}_i^\nu(s_i) \) be the formula obtained from \( \text{rat}_i^\nu(s_i) \) by substituting \( \text{Bst}_i(t_i; t_j) \) for all occurrences of each \( p_i(t_i; t_j) \) in \( \text{rat}_i^\nu(s_i) \), which is still in \( L_0 \). The rationalizability formula is given as \( \text{Rat}_i(s_i) := \land \langle \text{Rat}_i^\nu(s_i) : \nu \geq 0 \rangle \), which is in \( L_1 \). Again, we note that \( \text{Rat}_i(s_i) \) occurs in the scope of \( B_i(\cdot) \).

The formula \( \text{Rat}_i(s_i) \) is intended to be a solution of the inference process (17), i.e., \( \text{Ir}_i[D^R] \). However, the directions of predictions are opposite to (17); in (17), predictions go to deeper
layers along $\nu = 0, 1, \ldots$, but $\text{Rat}^\nu_i(s_i) = \forall t_j \in S_j (B_j(\text{Rat}^{\nu-1}_j(t_j)) \land \text{Bst}_i(s_i; t_j))$ has a prediction $B_j(\text{Rat}^{\nu-1}_j(t_j))$, and $\text{Rat}^{\nu-1}_j(t_j)$ has a prediction $B_i(\text{Rat}^{\nu-1}_i(t_i))$, and so on to $\nu = 0$. In the latter, we require $s_i$ to satisfy this backward argument for all $\nu \geq 0$. For this reason, it holds that $\text{Ir}_i[D^R] = \text{Ir}_i[D^R_1, D^R_2]$ with some additional axiom determines $I_i(s_i)$ to be equivalent to $\text{Rat}_i(s_i)$. The one direction is given by the following theorem, which will be proved later in this section.

**Theorem 5.1. (Necessity)** Let $s_i \in S_i, s_j \in S_j$ and $\{i, j\} = \{1, 2\}$. Then,

1. $\vdash \text{Ir}_i[D^R] \supset [B_i(I_i(s_i)) \supset B_i(\text{Rat}_i(s_i))]$ in $GL(L_1)$;
2. $\vdash \text{Ir}_i[D^R] \supset \text{Ir}_i[I_i(s_i) \supset \text{Rat}_i(s_i); I_j(s_j) \supset \text{Rat}_j(s_j)]$ in $GL(L_2)$.

In (1), $\text{Ir}_i[D^R]$ implies that if agent $i$ believes that $s_i$ is a final decision, then he believes the rationalizability property for $s_i$. In (2), the conclusions for both agents in (1) form an infinite regress. The epistemic logic $GL(L_1)$ is sufficient for (1), but $GL(L_2)$ is required for (2) since the infinitary formulae $\{\text{Rat}_i(s_i)\}_{s_i \in S_i}, i = 1, 2$ occur in the germinal form $\text{Ir}_i[\vdash]$ of infinite regress.

Consider the converse of the conclusions of Theorem 5.1. If we plug $\{\text{Rat}_i(s_i)\}_{s_i \in S_i}, i = 1, 2$ to $\{I_i(s_i)\}_{s_i \in S_i}, i = 1, 2$ in $\text{Ir}_i[D^R]$, they could be regarded as a solution for $D^R$. Formally, we substitute each $\text{Rat}_i(s_i)$ for the corresponding $I_i(s_i)$ in $D^R$ for $i = 1, 2$, and we denote the resulting formulae by $D^R(\text{Rat}) = [D^R_1(\text{Rat}), D^R_2(\text{Rat})]$. If $D^R(\text{Rat})$ is provable, then each $\text{Rat}_i(s_i)$ would be a candidate for $I_i(s_i)$. This argument is formulated as follows:

\[ V_i^R: D^R_i(\text{Rat}) \supset \forall t_i \in S_i (\text{Rat}_i(t_i) \supset I_i(t_i)). \]

We write $V^R = (V_1^R, V_2^R)$. In fact, we need the infinite regress $\text{Ir}_i[V^R]$ of $V^R = (V_1^R, V_2^R)$ in order to have the converse of the conclusions of Theorem 5.1. We have the following theorem, which will be proved below.

**Theorem 5.2. (Full Characterization)** Let $(s_1, s_2) \in S_1 \times S_2$ and $i = 1, 2$. Then, both hold in $GL(L_2)$:

1. $\vdash \text{Ir}_i[V^R] \supset \text{Ir}_i[\text{Rat}_i(s_1) \supset I_i(s_1), \text{Rat}_2(s_2) \supset I_2(s_2)];$
2. $\vdash \text{Ir}_i[D^R] \land \text{Ir}_i[V^R] \supset \text{Ir}_i[\text{Rat}_i(s_1) \equiv I_1(s_1), \text{Rat}_2(s_2) \equiv I_2(s_2)].$

The first is the converse of Theorem 5.1.(2). Combining this and Theorem 5.1.(2), we obtain the second assertion, the full characterization of $I_1(s_1)$ and $I_2(s_2)$, which is done in $GL(L_\alpha)$ with $\alpha = 2$. The infinitary logic $GL(L_2)$ is required and is sufficient to have these results.

Theorems 5.1 and 5.2 study the logical inferences required for decision making and possible final decisions. These are not about an actual play of a recommended action. The next stage for agent $i$ is the play of such an action. For this, the agent needs the detailed information about the payoff functions $g_1$ and $g_2$ of the game $G = (\{1, 2\}, S_1, S_2, g_1, g_2)$. The payoff function $g_i (i = 1, 2)$ is formulated in terms of atomic propositions as follows:

\[ \{\text{Pr}_i(s_1, s_2 : t_1, t_2) : g_i(s_1, s_2) \geq g_i(t_1, t_2)\} \cup \{\neg \text{Pr}_i(s_1, s_2 : t_1, t_2) : g_i(s_1, s_2) < g_i(t_1, t_2)\}, \]

which is denoted by $\Gamma_i$. We assume the infinite regress of these preferences, i.e., $\text{Ir}_i[\Gamma] = \text{Ir}_i[\Gamma_1 \cup \Gamma_2]$.

---

9In the infinitary logic, we can formulate the choice of weakest formulae enjoying the property $D^R$ as infinitary formulae without using inference rules.
Lemma 5.1. \( \text{the epistemic content} \) (for (3), but GL The proof of (20) is partially semantic since Lemma 3.6 is used. (proof-theoretic ways in GL (see Hu-Kaneko [14] within the framework of a fixed-point logic).

The parallel manner, with the use of only germinal forms of infinite regress, to the discourse in this becomes the decision making following the line of Nash’s [26] theory; specifically, however, so far, we do not know whether the rationalizability property is captured in terms of regress can be captured in terms of a fixed-point logic, similar to the common knowledge logic. the germinal forms \( G \) for infinite epistemic regress are conceptually not specific to the theory of rationalizability, but \( G \) and \( \mathcal{R} \) are specific to the theory of rationalizability. The above discourse is the very first attempt in this respect.

In the game theory literature, decision making and existence of a resulting outcome have been agents in his mind. The discourse of decision making is done within the infinitary logic GL (i

We remark that when “some prediction” in \( D_i^R \) is replaced by “all predictions”, the theory becomes the decision making following the line of Nash’s [26] theory; specifically, \( D_i^R \) is changed into \( \land_{s_i \in S_i} (I_i(s_i) \supset \land_{t_j \in S_j} [B_j(I_j(t_j)) \supset \text{Bst}_i(s_i; t_j)]) \). Then, we can develop the theory in a parallel manner, with the use of only germinal forms of infinite regress, to the discourse in this section, but this theory depends more upon the payoff structure and is more complex as a whole (see Hu-Kaneko [14] within the framework of a fixed-point logic).

Finally, we prove the above theorems and (20). All steps, except for Lemma 5.4, are done in proof-theoretic ways in GL(\( L_1 \)) and GL(\( L_2 \)). Lemma 5.4 is proved using the Kripke semantics. The proof of (20) is partially semantic since Lemma 3.6 is used.

Lemma 5.1 states various properties of infinitary regress formulae \( \text{Ir}_i[A] \). GL(\( L_2 \)) is required for (3), but GL(\( L_1 \)) is enough for the others as long as content formulae are in \( L_1 \). We define the epistemic content of \( \text{Ir}_i^n[A] \) by \( \text{Ir}_i^n[A] := A_i \land \text{Ir}_i[A] \).

**Lemma 5.1.** \( \vdash \text{Ir}_i[A] \equiv B_i(\text{Ir}_i^n[A]); \)
(2): if ⊢ A_k for k = 1, 2, then ⊢ Ir_i[A];
(3): ⊢ Ir_i[A] ⊃ [Ir_i[D^R] ⊃ Ir_i[Ir_i[D^R]]];
(4): ⊢ Ir_i[A_1 ⊃ C_1] ∧ Ir_i[A_2 ⊃ C_2] ∧ Ir_i[A_1, A_2] ⊃ Ir_i[C_1, C_2];
(5): ⊢ [Ir_i[A_1] ∧ Ir_i[A_2] ∧ Ir_i[C_1, C_2] ⊃ Ir_i[A_1 ∧ C_1, A_2 ∧ C_2].

Proof. We prove (1), (3), and (4).
(1): Recall Ir_i[A] = ⊢ A ∧ Ir_i[A:] v ≥ 0, where Ir_i[0][A] = B_i(A) and Ir_i[v][A] = B_i(A ∧ Ir_i[A]) for all v ≥ 0. Hence, ⊢ Ir_i[A] ⊃ B_i(A ∧ Ir_i[A]) for all v ≥ 0; so ⊢ Ir_i[A] ⊃ (B_i[Ir_i[A]] : v ≥ 0).

By ∧-Barcan, we have ⊢ Ir_i[A] ⊃ B_i(∧[Ir_i[A] : v ≥ 0]). Thus, ⊢ Ir_i[A] ⊃ B_i(A ∧ Ir_i[A]). The converse is similar.

(3): By (1), ⊢ Ir_i[A] ⊃ B_i[Ir_i[A]] for i = 1, 2. Suppose that ⊢ Ir_i[A] ⊃ [Ir_i[Ir_i[A]], Ir_i[Ir_i[A]]] for i = 1, 2. Since ⊢ B_i[Ir_i[A]] ⊃ [Ir_i[Ir_i[A]], Ir_i[Ir_i[A]]] by Nec and K, and since ⊢ Ir_i[Ir_i[A]] ⊃ B_i(A ∧ Ir_i[A]) by (1), we have ⊢ Ir_i[A] ⊃ Ir_i[Ir_i[A], Ir_i[Ir_i[A]]]. By ∧-rule, ⊢ Ir_i[A] ⊃ Ir_i[Ir_i[A], Ir_i[Ir_i[A]]].

(4): It suffices to show that ⊢ Ir_i[A_1 ⊃ C_1] ∧ Ir_i[A_2 ⊃ C_2] ∧ Ir_i[A_1, A_2] ⊃ Ir_i[C_1, C_2]. It is proved by induction over ν that ⊢ Ir_i[A_1 ⊃ C_1] ∧ Ir_i[A_2 ⊃ C_2] ∧ Ir_i[A_1, A_2] ⊃ Ir_i[C_1, C_2] for all v ≥ 0. By ∧-rule, we have the result. ■

Lemma 5.2. GL(L_1) ⊃ [Ir_i[D^R] ⊃ [I_i(s_i)] ⊃ Rat_i^v(s_i)] for all v ≥ 0, s_i ∈ S_i, i = 1, 2.

Proof. We show this by induction on v. Since ⊢ Ir_i[D^R] ⊃ D^R and ⊢ D^R ⊃ [I_i(s_i)] ⊃ ∀t_j ∈ S_j, Bst_i(s_j; t_j), we have the assertion for v = 0. Suppose the assertion for v. Then, ⊢ Ir_i[D^R] ⊃ [B_j(I_j(s_j)) ∧ Bst_i(s_i; s_j)] ⊃ B_j(Rat_j^v(s_j)) ⊃ Bst_i(s_i; s_j)]. Hence, we have ⊢ Ir_i[D^R] ⊃ [∀t_j ∈ S_j, B_j(I_j(s_j)) ∧ Bst_i(s_i; s_j)] ⊃ ∀t_j ∈ S_j, B_j(Rat_j^v(t_j)) ∧ Bst_i(s_i; s_j)]. Since ⊢ Ir_i[D^R] ⊃ Ir_j[D^R], we have ⊢ Ir_i[D^R] ⊃ [Ir_j[D^R] ⊃ ∀t_j ∈ S_j, B_j(I_j(t_j)) ∧ Bst_i(s_i; t_j)] ⊃ ∀t_j ∈ S_j, B_j(Rat_j^v(t_j)) ∧ Bst_i(s_i; t_j)]. Also, since ⊢ Ir_i[D^R] ⊃ [I_i(s_i)] ⊃ ∀t_j ∈ S_j, B_j(I_j(t_j)) ∧ Bst_i(s_i; t_j)], we have ⊢ Ir_i[D^R] ⊃ [I_i(s_i)] ⊃ ∀t_j ∈ S_j, B_j(Rat_j^v(t_j)) ∧ Bst_i(s_i; t_j)]. Thus, ⊢ Ir_i[D^R] ⊃ [I_i(s_i)] ⊃ Rat_i^v(s_i)]. Hence, we have the assertion for v + 1. ■

Proof of Theorem 5.1.(1): This is obtained by Lemma 5.2.

(2): Lemma 5.2 implies ⊢ [Ir_i[D^R] ⊃ [I_i(s_i)] ⊃ Rat_i^v(s_i)] for i = 1, 2. By Lemma 5.1.(2), we have ⊢ Ir_i[Ir_i[D^R] ⊃ [I_i(s_i)] ⊃ Rat_i^v(s_i)], Ir_i[Ir_i[D^R] ⊃ [I_2(s_2)] ⊃ Rat_2^v(s_2)]. Using Lemma 5.1.(4), we have ⊢ Ir_i[Ir_i[Ir_i[D^R]], Ir_i[Ir_i[D^R]] ⊃ [Ir_i[I_i(s_i)] ⊃ Rat_1^v(s_i), I_2(s_2)] ⊃ Rat_2^v(s_2)]. Since ⊢ Ir_i[D^R] ⊃ Ir_i[Ir_i[D^R]], Ir_i[Ir_i[D^R]] by Lemma 5.1.(3), we have the assertion. ■

To prove Theorem 5.2, we will show that ⊢ D^R(Rat) for i = 1, 2. Then, we have ⊢ Ir_i[D^R(Rat)] by Lemma 5.1.(2). By Lemma 5.1.(4), we have ⊢ Ir_i[V^R] ⊃ [Ir_i[Rat_1^v(s_1)] ⊃ I_i(s_1), Rat_1^v(s_1)] ⊃ I_2(s_2)]. This is Theorem 5.2.(1). Combining this with Theorem 5.1.(2) by Lemma 5.1.(5), we have Theorem 5.2.(2).

The first step for ⊢ D^R(Rat) for i = 1, 2 is the following lemma.

Lemma 5.3. (Monotonicity): GL(L_0) ⊃ [Rat_i^v+1(s_i) ⊃ Rat_i^v(s_i)] for all v ≥ 0, s_i ∈ S_i, i = 1, 2.

Proof. We prove the assertion by induction over v ≥ 0. Recall Rat_0^v(s_i) = ∀t_j ∈ S_j, Bst_i(s_j; t_j). Since Rat_1^v(s_i) = ∀t_j ∈ S_j, B_j(Rat_0^v(t_j)) ∧ Bst_i(s_i; t_j)), we have ⊢ Rat_1^v(s_i) ⊃ ∀t_j ∈ S_j, Bst_i(s_j; t_j), i.e., ⊢ Rat_1^v(s_i) ⊃ Rat_0^v(s_i). Suppose that ⊢ Rat_i^v+1(s_i) ⊃ Rat_i^v(s_i) for i = 1, 2. This implies ⊢
\[ \text{B}_j(\text{Rat}^{t_1+1}(s_j)) \wedge \text{Bst}_i(s_i; s_j) \supset \text{B}_j(\text{Rat}^{t_2}(s_j)) \wedge \text{Bst}_i(s_i; s_j), \text{ and then } \vdash \forall t_j \in S_j [\text{B}_j(\text{Rat}^{t_1+1}(t_j)) \wedge \text{Bst}_i(s_i; t_j)] \supset \forall t_j \in S_j [\text{B}_j(\text{Rat}^{t_2}(t_j)) \wedge \text{Bst}_i(s_i; t_j)], \text{ i.e., } \vdash \text{Rat}^{t_1+2}(s_i) \supset \text{Rat}^{t_2+1}(s_i). \]

Now, we prove \( \vdash D_i^R(\text{Rat}) \) for \( i = 1, 2 \). The proof of part (1) is based on the soundness/completeness (Theorem 3.1); the finiteness of \( S_1 \) and Lemma 5.3 are used. In the following lemma, we use the abbreviation \( \wedge \lambda \alpha \nu \text{A}_\nu \) of \( \wedge (A_\nu : \nu \geq 0) \).

**Lemma 5.4.** \( GL(L_1) \vdash \text{Rat}_i(s_i) \supset \forall t_j \in S_j [\text{B}_j(\text{Rat}^{t_j}(t_j)) \wedge \text{Bst}_i(s_i; t_j)]. \)

**Proof.** First, we recall \( \wedge \alpha \nu \text{Rat}^{t_j}_i(s_i) = \wedge \nu \forall t_j \in S_j [\text{B}_j(\text{Rat}^{t_j}(t_j)) \wedge \text{Bst}_i(s_i; t_j)]. \) We prove \( \vdash \wedge \nu \text{Rat}^{t_j}_i(s_i) \supset \forall t_j \in S_j [\wedge \nu \text{B}_j(\text{Rat}^{t_j}(t_j)) \wedge \text{Bst}_i(s_i; t_j)]. \) By rule Io2.\( (i) \) with \( \alpha = 1 \), \( \wedge \nu \text{B}_j(\text{Rat}^{t_j}(t_j)) \) is a permissible conjunction. Since \( \vdash \wedge \nu \text{B}_j(\text{Rat}^{t_j}(t_j)) \equiv \text{B}_j(\text{Rat}^{t_j}(t_j)), \) it follows that \( \vdash \wedge \nu \text{Rat}^{t_j}_i(s_i) \supset \forall t_j \in S_j [\text{B}_j(\text{Rat}^{t_j}(t_j)) \wedge \text{Bst}_i(s_i; t_j)], \) which is the assertion of the lemma.

Let \( M = (F, \tau) \) be a serial Kripke model, and \( w \) any possible world in \( W \). Suppose \( (M, w) \models \wedge \nu \forall t_j \in S_j [\text{B}_j(\text{Rat}^{t_j}(t_j)) \wedge \text{Bst}_i(s_i; t_j)]. \) Then, \((M, w) \models \forall t_j \in S_j [\text{B}_j(\text{Rat}^{t_j}(t_j)) \wedge \text{Bst}_i(s_i; t_j)]\) for any \( \nu \geq 0. \) Let

\[ T_j^\nu = \{ t_j \in S_j : (M, w) \models \text{B}_j(\text{Rat}^{t_j}(t_j)) \wedge \text{Bst}_i(s_i; t_j) \} \] for \( \nu \geq 0. \)

Since \((M, w) \models \forall t_j \in S_j [\text{B}_j(\text{Rat}^{t_j}(t_j)) \wedge \text{Bst}_i(s_i; t_j)], \) we have \( T_j^\nu \neq \emptyset \) for all \( \nu. \) Since \((M, w) \models \text{Rat}_{j_1}^{t_1}(s_j) \supset \text{Rat}_{j_2}^{t_1}(s_j) \) by Lemma 5.3 and Soundness, we have \( T_j^\nu \supset T_j^{\nu+1} \) for all \( \nu \geq 0. \) Since \( S_j \) is a finite set, there is some \( \nu_0 \) such that \( T_j^{\nu_0} \) is constant for all \( \nu \geq \nu_0. \) Hence, we find an \( s_j \in \cap_{\nu} T_j^\nu, \) which implies \((M, w) \models \langle \wedge \nu \text{B}_j(\text{Rat}^{t_j}(s_j)) \rangle \wedge \text{Bst}_i(s_i; s_j). \) Thus, \((M, w) \models \forall t_j \in S_j [\wedge \nu \text{B}_j(\text{Rat}^{t_j}(t_j)) \wedge \text{Bst}_i(s_i; t_j)]. \) Then, \((M, w) \models \langle \wedge \nu \text{Rat}^{t_j}_i(s_i) \supset \forall t_j \in S_j [\langle \wedge \nu \text{B}_j(\text{Rat}^{t_j}(t_j)) \rangle \wedge \text{Bst}_i(s_i; t_j)] \rangle \wedge \text{Bst}_i(s_i; t_j). \) Since \( F, \tau, w \in W \) are all arbitrary, we have \( \vdash \wedge \nu \text{Rat}^{t_j}_i(s_i) \supset \forall t_j \in S_j [\langle \wedge \nu \text{B}_j(\text{Rat}^{t_j}(t_j)) \rangle \wedge \text{Bst}_i(s_i; t_j)] \) by completeness.\]

**Proof of (20):** We use Lemma 3.6, which allows us to infer (20) from assertions about \( \text{Rat}^{t_j}_i(s_j) \) for finite \( \nu \)’s. Now, we work with \( GL(L_0). \) In fact, the main argument uses the technique that eliminates the belief operators \( \text{B}_1(\cdot) \) and \( \text{B}_2(\cdot) \) from \( \text{K}^\nu \) and hence we can work with finitary classical logic, whose provability relation is denoted by \( \vdash_0 \). Correspondingly, we denote, by \( \text{Nat}^{t_j}_i(s_j) \), the formula obtained from \( \text{Rat}^{t_j}_i(s_j) \) eliminating all \( \text{B}_1(\cdot) \) and \( \text{B}_2(\cdot). \) The set \( \langle \text{Nat}^{t_j}_i(s_j) \rangle \) is complete by (19) with respect to atomic preference propositions; for a finitary nonepistemic formula \( A \) containing only atomic preference propositions,

\[ \vdash_0 \wedge (\Gamma_1 \cup \Gamma_2) \supset A \text{ or } \vdash_0 \wedge (\Gamma_1 \cup \Gamma_2) \supset \neg A. \quad (21) \]

This is applied to \( \text{Nat}^{t_j}_i(s_j) \) for all \( i = 1, 2, s_i \in S_i, \) and \( \nu \geq 0. \) Also, when \( A \) contains only atomic preference propositions for agent \( i, \) the premise in (21) can be \( \wedge \Gamma_i. \)

We prove, by induction over \( \nu, \) that for \( i = 1, 2, s_i \in S_i, \) and \( \nu \geq 0, \)

if \( \vdash_0 \wedge (\Gamma_1 \cup \Gamma_2) \supset \text{Nat}^{t_j}_i(s_i), \text{ then } \vdash \text{Ir}_i[\Gamma] \supset \text{B}_i(\text{Rat}^{t_j}_i(s_i)); \quad (22) \)

if \( \vdash_0 \wedge (\Gamma_1 \cup \Gamma_2) \supset \neg \text{Nat}^{t_j}_i(s_i), \text{ then } \vdash \text{Ir}_i[\Gamma] \supset \text{B}_i(\neg \text{Rat}^{t_j}_i(s_i)). \quad (23) \)

For \( \nu = 0, \text{Nat}^{t_j}_i(s_i) = \text{Rat}^{t_j}_i(s_i) = \forall t_j \in S_j \text{ Bst}_i(s_i; t_j). \) Since \( \text{Ir}_i[\Gamma] = (\wedge \Gamma_i) \wedge \text{Ir}_i[\Gamma], \) we obtain (22) and (23) for \( \nu = 0 \) by applying Nec and K. Suppose that (22) and (23) hold for \( \nu. \) By (21), \( \vdash_0 \wedge (\Gamma_1 \cup \Gamma_2) \supset \text{Nat}^{t_j+1}_i(s_i) \text{ or } \vdash_0 \wedge (\Gamma_1 \cup \Gamma_2) \supset \neg \text{Nat}^{t_j+1}_i(s_i). \) First, let \( \vdash_0 \wedge (\Gamma_1 \cup \Gamma_2) \supset \text{Nat}^{t_j+1}_i(s_i); \) by definition, \( \text{Nat}^{t_j+1}_i(s_i) = \forall t_j \in S_j [\text{Nat}^{t_j}_i(t_j) \wedge \text{Bst}_i(s_i; t_j)], \) and hence, by (21) again, \( \vdash_0 \wedge (\Gamma_1 \cup \Gamma_2) \supset (\text{Nat}^{t_j}_i(t_j) \wedge \text{Bst}_i(s_i; t_j)) \) for some \( t_j \in S_j. \) For this \( t_j, \) it holds that \( \vdash \wedge \Gamma_i \supset \text{Bst}_i(s_i; t_j) \) and \( \vdash \text{Ir}_i[\Gamma] \supset \text{B}_i(\text{Rat}^{t_j}_i(t_j)) \) by (22) for \( \nu. \) Combining these, we have \( \vdash \text{Ir}_i[\Gamma] \wedge (\wedge \Gamma_i) \supset \text{Nat}^{t_j}_i(t_j) \wedge \text{Bst}_i(s_i; t_j) \).
\[ \mathbf{B}_j(\text{Rat}_j^\nu(t_j)) \land \text{Bst}_i(s_i) \}. \] Hence, \( \vdash \text{I}_j[\Gamma] \land (\land \Gamma_i) \supset \forall t_j \in S_j [\mathbf{B}_j(\text{Rat}_j^\nu(t_i)) \land \text{Bst}_i(s_i, t_j)]. \) Thus, \( \vdash \text{I}_j[\Gamma] \Rightarrow \text{Rat}_i^{\nu+1}(s_i), \) so, \( \vdash \text{I}_j[\Gamma] \Rightarrow \mathbf{B}_i(\text{Rat}_i^{\nu+1}(s_i)) \) by Nec and K.

Second, let \( \vdash_0 \land (\Gamma_1 \cup \Gamma_2) \supset \neg \text{Nat}_i^{\nu+1}(s_i). \) Again, by definition and (21), \( \vdash_0 \land (\Gamma_1 \cup \Gamma_2) \supset \neg (\text{Nat}_j^\nu(t_j) \land \text{Bst}_i(s_i, t_j)) \) for all \( t_j \in S_j. \) Let \( t_j \in S_j. \) Then, \( \vdash_0 \land (\Gamma_1 \cup \Gamma_2) \supset \neg \text{Nat}_j^\nu(t_j) \) or \( \vdash_0 \land (\Gamma_1 \cup \Gamma_2) \supset \neg \text{Bst}_i(s_i, t_j). \) Then, by (23) for \( \nu, \) we have \( \vdash \text{I}_j[\Gamma] \Rightarrow \mathbf{B}_j(\neg \text{Rat}_j^\nu(t_j)) \) or \( \vdash \land \Gamma_i \supset \neg \text{Bst}_i(s_i, t_j). \) Combining these, we have \( \vdash \text{I}_j[\Gamma] \land (\land \Gamma_i) \supset \mathbf{B}_j(\neg \text{Rat}_j^\nu(t_j)) \lor \neg (\text{Bst}_i(s_i, t_j)). \) This and Axiom D for \( \mathbf{B}_j(\cdot) \) imply \( \vdash \text{I}_j[\Gamma] \land (\land \Gamma_i) \supset \neg \text{B}_j(\text{Rat}_j^\nu(t_j)) \lor \neg (\text{Bst}_i(s_i, t_j)), \) i.e., \( \vdash \text{I}_j[\Gamma] \land (\land \Gamma_i) \supset \neg (\mathbf{B}_j(\text{Rat}_j^\nu(t_j)) \lor \text{Bst}_i(s_i, t_j)). \) Since \( t_j \) is arbitrary, we have \( \vdash \text{I}_j[\Gamma] \land (\land \Gamma_i) \supset \neg \forall t_j \in S_j (\mathbf{B}_j(\text{Rat}_j^\nu(t_j)) \land \text{Bst}_i(s_i, t_j)). \) That is, \( \vdash \text{I}_j[\Gamma] \supset \neg \text{Rat}_i^{\nu+1}(s_i), \) which, by Nec and K, implies (23) for \( \nu + 1. \)

Now, take any \( s_i \in S_i. \) Then, let \( s_i \) be rationalizable action. Then, \( \vdash_0 \land (\Gamma_1 \cup \Gamma_2) \supset \text{Nat}_i^\nu(s_i) \) for all \( \nu \geq 0. \) In this case, by (22), \( \vdash \text{I}_j[\Gamma] \supset \text{Rat}_i^\nu(s_i) \) for all \( \nu \geq 0. \) Thus, \( \vdash \text{I}_j[\Gamma] \supset \text{Rat}_i(s_i) \) by \( \land \)-rule. Hence, \( \vdash \text{I}_j[\Gamma] \supset \mathbf{B}_i(\text{Rat}_i(s_i)). \)

Let \( s_i \) be a non-rationalizable action. Then, \( \vdash_0 \land (\Gamma_1 \cup \Gamma_2) \supset \neg \text{Nat}_i^\nu(s_i) \) for some \( \nu \geq 0. \) In this case, by (23), \( \vdash \text{I}_j[\Gamma] \supset \neg \text{Rat}_i^\nu(s_i) \) for some \( \nu \geq 0. \) By Lemma 3.6, we have \( \vdash \text{I}_j[\Gamma] \supset \neg (\text{Rat}_j^\nu(s_j) : \nu \geq 0). \) Hence, \( \vdash \text{I}_j[\Gamma] \supset \mathbf{B}_i(\neg \text{Rat}_j(s_j)). \) These imply (20). \( \blacksquare \)

### 6 Proof of the Completeness of GL(\( L_\alpha \)) by Q-filters

We adopt the Q-filter method to prove completeness of GL(\( L_\alpha \)). First, we give a sketch of the proof, a summary of the concepts to be used, and then go to the main body of the proof.

#### 6.1 Sketch of the proof

As usual, we show that if a formula \( A \in L_\alpha \) is not provable, we find a Kripke model so that \( A \) is not true in some world. It is standard in the literature to construct maximal consistent sets as those possible worlds via the Henkin method (cf. Hughes and Cresswell [15]). This may appear to be applicable to our logics because the set of formulæ \( L_\alpha \) \((0 \leq \alpha \leq \omega)\) is kept countable. But this does not work in our case for two reasons. Since GL(\( L_\alpha \)) allows infinite conjunctions, the Henkin method to extend a consistent set does not fit our purpose; the infinitary approach from Karp [19] avoids this difficulty by requiring Axiom of Choice in the axiomatic system (cf. Heifetz [12] in the epistemic logic context). Instead, we adopt the Q-filter method, due to Rasiowa-Sikorski [29] for algebraic semantics and Tanaka-Ono [33] for Kripke semantics. A Q-filter is a strengthened version of a prime filter to deal with infinitary conjunctions. This method has been developed as an alternative to prove completeness for a first-order logic as well as for infinitary modal logics (cf., Tanaka [32]). We note that the countability of the language \( L_\alpha \) is crucial in applications of these lemmas.

The Q-filter method relies upon various concepts in Boolean algebra, though we deal with Kripke semantics rather than algebraic semantics. Utilizing the Q-filter method, we construct a counter-model. This is not the canonical model; instead, we start with the Lindenbaum algebra \( L_\alpha \equiv \equiv, \) where \( \equiv \equiv \) is the equivalence relation of provability in GL(\( L_\alpha \)). Then, a Q-filter is a subset of \( L_\alpha \equiv \equiv \) and is a possible world for the counter-model. A Q-filter is required to satisfy certain closure properties in addition to the prime filter condition. These closure properties are guaranteed by the formula construction steps, Ia2, (i) and (ii), for the definition of \( L_\alpha \). Once the
set of possible worlds is defined, accessibility relations $R_i, i \in N$ are defined in a similar manner as in the standard proof based on maximal consistent sets.

In Section 6.2, we provide a small summary of $\mathcal{Q}$-filters in a Boolean algebra. In Section 6.3, we define the Lindenbaum algebra based on $\text{GL}(L_\alpha)$, and prepare for applications of the Rasiowa-Sikorski and Tanaka-Ono lemmas. In Section 6.4, we construct a counter-model. A key step is the truth lemma that a formula $A$ is true in a world $w$ if and only if $[A] \in w$, where $[A]$ is the equivalence class including $A$. This step requires the Tanaka-Ono Lemma to deal with $\mathcal{B}_i(\cdot)$. Finally, we show that if $\not\models A$, there is a $\mathcal{Q}$-filter $w$ such that $[A] \notin w$; the existence of such a $\mathcal{Q}$-filter $w$ is guaranteed by the Rasiowa-Sikorski Lemma.

### 6.2 Boolean algebra and $\mathcal{Q}$-filters

We give basic definitions and relevant properties of a Boolean algebra (cf., Halmos [10] and Mendelson [23]). Consider a Boolean algebra $\mathcal{B} = (\mathcal{B}, \cap, \cup, -, 0, 1)$. We define $a \leq b$ iff $a \cap b = b$. Then $\leq$ is a lattice ordering on $\mathcal{B}$ (i.e., $a \cap b$ and $a \cup b$ are the greatest lower bound and least upper bound of $a, b$ with respect to $\leq$). We say that a nonempty subset $F$ of $\mathcal{B}$ is a filter iff $F_1$ (upward closed): $a \leq b$ and $a \in F \implies b \in F$; and $F_2(\cap$-closed): $a, b \in F \implies a \cap b \in F$. Also, we say that a filter $F$ is prime iff $P_1$(Non-triviality): $F \neq \mathcal{B}$; and $P_2(\cup$-property): $a \cup b \in F \implies a \in F$ or $b \in F$. We have the following fact on a prime filter $F$:

$$a \in F \iff (-a) \notin F.$$  \hspace{1cm} (24)

In the following, we write $a \rightarrow b$ for $-a \cup b = (-a) \cup b$. When $F$ is a prime filter, $a \rightarrow b \in F$ if and only if $a \notin F$ or $b \in F$, since $(-a) \cup a = 1 \in F$.

For any subset $S$ of $\mathcal{B}$, the greatest lower bound of $S$ in $(\mathcal{B}, \cap, \cup, -, 0, 1)$ is denoted by $\cap S$, and the least upper bound of $S$ is denoted by $\cup S$. Note that $\cap S$ and $\cup S$ may not exist, but if either exists, it is unique. Let $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ be a pair of countable sets of nonempty subsets of $\mathcal{B}$ so that

- $((\cap, \cup)$-closed): $\cap \mathcal{Q}_1$ and $\cup \mathcal{Q}_2$ exist for all $Q_1 \in \mathcal{Q}_1$ and $Q_2 \in \mathcal{Q}_2$.

We say that a prime filter $F$ is a $\mathcal{Q}$-filter iff

- $Q_1$: for any $Q_1 \in \mathcal{Q}_1$, $Q_1 \subseteq F \implies \cap Q_1 \in F$;
- $Q_2$: for any $Q_2 \in \mathcal{Q}_2$, $\cup Q_2 \in F \implies a \in Q_2$ for some $a \in F$.

These correspond to the conditions $F_2$ and $P_2$. The following is Rasiowa-Sikorski lemma (see also Tanaka-Ono [33]).

**Lemma 6.1.** (Rasiowa-Sikorski [29]) Let $\mathcal{B}$ be a Boolean algebra, and $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ a pair of countable sets of nonempty subsets of $\mathcal{B}$ with $(\cap, \cup)$-closedness. For any $a, b \in \mathcal{B}$, if $a \not\leq b$, then there is a $\mathcal{Q}$-filter $F$ such that $a \in F$ and $b \notin F$.

For a given $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$, we denote the set of all $\mathcal{Q}$-filters of $\mathcal{B}$ by $F_\mathcal{Q}(\mathcal{B})$. The nonemptiness of $F_\mathcal{Q}(\mathcal{B})$ follows from Lemma 6.1 if $0 \not\in \mathcal{Q}$. The set $F_\mathcal{Q}(\mathcal{B})$ will be adopted for the set of all possible worlds in our construction of a Kripke model.

Since the logic $\text{GL}(L_\alpha)$ has belief operators, Rasiowa-Sikorski lemma is not enough: We extend it, which is Tanaka-Ono lemma. We say that $\mathcal{B} = (\mathcal{B}, \cap, \cup, -, 0, 1, \Box_1, ..., \Box_n)$ is a multi-modal algebra iff
ma1: \((B, \cap, \cup, \neg, 0, 1)\) is a Boolean algebra;

ma2: for \(i \in N\), \(\Box_i\) is a unary operator on \(B\) satisfying the property that \(\Box_i 1 = 1\)
and \(\Box_i (a \cap b) = \Box_i a \cap \Box_i b\) for all \(a, b \in B\).

We define \(\Box_i^{-1} F = \{ x \in B : \Box_i x \in F \} \) for any \(F \subseteq B\).

Let \(B\) be a multi-modal algebra, and \(Q = (Q_1, Q_2)\) a fixed pair of countable sets of nonempty subsets of \(B\) satisfying \((\cap, \cup)\)-closedness. The following three conditions are crucial for the Tanaka-Ono Lemma: for all \(i \in N\),

q0: for all \(Q_1 \in Q_i\), \(\cap(\Box_i Q_1) := \cap\{\Box_i a : a \in Q_1\} \) exists and \(\cap(\Box_i Q_1) = \Box_i(\cap Q_1)\);

q1: \(\Box_i(a \rightarrow b) : b \in Q_1\) \(\in Q_i\) for all \(a \in B\) and all \(Q_1 \in Q_i\);

q2: \(\Box_i(b \rightarrow a) : b \in Q_2\) \(\in Q_1\) for all \(a \in B\) and all \(Q_2 \in Q_2\).

**Lemma 6.2. (Tanaka-Ono [33])** Let \(B = (B, \cap, \cup, \neg, 0, 1, \Box_1, \ldots, \Box_n)\) be a multi-modal algebra, and \(Q = (Q_1, Q_2)\) a fixed pair of countable sets of nonempty subsets of \(B\). Suppose that \(Q\) satisfies \((\cap, \cup)\)-closedness, and the conditions q0, q1, and q2 for \(i \in N\). Then, for any \(i \in N\), \(b \in B\), and \(F \in F_Q(B)\), if \(\Box_i b \notin F\), there exists a \(G \in F_Q(B)\) such that \(\Box_i^{-1} F \subseteq G\) and \(b \notin G\).

**6.3 Lindenbaum algebra**

Recall that for any \(A, B \in L_\alpha\), \(A \equiv B\) iff \(\vdash (A \supset B) \wedge (B \supset A)\) in \(GL(L_\alpha)\). We take the quotient set \(L_\alpha/\equiv\). For any \(A \in L_\alpha\), we denote, by \([A]\), the equivalence class in \(L_\alpha/\equiv\) including \(A\). In \(B := L_\alpha/\equiv\), we define elements \(0, 1\) and operations \(\cap, \cup, \neg\), and \(\Box_1, \ldots, \Box_n\) by

\[
\ell_1: 0 = [\neg p_0 \wedge p_0] \text{ and } 1 = [p_0 \supset p_0];
\]

\[
\ell_2: \text{ for any } A, B \in L_\alpha, [A] \cap [B] = [A \wedge B], [A] \cup [B] = [\neg(\neg A \wedge \neg B)], [\neg A] = [\neg A];
\]

\[
\ell_3: \text{ for any } A \in L_\alpha, \Box_i [A] = [B_i(A)] \text{ for } i \in N.
\]

Using these, we have, for any \(A, B \in L_\alpha\),

\[
[A] \rightarrow [B] = ([\neg A]) \cup [B] = [\neg A] \cup [B] = [\neg(\neg A \wedge \neg B)] = [A \supset B].
\]

(25)

It follows from this that \(\Box_i([A] \rightarrow [B]) = \Box_i([A \supset B]) = [B_i(A \supset B)]\).

**Lemma 6.3.** \(L = (B, 0, 1, \cap, \cup, \neg, \Box_1, \ldots, \Box_n)\) with \(B = (L_\alpha/\equiv)\) is a multi-modal algebra.

**Proof.** We can show in the standard manner that \((B, 0, 1, \cap, \cup, \neg)\) with \(B = (L_\alpha/\equiv)\) is a Boolean algebra. It remains to show condition ma2. Let \(i \in N\). Since \(\vdash (A \supset B) \supset B_i(A \supset A) \wedge B_i(A \supset D) \supset (A \supset A)\), we have \(\Box_i 1 = 1\). Since \(\vdash B_i(A \wedge C) \supset B_i(A) \wedge B_i(C) \supset B_i(A) \wedge B_i(C) \supset B_i(A \wedge C)\), we have \(\Box_i([A] \cap [C]) = [\Box_i[A] \cap \Box_i[C]]\).

In the following, we call \(L\) in Lemma 6.3 the *Lindenbaum algebra*. We prove the following lemma, which guarantees we can use Lemmas 6.1 and 6.2 in the proof of completeness.

**Lemma 6.4.** For any \(\wedge \Phi \in L_\alpha\) and \(i \in N\),

\((a): \Box_i \{[C] : C \in \Phi\} = [\wedge \Phi];\)

\((b): \Box_i \{\Box_i[C] : C \in \Phi\} = [B_i(\wedge \Phi)].\)
Proof. (a): First, let us see that $[\land \Phi]$ is a lower bound of $\{[C] : C \in \Phi\}$. Since $\vdash \land \Phi \supset C$ for all $C \in \Phi$ by L4, we have $(-[\land \Phi]) \cup [C] = 1$. Let $C \in \Phi$. Then, we have

$$[\land \Phi] = [\land \Phi] \cap 1 = [\land \Phi] \cap ([[-\land \Phi]] \cup [C])$$

$$= ([[-\land \Phi]] \cap [\land \Phi] \cap [C]) = 0 \cup ([\land \Phi] \cap [C]) = [\land \Phi] \cap [C].$$

Hence, $[\land \Phi] \leq [C]$. Since $C$ is arbitrary in $\Phi$, $[\land \Phi]$ is a lower bound of $\{[C] : C \in \Phi\}$.

It remains to show that $[\land \Phi]$ is the greatest lower bound of $\{[C] : C \in \Phi\}$. Now, let $[D]$ be a lower bound of $\{[C] : C \in \Phi\}$. This means $[D] \leq [C]$, i.e., $[D] \cup [C] = [C]$, for any $C \in \Phi$. Let $C \in \Phi$. Then $([D] \cup [C]) = ([D] \cup [C]) = 1$. This implies $\vdash D \supset C$. Since $C$ is arbitrary in $\Phi$, we have, by $\land$-rule, we have $\vdash D \supset \land \Phi$. This means that $[\land \Phi]$ is greater than or equal to $[D]$ in $L$. Thus, $[\land \Phi]$ is the greatest lower bound of $\{[C] : C \in \Phi\}$.

(b): Since $\vdash \Diamond_i (\land \Phi) \supset B_i(A)$ for all $A \in \Phi$, and since $\{\Box_i[A] : A \in \Phi\} = \{[B_i(A)] : A \in \Phi\}$, $[B_i(\land \Phi)]$ is a lower bound of $\{\Box_i[A] : A \in \Phi\}$. Now, let $[D]$ be a lower bound of $\{\Box_i[A] : A \in \Phi\}$. Using the same argument as in (a), we have $\vdash D \supset B_i(A)$ for all $A \in \Phi$. Thus, $\vdash D \supset \land \Phi_i(\land \Phi)$. By $\land$-Barcan, we have $\vdash D \supset B_i(\land \Phi)$. This means that $[B_i(\land \Phi)]$ is the greatest lower bound of $\{\Box_i[A] : A \in \Phi\}$. 

Now we define a pair $Q = (Q_1, Q_2)$ as follows:

$$Q_1 = \{\{[A] : A \in \Phi\} : \land \Phi \in L_\alpha\} \text{ and } Q_2 = \emptyset. \tag{26}$$

Then, $Q_1$ is a countable. Then, the following lemma holds.

Lemma 6.5. (1): $Q = (Q_1, Q_2)$ satisfies $(\cap, \sqcup)$-closedness.

(2): $Q = (Q_1, Q_2)$ satisfies the conditions $q0, q1, q2$.

Proof. Since $Q_2 = \emptyset$, the $(\cap, \sqcup)$-closedness for $\sqcup$ and $q2$ are vacuous.

(1): Let $Q \in Q_1$. This $Q$ is written as $\{[A] : A \in \Phi\}$ for some $\land \Phi \in L_\alpha$. Since $\cap Q = [\land \Phi]$ by Lemma 6.4.(a), $\cap Q$ belongs to $B = L_\alpha/\equiv$.

(2)(q0): We show that for any $Q \in Q_1$, $\cap(\Box a : a \in Q) := \cap \Box a : a \in Q$ exists and $\cap(\Box a : a \in Q) = \Box \cap(\Box a : a \in Q)$. Since $Q \in Q_1$, $\{\Box a : a \in Q\}$ is expressed as $\{[B_i(A)] : A \in \Phi\}$ for some $\land \Phi \in L_\alpha$. By I31-I32, $\land \Phi \in L_\alpha$ implies $\land \Phi_i(\land \Phi) \in L_\alpha$. Then, by Lemma 6.4.(b) and $\vdash \land \Phi_i(\land \Phi) \equiv B_i(\land \Phi)$, it holds that $\cap(\Box a : a \in Q) = \land \Phi_i(\land \Phi) = \Box \cap(\Box a : a \in Q)$.

(q1): Let $Q \in Q_1$ and $a \in B$. We show $\{\Box a : b \in Q\} \in Q_1$. Since $a = [A]$ for some $A \in L_\alpha$ and $Q$ is also expressed as $\{[B] : B \in \Phi\}$ for some $\land \Phi \in L_\alpha$, we have, by (25),

$$\{\Box a : b \in Q\} = \{[B_i(A \supset B)] : B \in \Phi\}. \tag{27}$$

Since $\land (A \supset B) : B \in \Phi \in L_\alpha$ by Ia2.(i), we have $\land \{B_i(A \supset B) : B \in \Phi\} \in L_\alpha$ by Ia2.(ii). Let $\Phi' = (B_i(A \supset B) : B \in \Phi)$. Then, since $\land \Phi' \in L_\alpha$, we have, by (27), $\{\Box a : b \in Q\} = \{[B_i(A \supset B)] : B \in \Phi\} \in Q_1$. 

6.4 Construction of a counter-model

Recall that $L = (B, 0, 1, \cap, \cup, -, \Box_i, \ldots, \Box_n)$ with $B = L_\alpha/\equiv$ is the Lindenbaum algebra given in Lemma 6.3. Also, let $Q = (Q_1, Q_2)$ be given by (26). Now, we define a Kripke frame
\[ K = (W; R_1, \ldots, R_n) \] and an assignment \( \tau \) as follows:

(i): \( W = \mathbb{F}_Q(L) \), where \( \mathbb{F}_Q(L) \) is the set of all \( Q \)-filters for \( L \);

(ii): for all \( i \in N, wR_iu \) if and only if \( \Box^{-1}_i w \subseteq u \); 

(iii): for any \( w \in W \) and any propositional variable \( p, \tau(w, p) = \top \) if and only if \([p]\) \( \in w \).

The nonemptiness of \( \mathbb{F}_Q(L) \) follows from Lemma 6.1 and the contradiction-freeness of \( GL(L_{\alpha}) \) noted after Theorem 3.1. Then, \( M = (K, \tau) = (W; R_1, \ldots, R_n, \tau) \) is a Kripke model.

**Lemma 6.6.** \( R_i \) is serial for each \( i \in N \).

**Proof.** Let \( w \in W \). Consider \( \Box, 0 = \Box_i[\neg p_0 \land p_0] \) Then, \( \Box, 0 = [B_i(\neg p_0 \land p_0)] = [\neg p_0 \land p_0] = 0 \) by \( \ell_1 \) and by Axiom D. Since \( w \) is a prime filter, we have \( \Box, 0 = 0 \notin w \). By Lemma 6.2 (Tanaka-Ono Lemma), we have \( u \in \mathbb{F}_Q(L) \) such that \( \Box^{-1}_i w \subseteq u \), i.e., \( wR_iu \), and \( 0 \notin u \).

The following lemma is central to the completeness theorem.

**Lemma 6.7.** (Truth lemma) For any \( A \in L_{\alpha} \) and \( w \in W, (K, \tau, w) \models A \) if and only if \([A] \in w \).

**Proof.** We prove the assertion by induction along the definition \( 1\beta_0-1\beta_2 (\beta \leq \alpha) \) of formulae. Consider a propositional variable \( p \). Then \( (K, \tau, w) \models p \iff \tau(w, p) = \top \iff [p] \in w \).

Now, consider a non-propositional formula \( A \) in \( L_{\alpha} \). Suppose that \( A \) is generated by \( 1\beta_1 \). Here, the induction hypothesis (abbreviated as \( IH \)), is simply that the assertion holds for any proper subformulae of \( A \). The case \( \land \) is applied to an infinitary conjunctive formula.

\( (\lor) \): Let \( (K, \tau, w) \models A \lor B \). Then \( (K, \tau, w) \not\models A \lor (K, \tau, w) \models B \). By the induction hypothesis, we have \([A] \notin w \) or \([B] \in w \). Since \([\neg A] \in w \) or \([B] \in w \), and since \([\neg A] \subseteq [A \supset B] \) and \([B] \subseteq [A \supset B] \), we have \([A \lor B] \in w \).

Let \([A \lor B] \in w \). Then \([\neg A \lor B] = [\neg A] \cup [B] \in w \). Since \( w \) is a prime filter, we have \([\neg A] \in w \) or \([B] \in w \). Hence \([A] \notin w \) or \([B] \in w \). By \( IH \), we have \((K, \tau, w) \not\models A \) or \((K, \tau, w) \models B \). Thus, \((K, \tau, w) \models A \lor B \).

\( (\neg) \): The proof is similar.

\( (\land) \): Let \( (K, \tau, w) \models B_i(A) \). Then \( (K, \tau, u) \models A \) for any \( u \) with \((w, u) \in R_i \). By \( IH \), \([A] \in u \) for any \( u \) with \((w, u) \in R_i \). Now, on the contrary, suppose that \( \Box_i[A] \notin w \). Then, by Lemma 6.2 (Tanaka-Ono Lemma), there is a \( u \in \mathbb{F}_Q(L) \) such that \( \Box^{-1}_i w \subseteq u \) and \([A] \notin u \). This is a contradiction. Hence, \([B_i(A)] = \Box_i[A] \in w \).

Let \([B_i(A)] = \Box_i[A] \in w \). Then \([A] \in u \) for all \( u \) with \( \Box_i^{-1} w \subseteq u \). By \( IH \), we have \((K, \tau, u) \models A \) for all \( u \) with \((w, u) \in R_i \). Hence, \((K, \tau, u) \models B_i(A) \).

\( (\lor) \): Let \( \land \Phi \) be a finite conjunctive formula generated by \( 1\beta_1 \), or an infinite conjunctive formula given from a germinal form. In the latter case, any \( A \in \Phi \) belongs to \( \cup_{\gamma<\beta} L_\gamma \). In either case, \( IH \) is that the assertion holds for any \( A \in \Phi \). In these cases, we have the following proof.

Let \( (K, \tau, w) \models \land \Phi \). Then \( (K, \tau, w) \models A \) for all \( A \in \Phi \). By \( IH \), \([A] \in w \) for all \( A \in \Phi \). Then \( \{[A] : A \in \Phi \} \) exists by Lemma 6.5.(1), and it belongs to \( w \) by \( Q_1 \). Hence, \([\land \Phi] = \land \{[A] : A \in \Phi \} \in w \).

Let \([\land \Phi] \in w \). Then \([\land \Phi] \leq [A] \) for all \( A \in \Phi \). Since \( w \) is a filter, we have \([A] \in w \) for all \( A \in \Phi \) by \( F_1 \). Hence \((K, \tau, w) \models A \) for all \( A \in \Phi \) by \( IH \), which implies \((K, \tau, w) \models \land \Phi \).
Now, consider the cases of I\(32.(i), I32.(ii), \) and I\(32.(iii)\). Suppose that \(\wedge \Phi = \wedge (D \supset C_\nu : \nu \ge 0), \wedge \Phi = \wedge (B_\nu(C_\nu) : \nu \ge 0), \) and \(\wedge \Phi = \wedge (C_\nu \land D_\nu : \nu \ge 0)\) be generated by I\(32.(i), I32.(ii), \) or I\(32.(iii)\) from \(D, \wedge (C_\nu : \nu \ge 0), \) and \(\wedge (D_\nu : \nu \ge 0)\). Here, IH is that the assertion holds form \(D, \wedge (C_\nu : \nu \ge 0), \) and \(\wedge (D_\nu : \nu \ge 0)\).

Let \((K, \tau, w) \models \wedge (D \supset C_\nu : \nu \ge 0)\). Then \((K, \tau, w) \models D \supset C_\nu, \) i.e., \((K, \tau, w) \not\models D\) or \((K, \tau, w) \models C_\nu, \) for all \(\nu \ge 0\). The latter part implies \((K, \tau, w) \models \wedge (C_\nu : \nu \ge 0)\). By IH, we have \([D] \not\subseteq w\) or \([\wedge (C_\nu : \nu \ge 0)] \subseteq w\). Since \(w\) is a prime filter, we have \([D] \rightarrow [\wedge (C_\nu : \nu \ge 0)] \subseteq w\), which implies \([D \supset C_\nu : \nu \ge 0)] \subseteq w\) by (25). Since \(\vdash (D \supset \wedge (C_\nu : \nu \ge 0)) \equiv \wedge (D \supset C_\nu : \nu \ge 0)\), we have \([\wedge (D \supset C_\nu : \nu \ge 0)] \subseteq w\). The converse can be obtained by tracing back this argument.

Let \((K, \tau, w) \models \wedge (B_\nu(C_\nu) : \nu \ge 0)\). This implies \((K, \tau, w) \models B_\nu(\wedge (C_\nu : \nu \ge 0))\). Let \(u\) be any world with \((w,u) \in R_i\). Then, \((K, \tau, u) \models \wedge (C_\nu : \nu \ge 0)\). By IH, we have \([\wedge (C_\nu : \nu \ge 0)] \subseteq w\). Now, on the contrary, suppose that \(\square_\nu [\wedge (C_\nu : \nu \ge 0)] \not\subseteq w\). Then, by Lemma 6.2 (Tanaka-Ono Lemma), there is a \(w' \in FQ(L)\) such that \(\square_\nu^{-1} w \subseteq w'\) and \([\wedge (C_\nu : \nu \ge 0)] \not\subseteq w'\). Since \((w,w') \in R_i\), this is a contradiction. Hence, \(\square_\nu [\wedge (C_\nu : \nu \ge 0)] \subseteq w\).

Conversely, let \([\wedge (B_\nu(C_\nu) : \nu \ge 0)] \subseteq w\). Then, \(\square_\nu [\wedge (C_\nu : \nu \ge 0)] = [\wedge (B_\nu(\wedge (C_\nu : \nu \ge 0)) = [\wedge (B_\nu(C_\nu) : \nu \ge 0)] \subseteq w\) using \(\Box\)-Barcan. Let \(u \in W\) be an arbitrary world with \(\square_\nu^{-1} w \subseteq u\). Then, \([\wedge (C_\nu : \nu \ge 0)] \subseteq u\). By IH, we have \((K, \tau, u) \models \wedge (C_\nu : \nu \ge 0)\). Since \(u\) is arbitrary with \((w,u) \in R_i\), we have \((K, \tau, u) \models B_\nu(C_\nu)\) for all \(\nu \ge 0\). Hence, \((K, \tau, w) \models \wedge (B_\nu(C_\nu) : \nu \ge 0)\).

Let \((K, \tau, w) \models \wedge (C_\nu \land D_\nu : \nu \ge 0)\). Then \((K, \tau, w) \models C_\nu \land D_\nu, \) i.e., \((K, \tau, w) \models C_\nu\) and \((K, \tau, w) \models D_\nu\) for all \(\nu \ge 0\). This implies \((K, \tau, w) \models \wedge (C_\nu : \nu \ge 0)\) and \((K, \tau, w) \models \wedge (D_\nu : \nu \ge 0)\). By IH, \([\wedge (C_\nu : \nu \ge 0)] \subseteq w\) and \([\wedge (D_\nu : \nu \ge 0)] \subseteq w\). Since \(w\) is a filter, we have \([\wedge (C_\nu : \nu \ge 0)] \subseteq w\). Since \(\vdash [C_\nu : \nu \ge 0] \land [D_\nu : \nu \ge 0] \equiv \wedge (C_\nu \land D_\nu : \nu \ge 0)\), we have \([\wedge (C_\nu \land D_\nu : \nu \ge 0)] \subseteq w\). The converse can be obtained by tracing back this argument.

The final step of completeness is to show that for any \(A \in L_\alpha, \) if \(A \not\models A\), then \((K, \tau, w) \not\models A\) for some world \(w \in W.\) Suppose \(A \not\models A\). This means \([A] \neq 1;\) hence, \([A] \neq 1.\) Applying Lemma 6.1 (Rasiowa-Sikorski lemma) to \([A]\) and 1, there is a \(\mathcal{Q}\)-filter \(F\) such that \(1 \in F\) and \([A] \not\in F.\)

Denote \(F\) by \(w.\) Then, by Lemma 6.7, we have \((K, \tau, w) \not\models A.\)
such as common knowledge; and it allows for the direct evaluation of depths of such infinitary concepts. Similar to the fixed-point approach, we can control infinitary expressions by imposing specific germinal forms. Moreover, we have shown that our completeness result holds for each layer, and our logics in different layers are connected by the conservative extension relation.

We provided two applications. The first is about explicit definabilities of epistemic axioms T, 4, and 5. Specifically, we showed that Axiom T can be captured in GL($L_\alpha$) for any $\alpha$ ($0 \leq \alpha \leq \omega$), Axiom 4 can be done in GL($L_\omega$), since it needs infinite iterations of the belief operator. Axiom 5 is not explicitly definable for any $\alpha$ ($0 \leq \alpha \leq \omega$). These results differentiate the three axioms. The second is for game theory: we considered an agent’s decision-making in a game, based on the idea of rationalizability. We gave a full epistemic characterization, which was done within GL($L_2$), a shallow part in the series in (1), and, based on this characterization, we obtained the playability result for an agent in a game.

Our approach gives rise to new open problems. As already stated, a full study of Table 3.1 is an open problem of great importance. As seen in Section 4.3, we showed that some known fixed-point logics such as common knowledge logic can be faithfully embedded into our system (Theorem 4.7). In recent years, the fixed-point approach has been extensively developed in modal $\mu$-calculus, and a natural question is whether such embedding results can be extended to (some specific fragments of) those logics, and what relationship exists between our system and modal $\mu$-calculus. A full answer to this question remains open, though we gave a summary of differences in our approach and modal $\mu$-calculus in the end of Section 4.3.

There are open problems related to explicit definability and embedding. We studied explicit definability and embedding for each of the three epistemic axioms and common knowledge. However, a general criterion for an infinitary (and/or finitary) concept to be explicitly definable in some GL($L_\alpha$) remains open.\(^\text{10}\) A related problem is to have a general understanding of when a fixed-point logic can be embedded into our system.

Our framework adopts the Hilbert-style proof theory. One alternative would be to formulate it in the Gentzen style sequence calculus. In particular, if cut-elimination is available, then one can discuss the sizes of proofs. For this purpose, there are two possibilities from the literature. One is to adopt Kaneko-Nagashima [21]’s formulation in the context of an infinitary logic, which is close to the original Gentzen formulation. Cut-elimination is available, while $\land$-Barcan prevents it from implying the full subformula property. Another is in the modal $\mu$-calculus, for which Brünnler-Studer [6] provided a different Gentzen style formulation, focusing on some shallow fragments for cut-elimination. A full study of these systems remains open.

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