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Uniqueness of degree-one Ginzburg-Landau vortex in the unit ball in dimensions \( N \geq 7 \)

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Abstract

For \( \varepsilon > 0 \), we consider the Ginzburg-Landau functional for \( \mathbb{R}^N \)-valued maps defined in the unit ball \( B^N \subset \mathbb{R}^N \) with the vortex boundary data \( x \) on \( \partial B^N \). In dimensions \( N \geq 7 \), we prove that for every \( \varepsilon > 0 \), there exists a unique global minimizer \( u_\varepsilon \) of this problem; moreover, \( u_\varepsilon \) is symmetric and of the form \( u_\varepsilon(x) = f_\varepsilon(|x|) \frac{x}{|x|} \) for \( x \in B^N \).

Keywords: uniqueness, symmetry, minimizers, Ginzburg-Landau.

MSC: 35A02, 35B06, 35J50.

1 Introduction and main results

In this note, we consider the following Ginzburg-Landau type energy functional

\[
E_\varepsilon(u) = \int_{B^N} \left[ \frac{1}{2} |\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1 - |u|^2) \right] \, dx,
\]

where \( \varepsilon > 0 \), \( B^N \) is the unit ball in \( \mathbb{R}^N \), \( N \geq 2 \), and the potential \( W \in C^1(\mathbb{R}) \) satisfies

\[
W(0) = 0, \quad W(t) > 0 \text{ for all } t \in (-\infty, 1) \setminus \{ 0 \}, \quad \text{and } W \text{ is convex.} \tag{1}
\]

We investigate the global minimizers of the energy \( E_\varepsilon \) in the set

\[
\mathcal{A} := \{ u \in H^1(B^N; \mathbb{R}^N) : u(x) = x \text{ on } \partial B^N = S^{N-1} \}.
\]
The requirement that \( u(x) = x \) on \( \mathbb{S}^{N-1} \) is sometimes referred in the literature as the vortex boundary condition.

We note that in our analysis the convexity of \( W \) needs not be strict; compare [6] where strict convexity is assumed.

The direct method in the calculus of variations yields the existence of a global minimizer \( u_\varepsilon \) of \( E_\varepsilon \) over \( \mathcal{A} \) for all range of \( \varepsilon > 0 \). Moreover, any minimizer \( u_\varepsilon \) belongs to \( C^1(B^N; \mathbb{R}^N) \) and satisfies \( |u_\varepsilon| \leq 1 \) and the system of PDEs (in the sense of distributions)

\[
- \Delta u_\varepsilon = \frac{1}{\varepsilon^2} W'(1 - |u_\varepsilon|^2) \quad \text{in} \quad B^N.
\]

The goal of this note is to give a short proof of the uniqueness and symmetry of the global minimizer of \( E_\varepsilon \) in \( \mathcal{A} \) for all \( \varepsilon > 0 \) in dimensions \( N \geq 7 \). We prove that, in these dimensions, the global minimizer is unique and given by the unique radially symmetric critical point of \( E_\varepsilon \) defined by

\[
u_\varepsilon(x) = f_\varepsilon(|x|) \frac{x}{|x|} \quad \text{for all} \quad x \in B^N,
\]

where the radial profile \( f_\varepsilon : [0, 1] \to \mathbb{R}_+ \) is the unique solution of

\[
\begin{align*}
- f_\varepsilon'' - \frac{N-1}{r} f_\varepsilon' + \frac{N-1}{r^2} f_\varepsilon &= \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2) \quad \text{for} \quad r \in (0, 1), \\
f_\varepsilon(0) &= 0, f_\varepsilon(1) = 1.
\end{align*}
\]

Moreover, \( f_\varepsilon > 0 \) and \( f_\varepsilon' > 0 \) in \((0, 1)\) (see e.g. [4]).

**Theorem 1.** Assume that \( W \) satisfies (1). If \( N \geq 7 \), then for every \( \varepsilon > 0 \), \( u_\varepsilon \) given in (3) is the unique global minimizer of \( E_\varepsilon \) in \( \mathcal{A} \).

To our knowledge, the question about the uniqueness of minimizers/critical points of \( E_\varepsilon \) in \( \mathcal{A} \) for any \( \varepsilon > 0 \) was raised in dimension \( N = 2 \) in the book of Bethuel, Brezis and Hélein [1, Problem 10, page 139], and in general dimensions \( N \geq 2 \) and also for the blow-up limiting problem around the vortex (when the domain is the whole space \( \mathbb{R}^N \) and by rescaling, \( \varepsilon \) can be assumed equal to 1) in an article of Brezis [2, Section 2].

It is well known that uniqueness is present for large enough \( \varepsilon > 0 \) for any \( N \geq 2 \). Indeed, for any \( \varepsilon > (W'(1)/\lambda_1)^{1/2} \) where \( \lambda_1 \) is the first eigenvalue of \(-\Delta\) in \( B^N \) with zero Dirichlet boundary condition, \( E_\varepsilon \) is strictly convex in \( \mathcal{A} \) and thus has a unique critical point in \( \mathcal{A} \) (that is the global minimizer of our problem).

For sufficiently small \( \varepsilon > 0 \) all results regarding uniqueness question available in the literature are in the affirmative. In particular, we have:

(i) Pacard and Rivièere [11] Theorem 10.2] showed in dimension \( N = 2 \) that, for small \( \varepsilon > 0 \), \( E_\varepsilon \) has in fact a unique critical point in \( \mathcal{A} \).

(ii) Mironescu [10] showed in dimension \( N = 2 \) that, when \( B^2 \) is replaced by \( \mathbb{R}^2 \) and \( \varepsilon = 1 \), a local minimizer of \( E_\varepsilon \) subjected to a degree-one boundary condition at infinity is
unique (up to translation and suitable rotation). This was generalized to dimension $N = 3$ by Millot and Pisante [9] and dimensions $N \geq 4$ by Pisante [12], also in the case of the blow-up limiting problem on $\mathbb{R}^N$ and $\varepsilon = 1$.

These results should be compared to those for the limit problem on the unit ball obtained by sending $\varepsilon \to 0$. In this limit, the Ginzburg-Landau problem converges to the harmonic map problem from $B^N$ to $S^{N-1}$. It is well known that, the vortex boundary condition gives rise to a unique minimizing harmonic map $x \mapsto \frac{x}{|x|}$ if $N \geq 3$; see Brezis, Coron and Lieb [3] in dimension $N = 3$, Jäger and Kaul [7] in dimensions $N \geq 7$, and Lin [8] in dimensions $N \geq 3$.

We highlight that, in contrast to the above, our result holds for all $\varepsilon > 0$, provided that $N \geq 7$. The method of our proof deviates somewhat from that in the aforementioned works. In fact it is reminiscent of our recent work [6] on the (non-)uniqueness and symmetry of minimizers of the Ginzburg-Landau functionals for $\mathbb{R}^M$-valued maps defined on $N$-dimensional domains, where $M$ is not necessarily the same as $N$. However we note that the results in [6] do not directly apply to the present context, as in [6] it is required that $W$ be strictly convex. Furthermore, a priori, it is not clear why non-strict convexity of the potential $W$ is sufficient to ensure uniqueness of global minimizers.

We exploit the convexity of $W$ to lower estimate the ‘excess’ energy by a suitable quadratic energy which can be handled by the factorization trick à la Hardy. Indeed, the positivity of the excess energy is then related to the validity of a Hardy-type inequality, which explains our restriction of $N \geq 7$. This echoes our observation made in [6] that a result of Jäger and Kaul [7] on the minimality of the equator map in these dimensions is related to a certain inequality involving the sharp constant in the Hardy inequality.

We expect that our result remains valid in dimensions $2 \leq N \leq 6$, but this goes beyond the scope of this note and remains for further investigation.

2 Proof of Theorem 1

Theorem 1 will be obtained as a consequence of a stronger result on the uniqueness of global minimizers of for the $\mathbb{R}^M$-valued Ginzburg-Landau functional with $M \geq N$. By a slight abuse of notation, we consider the energy functional

$$E_\varepsilon(u) = \int_{B^N} \left[ \frac{1}{2} |\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1 - |u|^2) \right] dx,$$

where $u$ belongs to

$$\mathcal{A} := \{ u \in H^1(B^N; \mathbb{R}^M) : u(x) = x \text{ on } \partial B^N = S^{N-1} \subset \mathbb{R}^M \}.$$

**Theorem 2.** Assume that $W$ satisfies (i). If $M \geq N \geq 7$, then for every $\varepsilon > 0$, $u_\varepsilon$ given in (3) is the unique global minimizer of $E_\varepsilon$ in $\mathcal{A}$.

When $W$ is strictly convex, the above theorem is proved in [6]; see Theorem 1.7. The argument therein uses the strict convexity in a crucial way.

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Proof. The proof will be done in several steps. First, we consider the difference between the energies of the critical point \( u_\varepsilon \), defined in (3), and an arbitrary competitor \( u_\varepsilon + v \) and show that this difference is controlled from below by some quadratic energy functional \( F_\varepsilon(v) \). Second, we employ the positivity of the radial profile \( f_\varepsilon \) in (4) and apply the Hardy decomposition method in order to show that \( F_\varepsilon(v) \geq 0 \), which proves in particular that \( u_\varepsilon \) is a global minimizer of \( E_\varepsilon \). Finally, we characterise the situation when this difference is zero and conclude to the uniqueness of the global minimizer \( u_\varepsilon \).

Step 1: Lower bound for energy difference. For any \( v \in H^1_0(B^N; \mathbb{R}^M) \), we have

\[
E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) = \int_{B^N} \left[ \nabla u_\varepsilon \cdot \nabla v + \frac{1}{2} |\nabla v|^2 \right] dx \\
+ \frac{1}{2\varepsilon^2} \int_{B^N} \left[ W(1 - |u_\varepsilon + v|^2) - W(1 - |u_\varepsilon|^2) \right] dx.
\]

Using the convexity of \( W \), we have

\[
W(1 - |u_\varepsilon + v|^2) - W(1 - |u_\varepsilon|^2) \geq -W'(1 - |u_\varepsilon|^2) (|u_\varepsilon + v|^2 - |u_\varepsilon|^2).
\]

The last two relations imply that

\[
E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) \geq \int_{B^N} \left[ \nabla u_\varepsilon \cdot \nabla v - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2) u_\varepsilon \cdot v \right] dx \\
+ \int_{B^N} \left[ \frac{1}{2} |\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - f_\varepsilon^2)|v|^2 \right] dx.
\]

Moreover, by (2), we obtain

\[
E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) \geq \int_{B^N} \left[ \frac{1}{2} |\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - f_\varepsilon^2)|v|^2 \right] dx =: \frac{1}{2} F_\varepsilon(v) \tag{5}
\]

for all \( v \in H^1_0(B^N; \mathbb{R}^M) \).

Step 2: A rewriting of \( F_\varepsilon(v) \) using the decomposition \( v = f_\varepsilon w \) for every scalar test function \( v \in C^\infty_c(B^N \setminus \{0\}; \mathbb{R}) \). We consider the operator

\[
L_\varepsilon := \frac{1}{2} \nabla L^2 F_\varepsilon = -\Delta - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2).
\]

Using the decomposition

\[
v = f_\varepsilon w
\]

for the scalar function \( v \in C^\infty_c(B^N \setminus \{0\}; \mathbb{R}) \), we have (see e.g. [5, Lemma A.1]):

\[
F_\varepsilon(v) = \int_{B^N} L_\varepsilon v \cdot v dx = \int_{B^N} w^2 L_\varepsilon f_\varepsilon \cdot f_\varepsilon dx + \int_{B^N} f_\varepsilon^2 |\nabla w|^2 dx \\
= \int_{B^N} f_\varepsilon^2 \left( |\nabla w|^2 - \frac{N - 1}{r^2} w^2 \right) dx,
\]

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because (1) yields \( L_\varepsilon f_\varepsilon \cdot f_\varepsilon = -\frac{N-1}{r} f_\varepsilon^2 \) in \( B^N \).

**Step 3:** We prove that \( F_\varepsilon(v) \geq 0 \) for every scalar test function \( v \in C^\infty_c(B^N \setminus \{0\}; \mathbb{R}) \). Within the notation \( v = f_\varepsilon w \) of Step 2 with \( v, w \in C^\infty_c(B^N \setminus \{0\}; \mathbb{R}) \), we use the decomposition

\[
w = \varphi g
\]

with \( \varphi = |x|^{-\frac{N-2}{2}} \) being the first eigenfunction of the Hardy's operator \( -\Delta - \frac{(N-2)^2}{4|x|^2} \) in \( \mathbb{R}^N \setminus \{0\} \) and \( g \in C^\infty_c(B^N \setminus \{0\}; \mathbb{R}) \). We compute

\[
|\nabla w|^2 = |\nabla \varphi|^2 g^2 + |\nabla g|^2 \varphi^2 + \frac{1}{2} \nabla(\varphi^2) \cdot \nabla(g^2).
\]

As \( |\nabla \varphi|^2 = \frac{(N-2)^2}{4|x|^2} \varphi^2 \) and \( \varphi^2 \) is harmonic in \( B^N \setminus \{0\} \), integration by parts yields

\[
F_\varepsilon(v) = \int_{B^N} f_\varepsilon^2 \left( |\nabla g|^2 \varphi^2 + \frac{(N-2)^2}{4r^2} \varphi^2 g^2 - \frac{N-1}{r^2} \varphi^2 g^2 \right) dx - \frac{1}{2} \int_{B^N} \nabla(\varphi^2) \cdot \nabla(f_\varepsilon^2) g^2 dx
\]

\[
\geq \int_{B^N} f_\varepsilon^2 |\nabla g|^2 \varphi^2 dx + \left( \frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{f_\varepsilon^2}{r^2} \varphi^2 g^2 dx
\]

\[
\geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} dx \geq 0,
\]

where we have used \( N \geq 7 \) and \( \frac{1}{2} \nabla(\varphi^2) \cdot \nabla(f_\varepsilon^2) = 2\varphi \varphi' f_\varepsilon f'_\varepsilon \leq 0 \) in \( B^N \setminus \{0\} \).

**Step 4:** We prove that \( F_\varepsilon(v) \geq 0 \) for every \( v \in H^1_0(B^N; \mathbb{R}^M) \) meaning that \( u_\varepsilon \) is a global minimizer of \( E_\varepsilon \) over \( \mathcal{A} \); moreover, \( F_\varepsilon(v) = 0 \) if and only if \( v = 0 \). Let \( v \in H^1_0(B^N; \mathbb{R}^M) \). As a point has zero \( H^1 \) capacity in \( \mathbb{R}^N \), a standard density argument implies the existence of a sequence \( v_k \in C^\infty_c(B^N \setminus \{0\}; \mathbb{R}^M) \) such that \( v_k \to v \) in \( H^1(B^N, \mathbb{R}^M) \) and a.e. in \( B^N \). On the one hand, by definition (5) of \( F_\varepsilon \), since \( W'(1-f_\varepsilon^2) \in L^\infty \), we deduce that \( F_\varepsilon(v_k) \to F_\varepsilon(v) \) as \( k \to \infty \). On the other hand, by (5) and Fatou's lemma, we deduce

\[
\liminf_{k \to \infty} F_\varepsilon(v_k) \geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \liminf_{k \to \infty} \int_{B^N} \frac{v_k^2}{r^2} dx
\]

\[
\geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} dx.
\]

Therefore, we conclude that

\[
F_\varepsilon(v) \geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} dx \geq 0, \quad \forall v \in H^1_0(B^N; \mathbb{R}^M),
\]

implying by (5) that \( u_\varepsilon \) is a minimizer of \( E_\varepsilon \) over \( \mathcal{A} \). Moreover, \( F_\varepsilon(v) = 0 \) if and only if \( v = 0 \).

**Step 5:** Conclusion. We have shown that \( u_\varepsilon \) is a global minimizer. Assume that \( \tilde{u}_\varepsilon \) is another global minimizer of \( E_\varepsilon \) over \( \mathcal{A} \). If \( v := \tilde{u}_\varepsilon - u_\varepsilon \), then \( v \in H^1_0(B^N; \mathbb{R}^M) \) and by Steps 1 and 4, we have that \( 0 = E_\varepsilon(\tilde{u}_\varepsilon) - E_\varepsilon(u_\varepsilon) \geq F_\varepsilon(v) \geq 0 \), which yields \( F_\varepsilon(v) = 0 \). Step 4 implies that \( v = 0 \), i.e., \( \tilde{u}_\varepsilon = u_\varepsilon \). \( \Box \)
Remark 3. Recall that in the case \( M \geq N \geq 7 \), Jäger and Kaul \([7]\) proved the uniqueness of global minimizer for harmonic map problem

\[
\min_{u \in \mathcal{A}} \int_{B^N} |\nabla u|^2 \, dx,
\]

where \( \mathcal{A}_* = \{ u \in H^1(B^N; S^{M-1}) : u(x) = x \text{ on } \partial B^N = S^{N-1} \subset S^{M-1} \} \). This can also be seen by the method above as observed in our earlier paper \([6]\). We give the argument here for readers’ convenience: Take a perturbation \( v \in H^1_0(B^N, \mathbb{R}^M) \) of the harmonic map \( u_*(x) = \frac{x}{|x|} \) such that \(|u_*(x) + v(x)| = 1 \) a.e. in \( B^N \). Then, by \([6, \text{ Proof of Theorem 5.1}]\),

\[
\int_{B^N} |\nabla (u_* + v)|^2 - |\nabla u_*|^2 \, dx = \int_{B^N} |\nabla v|^2 - |\nabla u_*|^2 |v|^2 \, dx = \int_{B^N} |\nabla v|^2 -(N-1) \frac{|v|^2}{|x|^2} \, dx.
\]

Using Hardy’s inequality in dimension \( N \) we arrive at

\[
\int_{B^N} |\nabla (u_* + v)|^2 - |\nabla u_*|^2 \, dx \geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{|v|^2}{|x|^2} \, dx.
\]

The result follows since \( N \geq 7 \).

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