Desroches, MF., Krauskopf, B., & Osinga, HM. (2007). *The geometry of slow manifolds near a folded node.*

Early version, also known as pre-print

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THE GEOMETRY OF SLOW MANIFOLDS NEAR A FOLDED NODE
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Abstract.
This paper is concerned with the geometry of slow manifolds of a dynamical system with two slow and one fast variable. Specifically, we study the dynamics near a folded node singularity, which is known to give rise to so-called canard solutions. Geometrically, canards are intersection curves of two-dimensional attracting and repelling slow manifolds, and they are a key element of slow-fast dynamics. For example, canard solutions are associated with mixed-mode oscillations, where they organize regions with different numbers of small oscillations.

We perform a numerical study of the geometry of two-dimensional slow manifolds in the normal form of a folded node in \( \mathbb{R}^3 \). Namely, we view the part of a slow manifold that is of interest as a one-parameter family of orbit segments up to a suitable cross-section. Hence, it is the solution of a two-point boundary value problem, which we solve by numerical continuation with the package AUTO. The computed family of orbit segments is used to obtain a mesh representation of the manifold as a surface. With this approach we show how the attracting and repelling slow manifolds change in dependence on the eigenvalue ratio \( \mu \) of the reduced flow. At \( \mu = 1 \) two primary canards bifurcate and secondary canards are created at odd integer values of \( \mu \). We compute 24 secondary canards to investigate how they spiral more and more around one of the primary canards. The first twelve secondary canards are continued in \( \mu \) to obtain a numerical bifurcation diagram.

Key words. singular perturbations, canard solution, boundary value problem, slow-fast systems, invariant manifolds

AMS subject classifications. 34E15, 34C30, 37C10, 65L10

1. Introduction. Multiple time scale systems are characterized by the property that certain variables evolve on vastly different time scales, which means that the systems may display dynamics that is composed of slow and fast elements. The occurrence of different time scales is quite natural in many applications, including chemical reaction dynamics [9, 32, 34, 36, 38], cell modelling [14, 42, 43], electronic circuits [12, 46, 47], and laser dynamics [15, 20]. The first example of slow-fast dynamics was discovered by Van der Pol [46, 47] in the 1920s. He considered an electrical circuit with a triode valve where the current is a cubic function of the voltage. The mathematical model of this circuit — known today as the Van der Pol equations — has the form of a planar vector field model of an oscillator with a cubic nonlinear damping term. The Van der Pol equations show sustained oscillations, which are harmonic for small damping parameter but very non-harmonic when the damping parameter is large. In the latter case the periodic solution is composed of slow motion that closely follows attracting segments of the underlying cubic curve (which forms one of the nullclines), followed by fast jumps as the trajectory reaches either of the two folds of this curve. At the jumps one of the variables barely changes until the trajectory reaches another attracting segment of cubic curve. Van der Pol called these periodic solutions relaxation oscillations. Models showing relaxation oscillations quite similar to that of the Van der Pol equations have been found in other application areas. A well known example is the FitzHugh-Nagumo system, which also has two time scales and a cubic nonlinearity. It was derived independently by FitzHugh [23] and Nagumo [37] as a simplified planar version of the famous Hodgkin-Huxley equations for the action potential of the giant axon of a squid in terms of transmembrane currents [30].

At the end of the 1970s, Benoît et al. [5] found and analyzed even more unusual periodic solutions in the Van der Pol system, which they called canards. A canard orbit has the spe-
cial property that it follows at least one unstable segment of the underlying cubic nullcline. In other words, the trajectory does not jump at the respective fold. Canard orbits account for a sudden increase in the amplitude of the attracting periodic orbit in the transition from harmonic oscillations to relaxation oscillations. This change is known as a canard explosion, and it occurs in an exponentially small interval of the damping parameter. For this reason, the associated canards are extremely difficult to observe in two-dimensional slow-fast systems.

The situation is very different in three-dimensional slow-fast systems, where canards may occur in much larger regions of system parameters. They have been found in numerous slow-fast systems. For example, canards explain sudden changes in amplitude and period of oscillatory behavior in chemical reactions [9, 32, 34, 36, 38]; they organize the dynamics in models of coupled neurons and also play an important role in intracellular activities [14, 42, 43]; canards have been studied for their role in diffusion-induced instabilities [9, 40]; and Guckenheimer et al. [6, 28] recently performed an extensive study of a reduced hybrid model of the forced Van der Pol system, which revealed relaxation oscillations and canard orbits of different types. A related phenomenon in slow-fast systems are mixed-mode oscillations, which consists of large-amplitude excursions followed by small-amplitude motions that are typically of (relatively) high frequency. This type of oscillations have been found in chemical and biological systems, and the connection between mixed-mode oscillations and canards has been clarified recently [8, 34, 43, 48]; see also the special issue [7].

For the theoretical study of canards in three-dimensional phase space one considers a dynamical system with two slow and one fast variable of the form

\[
\begin{align*}
\dot{u}_1 &= g_1(u_1, u_2, v, \varepsilon), \\
\dot{u}_2 &= g_2(u_1, u_2, v, \varepsilon), \\
\varepsilon \dot{v} &= f(u_1, u_2, v, \varepsilon).
\end{align*}
\]

(1.1)

Here \(g_1, g_2,\) and \(f\) are sufficiently smooth functions, and \(\varepsilon > 0\) is a small parameter that separates the different time scales. Since \(\varepsilon\) is small, the variables \(u_1\) and \(u_2\) move on a slower time scale than the fast variable \(v\). The equivalent of the cubic nullcline of the Van der Pol equations is now the surface — called the critical manifold — that is given as the \(v\)-nullcline of (1.1). The critical manifold has repelling parts and attracting parts, which meet along one-dimensional fold curves. As for the Van der Pol equations, the slow dynamics takes place close to the critical manifold. When a fold is reached two things can happen. The trajectory may jump at the fold curve towards another attracting sheet of the critical manifold. If a global return mechanism is present, e.g., the critical manifold is S-shaped with two fold curves, such jumps give rise to classical relaxation oscillations [35, 45]. The other possibility is that the trajectory is a canard solution that does not jump at the fold curve, but instead stays near the repelling part of the critical manifold for a certain amount of time.

Since the beginning of the 1980s, when canard solutions were first discovered, different analytical techniques have been applied to their study. Initially, non-standard analysis [3, 4, 5] and matched asymptotic expansions [35, 18] were used, but more recently canards have been studied with tools from geometric singular perturbation theory [22, 31]. Underlying this geometric approach is the realization that the study of canard solutions comes down to understanding the dynamics of the system near folds of the critical manifold. Of specific interest here are (isolated) points on fold curves where the direction of flow changes from pointing towards the fold to pointing away from the fold. These points, called folded singularities, are key to the understanding of canard solutions.

We are concerned here with the case of a folded node, which is a type of folded singularity that has been identified as an organizing center for the creation of canard solutions. Figure 1 shows the dynamics for \(\varepsilon = 0\) near a folded node on a regular fold curve \(F\) along
which the attracting sheet \( S^a \) and the repelling sheet \( S^r \) of the critical manifold join. At the folded node (black dot), the direction of the flow on the critical manifold changes. This allows for the existence of canard solutions on the critical manifold that cross \( F \) at the folded singularity with nonzero speed and then follow the repelling sheet \( S^r \). These canards for \( \epsilon = 0 \), which are referred to as singular canards, occur in an entire region that is bounded by the repelling part of the fold curve \( F \) and the special canard solution \( \tilde{\gamma}_s \) in Fig. 1. The main question is now what can be said about the dynamics near a folded node when \( \epsilon \neq 0 \). According to a well-known result by Fenichel [21, 22], away from the fold curve \( F \) the attracting and repelling sheets give rise for \( \epsilon \neq 0 \) to an attracting slow manifold \( S^a_{\epsilon} \) and a repelling slow manifold \( S^r_{\epsilon} \), respectively. Importantly, the two surfaces \( S^a_{\epsilon} \) and \( S^r_{\epsilon} \) do not connect along the fold curve for \( \epsilon \neq 0 \), but rather intersect, generically transversely, in one-dimensional solution curves. Hence, these curves are referred to as maximal canards, as they stay close to the repelling part of the critical manifold for a certain amount of time. In other words, finding the structure of canard solutions near the folded node is equivalent to understanding the geometry of the two-dimensional surfaces \( S^a_{\epsilon} \) and \( S^r_{\epsilon} \).

Canards near a folded node are best analyzed in a normal form setting. Normal forms for all types of folded singularities, including the folded node, were derived by Il’yashenko [2, part I, chapter 4]. Benoît [4] completely analyzed the case of a folded saddle and also considered the case of a folded node. He proved the existence of two maximal canards under a non-resonance condition with tools of nonstandard analysis and found secondary canards with spiralling behavior by numerical integration. Szmolyan and Wechselberger [44] consider all cases of folded singularities. For the folded node they proved the existence of primary canards by using geometric singular perturbation theory in combination with blow-up transformations. Guckenheimer and Haiduc [27] proved that there are infinite many secondary canards near a folded node. Wechselberger [48] studied how secondary canards bifurcate
from the weak primary canard and he sketched the underlying bifurcation structure. Both
Guckenheimer and Haiduc [27] and Wechselberger [48] find canards numerically by comput-
ing the one-dimensional intersection curves of the attracting and repelling slow manifolds in
a two-dimensional cross section with a shooting approach (integration from a line of initial
conditions far from the fold curve); the canards are then identified as crossings of the two
curves.

Our goal is to compute and visualize the slow manifolds as surfaces and the canards
as trajectories in \( \mathbb{R}^3 \) to obtain additional geometric insight into the dynamics near a folded
node. To this end, we consider the normal form as used in [48], which is given as the three-
dimensional vector field

\[
\begin{align*}
\dot{x} &= \frac{1}{2} \mu y - (\mu + 1) z, \\
\dot{y} &= 1, \\
\dot{z} &= x + z^2.
\end{align*}
\]

(1.2)

In (1.2) the folded node is at the origin and the parameter \( \mu \) is the ratio of the eigenvalues of
the Jacobian matrix of the so-called desingularized reduced flow evaluated at the folded node.
Note that, as a result of a parameter dependent blow-up procedure, (1.2) does not depend on
\( \varepsilon \). Nevertheless, this normal form describes how the attracting and repelling slow manifolds
intersect near a folded node. Specifically, for any \( \mu \) (1.2) has an attracting slow manifold
\( C^- \) and a repelling slow manifold \( C^+ \) that intersect in the maximal canards one wants to study;
see Sec. 2 for more information on the derivation and meaning of the normal form.

Specifically, we compute in this paper the slow manifolds \( C^- \) and \( C^+ \) of (1.2) as surfaces
in \( \mathbb{R}^3 \). This is achieved by representing a piece of interest of a slow manifold as a one-
parameter family of orbit segments that satisfy suitably chosen boundary conditions. The
resulting boundary value problem is solved with the continuation and collocation routines of
the package AUTO [13]. This setup is very flexible and accurate, because it is based on the
continuation of two-point boundary value problems [33]. It allows us to adjust the boundary
conditions to emphasize certain local features of the underlying dynamics in the vicinity of
the folded node. In particular, we are able to compute the slow manifolds past the folded
node and show their exponential growth on the other side of it. Our method allows for a very
precise visualization of the behavior of the slow manifolds in the vicinity of the folded node.
In particular, we are able to capture in detail the complexity of their intersections, that is, the
canards. For increasing \( \mu \), the two surfaces rotate more and more around the weak primary
canard (the perturbation of \( \hat{\gamma}_w \) in Fig. 1), which leads to the creation of secondary canards
at odd integer values of \( \mu \). Our method allows us to detect canards and to continue them as
solutions of a boundary value problem in the parameter \( \mu \). We compute up to 24 secondary
canards and show how they spiral around the weak primary canard. Furthermore, we continue
12 secondary canards in \( \mu \) to obtain a numerical bifurcation diagram.

Note that very few attempts have been made so far to produce accurate computations of
slow manifolds as surfaces. Milik et al. [34] visualize slow manifolds in the normal form
of a folded saddle-node; in this special case (of codimension one) the system decouples
into a one-parameter family of orbit segments that satisfy suitably chosen boundary conditions. The
resulting boundary value problem is solved with the continuation and collocation routines of
the package AUTO [13]. This setup is very flexible and accurate, because it is based on the
continuation of two-point boundary value problems [33]. It allows us to adjust the boundary
conditions to emphasize certain local features of the underlying dynamics in the vicinity of
the folded node. In particular, we are able to compute the slow manifolds past the folded
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solutions of a boundary value problem in the parameter \( \mu \). We compute up to 24 secondary
canards and show how they spiral around the weak primary canard. Furthermore, we continue
12 secondary canards in \( \mu \) to obtain a numerical bifurcation diagram.
and follow them in system parameters. More generally, computing invariant manifolds by
continuation and collocation as solution families subject to suitable boundary conditions is a
very flexible method [33], which is particularly well suited for slow-fast systems [19]. Our
method is not restricted to the normal form setting but can be used more widely, for example,
for the study of canards in the self-coupled FitzHugh-Nagumo system [10].

The outline of this paper is as follows. In the next section we present some technical
background material. We briefly review slow-fast systems in Sec. 2.1, explain the blow-up
method in Sec. 2.2, and then discuss some known properties of the normal form (1.2) in
Sec. 2.3. Section 3 explains in detail how to compute slow manifolds as collections of orbit
segments. In Sec. 4 we show how the slow manifolds change with \( \mu \), and Sec. 5 is devoted
to the detection and continuation of the secondary canards. In Sec. 6 we show that the geometry
of the slow manifolds does not change topologically when the normal form is perturbed. We
conclude with a summary and outlook in Sec. 7.

2. Background on the folded node. In this section we recall some basic facts about
singularly perturbed dynamical systems, folded singularities and the blow-up method to ana-
lyze them. We consider the three-dimensional normal form (1.2), as studied, for example, by
Benoît [3, 4], Guckenheimer and Haiduc [27], and Wechselberger [48]. Following [48] we
show how the normal form is derived from a generic three-dimensional system with a folded
node at the origin and present some useful properties of (1.2).

2.1. Slow-fast dynamical systems. The slow-time system (1.1) defines a vector field
using the slow time \( t \). An alternative way of writing (1.1) is to introduce the fast time
\( \tau = \frac{t}{\varepsilon} \), which gives the fast-time system

\[
\begin{align*}
\dot{u}_1' &= \varepsilon g_1(u_1, u_2, v, \varepsilon), \\
\dot{u}_2' &= \varepsilon g_2(u_1, u_2, v, \varepsilon), \\
v' &= f(u_1, u_2, v, \varepsilon),
\end{align*}
\]

(2.1)

where the prime indicates the derivative with respect to the fast time \( \tau \). This rescaling of
time is valid only for \( \varepsilon \neq 0 \) and does not modify the geometry of the trajectories. The main
question is whether it is possible to understand the dynamics for small \( \varepsilon > 0 \) by considering
the two limits of (1.1) and (2.1) given by \( \varepsilon = 0 \).

The limit of the slow-time system (1.1) for \( \varepsilon = 0 \) is known as the reduced system or slow
subsystem

\[
\begin{align*}
\dot{u}_1 &= g_1(u_1, u_2, v, 0), \\
\dot{u}_2 &= g_2(u_1, u_2, v, 0), \\
0 &= f(u_1, u_2, v, 0).
\end{align*}
\]

(2.2)

System (2.2) is a set of differential algebraic equations (DAE) on the slow time scale, namely,
two differential equations constrained by the algebraic equation \( f = 0 \). This condition defines
the critical manifold

\[
S := \{(u_1, u_2, v) \in \mathbb{R}^3 \mid f(u_1, u_2, v, 0) = 0\},
\]

(2.3)
on which the dynamics of the reduced system (2.2) takes place. We obtain differential equations
that describe the flow of the reduced system (2.2) on \( S \) as follows. Differentiating the
algebraic equation \( f = 0 \) with respect to time yields

\[
\begin{align*}
\ddot{u}_1 &= g_1(u_1, u_2, v, 0), \\
\ddot{u}_2 &= g_2(u_1, u_2, v, 0), \\
-f'_v \dot{v} &= f_{u_1} \dot{u}_1 + f_{u_2} \dot{u}_2.
\end{align*}
\]

(2.4)
We then rescale \((2.4)\) by \(-f_v\) to obtain the vector field
\[
\begin{align*}
\dot{u}_1 &= -f_v g_1 (u_1, u_2, v, 0), \\
\dot{u}_2 &= -f_v g_2 (u_1, u_2, v, 0), \\
\dot{v} &= f_{u_1} g_1 + f_{u_2} g_2,
\end{align*}
\] (2.5)
that generates a slow flow on \(S\).

The limit of the fast-time system \((2.1)\) for \(\varepsilon = 0\) is known as the layer system or fast subsystem
\[
\begin{align*}
u_1' &= 0, \\
u_2' &= 0, \\
v' &= f(u_1, u_2, v, 0).
\end{align*}
\] (2.6)
The variables \(u_1\) and \(u_2\) are constants in system \((2.6)\) and enter the equation for \(v\) as parameters. Hence, the layer system is a two-parameter family of differential equations on the fast time scale. The critical manifold \(S\) plays a key role in the layer system as well, namely, as a manifold of equilibria.

The idea of geometric singular perturbation theory \([22, 31]\) is to understand the dynamics of system \((1.1)\) with \(\varepsilon > 0\) sufficiently small by splitting the motion into its fast and slow components. The fast dynamics of \((1.1)\) is described by the layer system \((2.6)\), meaning that trajectories behave like solutions of \((2.6)\) until they get close to \(S\). On the slow time scale, solutions are well approximated by the reduced system \((2.2)\); in particular, trajectories remain confined to an \(\varepsilon\)-neighborhood of \(S\). The overall dynamics can indeed be understood in this way if the critical manifold \(S\) is normally hyperbolic. This means that the dynamics in the normal direction to the manifold dominates the dynamics in the tangent direction \([29]\).

Due to results by Fenichel \([21, 22]\), a normally hyperbolic critical manifold \(S\) persists under small perturbations as a nearby normally hyperbolic invariant manifold \(S\), for the singularly perturbed system \((1.1)\).

In the vicinity of points where normal hyperbolicity fails, the singularly perturbed problem can give rise to very complex dynamics. In order to study what happens when \(S\) is not normally hyperbolic, we consider the projection of \(S\) onto the \((u_1, u_2)\)-plane of slow variables. The critical manifold \(S\) consists of regular points where \(f_v \neq 0\) and critical points of the projection where \(f_v = 0\). According to singularity theory \([1]\), regular points are generic and they correspond to points where \(S\) is normally hyperbolic. Moreover, a generic critical point is a fold point, and together the fold points form a codimension-one submanifold \(F\) of \(S\). Along \(F\) two sheets of \(S\) meet. In \(\mathbb{R}^3\) there may be cusp points (degenerate folds), but they are generically isolated. We focus here on fold points and their influence on the dynamics of system \((1.1)\).

Specifically, we consider a critical manifold \(S\) with (locally) a nonempty fold curve that does not contain cusp points. Therefore, \(S\) can be written as \(S = S^a \cup F \cup S^r\), where \(S^a\) and \(S^r\) refer to the attracting and repelling sheets of \(S\), respectively, that meet at \(F\); formally
\[
\begin{align*}
S^a &= \{(u_1, u_2, v) \in S \mid f_v (u_1, u_2, v) < 0\}, \\
F &= \{(u_1, u_2, v) \in S \mid f_v (u_1, u_2, v) = 0\}, \\
S^r &= \{(u_1, u_2, v) \in S \mid f_v (u_1, u_2, v) > 0\}.
\end{align*}
\] (2.7)
System \((2.4)\) is singular along \(F\) while the desingularized system \((2.5)\) governs the dynamics in the vicinity of the critical manifold \(S\). Note that the rescaling by \(-f_v\) that achieves this desingularization changes the direction of time where \(f_v > 0\), that is, on the repelling sheet \(S^r\) of the critical manifold \(S\).
Roughly speaking, the original system (1.1) is governed by system (2.5) when it evolves almost on the attracting sheet $S^a$. The situation changes when a trajectory reaches the fold curve $F$, that is, when $\dot{v}$ becomes zero. In the generic situation, that is, if $\dot{v} \neq 0$ the prominent dynamics switches to the fast dynamics (2.6) and the trajectory escapes from $S$ along a fast fiber parallel to the $v$-axis. The condition $\dot{v} = f_{u_1} g_1 + f_{u_2} v_2 \neq 0$ is called the normal switching condition [35] and means geometrically that the reduced flow projected onto the $(u_1, u_2)$-plane is not tangent to the fold curve $F$. The point on the fold curve $F$ where the change of dynamics occurs is called a jump point. If $S$ is S-shaped, that is, there are two separate attracting sheets and two fold curves connected to a repelling sheet, then the presence of jump points typically leads to the existence of relaxation oscillations [35, 45], that is, periodic solutions that involve both fast and slow segments; a detailed discussion of this phenomenon is given in [28] for the forced Van der Pol system.

If $f_{u_1} g_1 + f_{u_2} v_2 = 0$ then (2.5) has an equilibrium on the fold curve $F$ and there is no jump. Such an equilibrium is called a folded singularity. According to the topological type of the singularity as an equilibrium of system (2.6), one has generically folded nodes, folded saddles, and folded foci. Locally, the dynamics near a folded singularity can be described by normal forms. Normal forms for planar or three-dimensional slow-fast systems were first derived by Il’yashenko [2]. These normal forms were subsequently used by Benoît [4], Shmolyan and Wechselberger [44], and Guckenheimer and Haiduc [27], among others, to obtain analytical results concerning the possible bifurcations near such folded singularities.

The case of a folded-node singularity is sketched in Fig. 1 and several trajectories of the slow flow associated with the normal form (1.2) are shown. Notice the change of direction of the slow flow across the fold curve $F$. As a result, some initial conditions on $S^a$ are attracted to regular points on $F$, which leads to a jump. However, an entire wedge exists, bounded by the singular canard $\gamma_s$ and a half-line on $F$ that ends at the folded node, where trajectories converge to the folded node and pass through to follow the repelling sheet $S^r$. This wedge is called the funnel region [48] and it is responsible for the generic existence of canard solutions in systems with $\varepsilon \neq 0$. Note that Fenichel theory cannot be invoked to describe the geometry of the slow manifolds in this region of the phase space simply by using the unperturbed limiting problems (2.2) and (2.6). Instead, blow-up methods are used to obtain the dynamics through the funnel region.

2.2. Blow-up of the folded node. One can apply the method of blow-up in the setting of geometric singular perturbation theory [16, 41]. The general idea is to rescale the variables of the original problem together with the singular parameter $\varepsilon$. In this way, one can transform a singularly perturbed system into a regularly perturbed system that is defined on a higher-dimensional phase space.

The slow manifolds $S^a_\varepsilon$ and $S^r_\varepsilon$ correspond to $\varepsilon$-leaves of three-dimensional attracting and repelling center manifolds $M_a$ and $M_r$ of the extended system

$$
\begin{align*}
\dot{u}_1' &= \varepsilon g_1(u_1, u_2, v, \varepsilon), \\
\dot{u}_2' &= \varepsilon g_2(u_1, u_2, v, \varepsilon), \\
\dot{v}' &= f(u_1, u_2, v, \varepsilon), \\
\varepsilon' &= 0.
\end{align*}
$$

(2.8)

The linearization of the extended system (2.8) has all eigenvalues equal to zero at the folded node. Hence, one cannot apply center manifold theory at points on $F$ and describe the behavior of the slow manifolds in a neighborhood of $F$. A good way of overcoming this difficulty is to apply a blow-up transformation at the folded node, which is a degenerate singularity of system (2.8). Roughly speaking, the blow-up method is a well-chosen coordinate transformation that desingularizes such a degenerate singularity. It was originally developed for planar
vector fields [17], but has been adapted to the case of three-dimensional singularly perturbed systems [44, 48]. The change of coordinates transforms the degenerate singularity at the origin into a sphere $S^3$ that contains points with (at least) one non-zero eigenvalue. Then the general methods of dynamical systems are applicable, in particular, center manifold theory; see [44] for a detailed exposition of the blow-up in this specific context.

In the case of a three-dimensional singularly perturbed system with a folded node at the origin, the desingularizing transformation is defined by

$$u_1 = \rho^2 x, \quad u_2 = \rho y, \quad v = \rho z, \quad \varepsilon = \rho^2 \bar{\varepsilon},$$

where $(x, y, z, \varepsilon) \in S^3$ and $\rho \in [0, \rho_0]$ is a new radial parameter. As is explained in [48], two charts $\kappa_1$ and $\kappa_2$ of the sphere $S^3$ suffice to understand the entire extended phase space. These charts yield so-called directional rescalings obtained by setting one coordinate in $S^3$ equal to $\pm 1$ and desingularizing the vector field in both charts, where chart $\kappa_1$ is defined by $x = -1$ and chart $\kappa_2$ by $\xi = 1$.

To investigate the possible intersections between the slow manifolds, that is, the existence of maximal canards, one only needs to look at chart $\kappa_2$, which describes the situation on the blown-up locus. As a main result, the blow-up extends the normal hyperbolicity of the slow manifolds $S^a_\varepsilon$ and $S^b_\varepsilon$ to the blown-up sphere. After desingularization, the system in $\kappa_2$ is given by

$$\begin{cases}
\dot{x} = \frac{1}{2} \mu y - (\mu + 1) z + O(\rho), \\
\dot{y} = 1, \\
\dot{z} = x + z^2 + O(\rho).
\end{cases} \tag{2.9}$$

By definition of the directional rescaling in chart $\kappa_2$ we have $\rho = \sqrt{\varepsilon}$. The key idea is now to study system (2.9) for $\rho = 0$, which is the normal form (1.2).

We now explain briefly why it is sufficient to consider system (1.2) to obtain all the information concerning the original singularly perturbed system (1.1). To avoid technical details about the blow-up, which can be found in [44, 48], we follow here the intuitive point of view (and the underlying notations) developed in [8]. By Fenichel’s theorems, for any fixed $\varepsilon > 0$ sufficiently small, the slow manifolds $S^a_\varepsilon$ and $S^b_\varepsilon$ exist outside a neighborhood of the fold curve $F$, that is, for $u_1 < -\delta$ (where $\delta > 0$ depends on $\varepsilon$). It is possible to extend $S^a_\varepsilon$ and $S^b_\varepsilon$ up to section $\sigma' := \{u_1 = -\delta' \varepsilon\}$ ($\delta' > 0$ such that $-\delta < -\delta' \varepsilon$) by considering orbit segments starting in section $\sigma := \{u_1 = -\delta\}$ and ending in section $\sigma'$. Note that these cross-sections are well defined as long as the flow is not tangent to the $u_2$-axis, which is satisfied in the vicinity of a folded node. Hence, $\sigma$ and $\sigma'$ are defined on intervals $I_{u_2}$ and $I'_{u_2}$ of $u_2$, respectively. Then we define, still following [8], the sets $S^a_\rho$ and $S^b_\rho$ as collections of orbit segments starting in $I_{u_2}$ and ending in $I'_{u_2}$. One can prove [48] that $S^a_\rho$ and $S^b_\rho$ are smooth perturbations of the critical manifold $S$ outside a neighborhood of the fold curve $F$ of size $(O(\varepsilon), O(\rho), O(\varepsilon))$ in $(u_1, u_2, v)$. The great benefit of the blow-up transformation in chart $\kappa_2$ is that for $\rho \to 0$ the slow manifolds $S^a_\rho$ and $S^b_\rho$ tend to invariant sets $C^-$ and $C^+$ of (1.2), respectively. This is possible because in $(x, y, z)$, the section $\sigma$ is given by $x = -\delta/\varepsilon$, hence by $x = -\delta/\rho^2$, and $\sigma'$ by $x = -\delta'$. Therefore, $S^a_\mu$ and $S^b_\mu$ can be seen as smooth $O(\rho)$ perturbations of $C^+$ and $C^-$, respectively.

In other words, we can study and understand the dynamics of $C^\pm$ up to the folded node; in particular, we can investigate their transverse intersections and deduce information about the singularly perturbed system. Indeed, by the Implicit Function Theorem, transverse intersections of $C^-$ and $C^+$ persist under small perturbations $\rho > 0$, as system (2.9) is a regularly perturbed system with respect to $\rho$. Consequently, the geometry of the manifolds $C^\pm$, which
we refer to as the repelling and attracting slow manifolds from now on, contains all the relevant information regarding the corresponding “true” slow manifolds \( \mathcal{S}_c^\pm \) and \( \mathcal{S}_s^\pm \) of system \((1.1)\), respectively. Indeed, the study of the geometry of \( C^\pm \) is the main topic of this paper.

2.3. Properties of the normal form. The normal form \((1.2)\) was already studied, in an equivalent form, by Benoît in \([4]\) and Guckenheimer and Haiduc in \([27]\). By looking at the equation for \( \dot{z} \) in \((1.2)\), it is clear that the critical manifold associated with this normal form is the parabolic cylinder

\[
S := \{(x, y, z) \in \mathbb{R}^3 \mid x + z^2 = 0\},
\]

that is folded along the \( y \)-axis, that is,

\[
F := \{(0, y, 0) \in \mathbb{R}^3\}.
\]

This is exactly the picture in Fig. 1. An important property of the normal form \((1.2)\) is its invariance under the time-reversing symmetry \((x, y, z, t) \mapsto (x, -y, -z, -t)\). Therefore, it suffices to concentrate on the attracting slow manifold \( C^-\); the repelling slow manifold \( C^+\) is given by the symmetry. A main advantage of \((1.2)\) is that it possesses two explicit canard solutions of algebraic growth, \( \gamma_s \) and \( \gamma_w \), given as

\[
\begin{align*}
\gamma_s(t) &= \left( -\frac{\mu^2}{4}t^2 + \frac{\mu}{2}, \, t, \, \frac{\mu}{2}t \right), \\
\gamma_w(t) &= \left( -\frac{1}{4}t^2 + \frac{1}{2}, \, t, \, \frac{1}{2}t \right).
\end{align*}
\]

We refer to \( \gamma_s \) as the strong canard and to \( \gamma_w \) as the weak canard, because they correspond to the strong and the weak eigendirection of the linearization of system \((1.2)\) at the folded node, respectively. Note that the geometry of the critical manifolds for \( \mu > 1 \) and for \( 1/\mu \in [0, 1] \) is topologically the same (where the roles of \( \gamma_s \) and \( \gamma_w \) are interchanged); recall that \( \mu \) denotes the ratio of the eigenvalues of system \((2.5)\) projected onto the \((y, z)\)-plane and linearized at the origin. Therefore, we consider here the changes of the slow manifolds \( C^\pm \) as a function of \( \mu \) only for \( \mu \geq 1 \).

It has been proved that \( C^\pm \) intersect transversely along \( \gamma_s \) and \( \gamma_w \) when \( \mu \) is not an integer \([4, 44]\). For integer values of \( \mu \) the manifolds \( C^+ \) and \( C^- \) intersect transversely along \( \gamma_s \) and tangentially along \( \gamma_w \). A new canard is created from the weak canard \( \gamma_w \) at every odd integer value of the parameter \( \mu \) (for \( \mu \geq 3 \)) \([48]\). The bifurcating canards \( \eta_i \) are called secondary canards. It was analytically proved in \([44]\) that the slow manifolds \( C^- \) and \( C^+ \) spiral \( |\mu| \) times around the weak canard \( \gamma_w \) (here \(|q|\) denote the integer part of the real number \( q \)). Due to the time-reversing symmetry of the normal form, this implies that \( C^- \) and \( C^+ \) make \( \lfloor \frac{1}{2\mu} \rfloor \) full rotations around each other. Each one of these full rotations ends in a transverse intersection along a secondary canard. Hence, there are \( \lfloor \frac{1}{2\mu} \rfloor \) secondary canards that successively make one additional complete revolution around \( \gamma_w \). More precisely, \( \eta_i \) makes \( i + \frac{1}{2} \) rotations around \( \gamma_w \); see already Sec. 5. Geometrically, we can think of the strong canard \( \gamma_s \) as the secondary canard \( \eta_0 \), because \( \gamma_s \) makes a half rotation around \( \gamma_w \). Due to the symmetry of \((1.1)\), the intersection points of the canard solutions with the section \( \Sigma_0 = \{y = 0\} \) lie on the \( x \)-axis, and their order is such that \( \gamma_w \) is always located between \( \eta_{\lfloor \frac{1}{2\mu} \rfloor} \) and \( \eta_{\lfloor \frac{1}{2\mu} \rfloor - 1} \).

3. Computing slow manifolds. To get an insight into the geometry of the attracting and repelling slow manifolds \( C^\pm \) we developed a computational technique for their numerical approximation and visualization. The main underlying idea of our approach is that one
can compute (a finite part of) a two-dimensional invariant manifold of a system of ordinary
differential equations as a collection of orbit segments by numerical continuation of a one-
parameter family of two-point boundary value problems. This idea can be applied in a wide
variety of contexts [33]. Below we explain how we use this general idea to compute $C^\pm$
for the normal form; see [10] for details on how to compute $C^\pm$ for systems not in normal form.

3.1. Slow manifolds as collections of orbit segments. As is common in numerical
continuation, we consider a vector field of the form

$$\dot{u} = T g(u, \lambda),$$

where $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is sufficiently smooth and $T \in \mathbb{R}$ and $\lambda \in \mathbb{R}^p$ are parameters.
The parameter $T$ is the total integration time. It appears explicitly as a free parameter on
the right-hand side of (3.1), so that an orbit segment $u(t)$ is always represented over the interval
$[0, 1]$. By imposing suitable boundary conditions on solutions of (3.1) we can characterize
any $k$-dimensional invariant submanifold in $\mathbb{R}^n \times \mathbb{R}^p$. To be more precise, we consider the
boundary conditions

$$\begin{cases}
  u(0) \in L, \\
  u(1) \in \Sigma,
\end{cases}$$

where $L$ is a one-dimensional submanifold and $\Sigma$ a codimension-one submanifold of $\mathbb{R}^n$.
One needs $(n - 1)$ boundary conditions to restrict $u(0)$ to the curve $L$ and one boundary
condition to restrict $u(1)$ to the $(n - 1)$-dimensional manifold $\Sigma$. Hence, the total number of
boundary conditions in (3.2) is $n$. That is, with $T$ as a free parameter Eqs. (3.1)–(3.2) define
a one-parameter family of well-posed two-point boundary value problems that represent orbit
segments starting at $L$ and ending in the section $\Sigma$; see, for example, [11]. The family is
parametrized by the position of $u(0)$ on $L$, and $T$ is the integration time to reach $\Sigma$ from
$L$. Depending on the choice of $L$ and $\Sigma$, this general setup can be used to compute different
types of dynamical objects, including two-dimensional invariant manifolds [33] and their
one-dimensional intersection curves with the section $\Sigma$ [19].

In the present setting the phase space is three dimensional and our goal is to find appro-
priate definitions for $L$ and $\Sigma$ so that (3.1)–(3.2) define $C^\pm$ as surfaces in a region of interest.
Since $C^+$ can be found from $C^-$ by symmetry in (1.2), we only explain here the computation
of $C^-$. The family of orbit segments obtained by continuation of (3.1)–(3.2) defines (part
of) the two-dimensional manifold $C^-$, provided the one-dimensional submanifold $L$ satisfies
$L \subset C^-$ [22, 48]. Since we do not know $C^-$, we choose $L$ on the attracting sheet $S^a$ of
the critical manifold $S$ sufficiently far away from the fold curve $F$, that is in a region where $S^a$
is a good approximation of $C^-$. Then the boundary value problem defines an approximation of
$C^-$. From (1.2) and (2.10) we know that

$$S^a = \{(x, y, z) \in \mathbb{R}^3 \mid x + z^2 = 0, \; z < 0\}.$$

Since the fold curve $F$ is the $y$-axis, we define

$$L = L^\xi := \{(-\xi, s, -\sqrt{\xi}) \mid s \in \mathbb{R}\},$$

which is the line on $S^a$ with $x = -\xi$ parallel to $F$ at ‘distance’ $\xi > 0$ from $F$. The inter-
esting dynamics takes place near the folded node on $F$, so a suitable choice for $\Sigma$ is a plane
transverse to $F$ near the origin. We define

$$\Sigma = \Sigma_\alpha := \{y = \alpha\},$$

where $\alpha \geq 0$. The two-point boundary value problem (3.1)–(3.2) for the choices (3.3) and
(3.4) defines a one-parameter family of orbit segments that lie on $C^-$ in good approximation,
provided $\xi$ is large enough.
3.2. Finding a first orbit segment on the slow manifold. To start the continuation we must provide a first orbit segment that solves (3.1) subject to the boundary conditions (3.2). For the normal form (1.2) two explicit canard solutions are known, which we can use as seed solutions. Note that neither one of the two explicit solutions has a point in common with the attracting sheet \( S^a \) of the critical manifold, which means that it is not possible to choose \( \xi \) such that the two explicit canard solutions contain segments that start on \( L^- \xi \) and solve the boundary value problem (3.1)-(3.2). However, we only use the explicit solutions as a seed for the Newton iteration, that is, we must find a suitable solution segment such that Newton’s method converges to a solution of (3.1)-(3.2).

To be concrete, we start from the strong canard \( \gamma_s \) given in (2.12) and consider the initial orbit segment

\[
u(t) = \gamma_s(tT + t_0), \quad 0 \leq t \leq 1,
\]

(3.5)

for some start time \( t_0 \) and total integration time \( T \). We choose \( t_0 < 0 \) such that the \( x \)-coordinate of \( \gamma_s(t_0) \) is equal to \( -\xi \), that is,

\[
-\frac{\mu^2}{4} t_0^2 + \frac{\mu}{2} = -\xi.
\]

In order to satisfy the second boundary condition \( u(1) \in \Sigma_\alpha \), we need

\[
\gamma_s(T + t_0) \in \Sigma_\alpha \iff T + t_0 = \alpha.
\]

Note that the start time \( t_0 \) must be negative, because the \( y \)-coordinate acts as time in the normal form (1.2) and we wish to preserve the direction of time. Therefore, we have

\[
t_0 = -\sqrt{\frac{2\mu + 4\xi}{\mu^2}} \quad \text{and} \quad T = \alpha + \sqrt{\frac{2\mu + 4\xi}{\mu^2}}
\]

(3.6)

The solution segment (3.5) only approximately satisfies the boundary condition \( u(0) \in L^- \xi \), namely, the difference between the \( z \)-coordinates of \( u(0) = \gamma_s(t_0) \) and the point on \( L^- \xi \) at \( s = t_0 \) is

\[
\frac{\mu}{2} \sqrt{\frac{2\mu + 4\xi}{\mu^2}} - \sqrt{\xi} = \frac{\mu}{2 (\sqrt{\frac{\mu}{2}} + \xi + \sqrt{\xi})}.
\]

(3.7)

This difference is small, provided \( \xi \gg \mu \), and decreases as \( \xi \to \infty \). Hence, if \( \xi \) is large enough, we expect that Newton’s method converges and the first correction step of the continuation leads to a solution of (3.1)-(3.2). We remark that for a slow-fast system that is not in normal form an explicit solution is generally not known. This difficulty can be overcome with a homotopy approach, as is demonstrated in [10].

3.3. Computation of \( C^- \) from \( L^- \xi \) to \( \Sigma_\alpha \). The computed part of \( C^- \) depends on the two user-specified parameters \( \xi \) and \( \alpha \) that define \( L^- \xi \) and \( \Sigma_\alpha \), respectively. The parameter \( \xi \) controls the accuracy of the computation in that it determines the initial distance between \( C^- \) and the critical manifold \( S^a \). By construction, an orbit segment satisfying (3.1)-(3.2) converges to an actual orbit segment on \( C^- \) in the limit \( \xi \to \infty \). It is a very difficult task beyond the scope of this paper to find an explicit \( \xi \)-dependent error bound for the approximation of \( C^- \) and how it depends on \( \mu \). To derive a practical measure for the accuracy of the computations presented here we make use of the fact that the strong canard \( \gamma_s \) is given as an explicit
solution (2.12). As mentioned in Sec. 3.2, the orbit segment (3.5) of \( \gamma_s \) is the seed solution for Newton’s method, and we use the difference between \( \gamma_s(tT + t_0) \) and the approximate solution \( u^*(t) \) as an indication of the overall approximation error. Namely, we consider the pointwise difference between \( \gamma_s(tT + t_0) \) and \( u^*(t) \) with \( 0 \leq t \leq 1 \) and ensure that it is sufficiently small. At \( t = 0 \) this difference is given by (3.7), and it decreases exponentially for \( t > 0 \); this decrease is particularly fast due to the difference in time scales. Specifically, we only consider the difference in the \( z \)-coordinate and require that

\[
| \gamma_s(tT + t_0) - u^*(t) |_z < 10^{-5}
\]

for all \( t \) with \( u^*_s(t) > -\frac{1}{2} \xi \). In other words, this condition ensures the accuracy of the second and relevant part of the orbit segment, and we found that it is satisfied with \( \xi = 100 \) for all \( \mu \leq 5.5 \) that we consider in this paper. For larger \( \mu \) also \( \xi \) needs to be increased; namely, we use \( \xi = 200 \) for \( \mu = 14.5 \), \( \xi = 400 \) for \( \mu = 25.5 \), and \( \xi = 1000 \) for \( \mu = 49.5 \).

The parameter \( \alpha \) determines the location of the section \( \Sigma_{\alpha} \). Its choice depends on which aspect of \( C^\pm \) is of interest. A natural choice is \( \alpha = 0 \), such that the folded node (the origin) is contained in \( \Sigma_{\alpha} \). This means that \( C^- \) and \( C^+ \) are computed both up to \( \Sigma_0 \), which emphasizes their intersection curves \( C^\pm_0 = C^\pm \cap \Sigma_0 \). Note that these intersection curves have been computed before by shooting methods (numerical integration of initial values of \( S^\pm \)); see for example, [27, 26, 48]. By contrast, we compute the curves \( C^\pm_0 \) as well as the surfaces \( C^\pm \) themselves with the collocation and continuation routines of the package AUTO [13]. The main advantage of using collocation, as opposed to a shooting method, is that the size of the continuation step is determined as a variation along the entire orbit segment instead of the initial condition alone. This feature is particularly useful for slow-fast systems, which are extremely sensitive to variations in the initial condition [19]. Specifically, we compute the one-dimensional curves \( C^\pm_0 \) with an adaptation of the software MANBVP [19], where orbit segments are generated according to the local curvature of \( C^\pm_0 \). The two-dimensional surfaces \( C^\pm \) are computed with an AUTO run with a fixed continuation step size, which ensures a uniform distribution of mesh points on the surface.

In order to investigate how \( C^- \) and \( C^+ \) intersect near the folded node at the origin, we consider orbit segments computed up to \( \Sigma_{\alpha} \) with \( \alpha > 0 \). By symmetry \( C^+ \) ends in \( \Sigma_{-\alpha} \), so that the two slow manifolds are seen to interact in the region \(-\alpha \leq y \leq \alpha \). To visualize the geometry of this interaction it is convenient to show only the “ribbons” of \( C^- \) and \( C^+ \) in between the planes \( \Sigma_{-\alpha} \) and \( \Sigma_{\alpha} \). To this end, we clip each orbit segment of the computed manifold \( C^- \) where it intersects \( \Sigma_{-\alpha} \). We then determine a mesh with a fixed number of mesh points that are uniformly distributed according to arclength along clipped orbit segments. The resulting ribbons of \( C^\pm \) can be readily visualized, which provides insight into the interaction of the slow manifolds; see already Sec. 4.

3.4. Illustration of the method. Figure 2 illustrates our method for the normal form (1.2) with \( \mu = 1.2 \). Figure 2(a) shows an approximation of the attracting slow manifold \( C^- \) for \( \alpha = 0 \), that is, we computed a collection of orbit segments that start on \( L^-_{100} \), the straight red line in Fig. 2(a), and end in \( \Sigma_0 \), the \( (x, z) \)-plane shown in green. The bold red curve in \( \Sigma_0 \) is the intersection \( C^-_0 \) of \( C^- \) with \( \Sigma_0 \), and a small segment of its symmetrical image \( C^+_0 \) (blue curve, not labeled) is also shown. We started the continuation from the solution segment (bold black curve) \( \gamma_s(tT + t_0) \), \( t \in [0, 1] \), of the explicitly known strong canard \( \gamma_s \) with \( t_0 \) and \( T \) as defined in (3.6). The continuation is done in two directions parametrized by the \( y \)-coordinate along \( L_{100} \), where we start at \( y = t_0 < 0 \).

Let us first focus on the continuation run where \( y \) increases, because this part generates most of the intersection curve \( C^-_0 \). Note that it is natural to stop the continuation when \( y = 0 \) is reached. In practice, we use a user-defined function in AUTO to stop at \( y = 0 \). (The
solution family exists for $y > 0$, but then $T$ becomes negative.) The result from the first continuation run, as shown in Fig. 2(a), is an excellent illustration of Fenichel theory [22], which states that the attracting and repelling sheets of the critical manifold perturb smoothly to locally invariant manifolds for small $\varepsilon \neq 0$ outside a neighborhood of the fold curve $F$. In fact, outside a neighborhood of the origin, $C_{0}^{-}$ is almost identical to the intersection of the critical manifold $S$ with $\Sigma_{0}$. At the scale of Fig. 2(a) it is difficult to see what happens near
the folded node; an enlarged view of $C_0^\pm$ is presented in the next section.

During the second continuation run, when $y$ is decreasing, we encounter the orbit segment between $L_{100}$ and $\Sigma_0$ that corresponds to the weak canard $\gamma_w$, for which we have an explicit expression given in (2.12). Note that the symmetry of the normal form forces the attracting and repelling slow manifolds to intersect exactly on the line $z = 0$ in the section $\Sigma_0$. Hence, by imposing the user-defined function $u_1(1) = 0$ as part of the continuation of the one-parameter family that solves (3.1)–(3.2), AUTO [13] automatically detects all canard solutions of the normal form (1.2); a more detailed discussion is provided in Sec. 5. The actual canard solutions are obtained by concatenating the detected orbit segment on $C^+$, which ends in $\Sigma_0$ on the line $z = 0$, with its symmetrical copy on $C^+$ on the other side of $\Sigma_0$.

As $y$ decreases further, we encounter another special solution during the continuation, which is shown as the bold red curve in Fig. 2(a) that starts at the point labeled $y_{LP}$ on $L_{100}$. This point $y_{LP}$ is the unique point

$$y_{LP} = \frac{2(\mu + 1)}{\mu} \sqrt{\xi},$$

where the $x$-direction of the vector field (1.2) vanishes on $L_{100}^-$; that is, $y_{LP} \approx 36.777$ is the intersection of $L_{100}$ with the $x$-nullcline for $\mu = 1.2$. We know from Fenichel theory [22] that, away from $F$, the attracting slow manifold $C^-$ is a graph over the attracting sheet $S^a$ of the critical manifold. Hence, if $L_{100}$ were exactly on $C^-$ then all initial conditions on $L_{100}$ beyond $y_{LP}$, that is, with $y$-coordinates less than $y_{LP}$, would lie on (backward-extended) orbit segments that intersect $L_{100}^-$ at $y$-coordinates larger than $y_{LP}$. This behavior corresponds exactly to the case that an (un)stable manifold in a Poincaré section crosses the locus where the flow is tangent to the section; see [19] for more details.

In practice $L_{100}$ only lies approximately on $C^-$ and solutions beyond $y_{LP}$ do not lie exactly on the computed approximation of $C^-$, but still very close to it. Hence, it appears as though a new part of $C^-$ is obtained, which manifests itself as a very sharp fold, or ‘crease,’ on the approximation of $C^-$. From a computational point of view, the continuation makes sense only for $u_1(0) \in [0, y_{LP}]$, because continuation beyond $y_{LP}$ produces a second approximation of the same part of $C^-$. Figure 2(a) does show a computation of $C^-$ past $y_{LP}$ to illustrate what happens. It was reported in [48] that the turning point $y_{LP}$ gives rise to additional canard solutions that persist only for a very short interval of $\mu$-values. The discussion above, however, implies that these additional ‘canards’ appear due to the numerical error of the approximation and, hence, are spurious. Note further that $y_{LP} \rightarrow -\infty$ as $\xi \rightarrow \infty$, that is, in the limit where $L_{\xi}$ converges to a line on $C^-$, the point $y_{LP}$ no longer exists.

Figure 2(b) demonstrated how the ribbons of $C^\pm$ in between $\Sigma_1$ and $\Sigma_{-1}$ can be used as a means of visualizing the interaction of the two manifolds. For clarity, the intersection curves $C^\pm_1$ and $C^\pm_{-1}$ are shown as well. The geometry of $C^\pm$ is further enhanced by including the strong and weak canards $\gamma_s$ and $\gamma_w$, respectively.

4. Geometry of the slow manifolds. We now study the slow manifolds $C^\pm$ for different values of the parameter $\mu$. We use both $\alpha = 0$ and $\alpha > 0$ in the method from Sec. 3 to illustrate not only the intersection curves $C_0^\pm$ of $C^\pm$ with $\Sigma_0$, as was done in [48], but also the geometry of the two-dimensional slow manifolds $C^\pm$ themselves. A main goal is to see how maximal canards arise as new intersection curves between $C^-$ and $C^+$. In all figures the attracting slow manifold $C^-$ is colored red, the repelling slow manifold $C^+$ blue, the section $\Sigma_{-1}$ is green, and the strong canard $\gamma_s$ and the weak canard $\gamma_w$ are black. As $\mu$ is increased, secondary canards appear, which we label successively as $\eta_i$. We adopted a particular color coding for these secondary canards: $\eta_1$ is orange, $\eta_2$ is magenta, $\eta_3$ is cyan, and we repeat these successive colors for each group of three consecutive secondary canards after $\eta_3$. 
Fig. 3. Local three-dimensional views of the slow manifolds $C^\pm$ computed up to section $\Sigma_0$ (panels (a1) and (b1)), together with two-dimensional illustrations of their intersections $C^\pm_0$ with $\Sigma_0$. Panel (a) corresponds to $\mu = 1$ and panel (b) to $\mu = 1.2$. The weak canard $\gamma_w$ and the strong canard $\gamma_s$ are shown as black curves and their intersections with $\Sigma_0$ are denoted by black dots. See also Figs. 4 and 5.

4.1. Geometry of $C^\pm$ up to $\Sigma_0$. We begin with a series of images for $\mu = 1$, $\mu = 1.2$, $\mu = 2.5$, $\mu = 3.5$, and $\mu = 8.5$ that illustrate the behavior of $C^\pm$ up to the section $\Sigma_0$; see Figs. 3(a),(b), 4(a),(b), and 5(a), respectively. Each figure shows a three-dimensional view in a neighborhood of the folded node of $C^+$ and $C^-$, computed up to $\Sigma_0$, together with the corresponding intersections in the plane $\Sigma_0$. To facilitate comparison and analysis, the viewpoint and aspect ratio is identical for all three-dimensional pictures, although the ranges along the axes vary.

Figure 3(a) shows the case $\mu = 1$, which acts as the starting point where $C^\pm$ have a
non-transverse tangent intersection along a single orbit. The first two (maximal) canards $\gamma_s$ and $\gamma_w$ are created as $\mu$ is increased. They are shown in Fig. 3(b) for $\mu = 1.2$; see also Fig. 2. Note that $\gamma_s$ and $\gamma_w$ are now two distinct orbits of the normal form (1.2) in which the slow manifolds $C^\pm$ intersect transversely. The case $\mu = 1.2$ is representative for all values $1 < \mu < 2$.

A qualitative change occurs at $\mu = 2$. Figure 4(a) shows the situation for $\mu = 2.5$. As can be observed particularly in panel (a2), the tips of $C^\pm_0$ have rotated around so that they now

Fig. 4. Local three-dimensional views of the slow manifolds $C^\pm$ computed up to section $\Sigma_0$ (panels (a1) and (b1)), together with two-dimensional illustrations of their intersections $C^\pm_0$ with $\Sigma_0$. Panel (a) corresponds to $\mu = 2.5$ and panel (b) to $\mu = 3.5$. Note the existence of the first secondary canard $\eta_1$ (orange dot) in panel (b2), which appears in a transcritical bifurcation at $\mu = 3$. See also Figs. 3 and 5.
point inside the region delimited by $\gamma_s$ and $\gamma_w$. Indeed, these tips rotate continuously with $\mu$. At $\mu = 2$ the tangent bundles $T_{\gamma_w} C^\pm$ coincide and the directions in $\Sigma$ are parallel to the $z$-axis, that is, $C^+_0$ both have a vertical tangency vector at $\gamma_w(0)$ and $C^-_0$ continues smoothly as $C^+_0$ past $\gamma_w(0)$. For $\mu = 3$ the tips have rotated so that the tangent bundles $T_{\gamma_w} C^\pm$ coincide again, but $C^-_0$ now both have a tangency vector parallel to the $x$-axis and the manifolds $C^\pm$ meet in a cusp. This cusp-shaped tangency is one of the transcritical bifurcations that occur.
Fig. 6. Three-dimensional views of the ribbons of $C^{\pm}$ in between $\Sigma_{-0.5}$ and $\Sigma_{0.5}$, together with all maximal canards and the intersection curves $C_{\pm}^{\mp}.5$. Panel (a) corresponds to $\mu = 25.5$, with 12 secondary canards, and panel (b) corresponds to $\mu = 49.5$, with 24 secondary canards. See also Figs. 2(a), 5(b) and the accompanying animation dko_fn_a03.gif.

for all odd integer values of $\mu \geq 3$ and results in a secondary canard. Figure 4(b) shows a phase portrait for $\mu = 3.5$ with the first secondary canard $\eta_1$ (orange curve). As can be seen clearly in Fig. 4(b2), there are now three intersection points of $C_{0.5}^{\pm}$ in $\Sigma_0$. Note that $\gamma_w$ is located between $\eta_1$ and $\gamma_s$ on the $x$-axis.

Figure 5(a) shows the situation for $\mu = 8.5$, where we have three secondary canards, denoted $\eta_1$ (orange), $\eta_2$ (magenta), and $\eta_3$ (cyan). It gives an idea of how the secondary canards appear as a result of the spiralling motion of $C^{\pm}$ around the weak canard $\gamma_w$; this is particularly visible for $C_{0.5}^{\pm}$ in $\Sigma_0$, shown in Fig. 5(a2) and in the accompanying animation dko_fn_a02.gif. The figure also illustrates the fact that $C^{\pm}$ make $\left\lfloor \frac{\mu-1}{2} \right\rfloor$ full rotations
around $\gamma_w$.

4.2. Ribbons of $C^\pm$ near the folded singularity. Figures 3, 4, and 5(a) give a good insight into the topological changes of the geometry of $C^\pm$. After the bifurcation of maximal canards at $\mu = 1$, all secondary maximal canards bifurcate from $\gamma_w$ in transcritical bifurcations at odd integer values of $\mu$. To bring out this behavior more clearly, we also compute ribbons of $C^\pm$ in between $\Sigma_{-\alpha}$ and $\Sigma_{\alpha}$ for suitable $\alpha > 0$. Figure 5(b) shows $C^\pm$ in between $\Sigma_{-1}$ and $\Sigma_1$, along with the intersection curves $\gamma_s$, $\gamma_w$ (black curves), $\eta_1$ (orange), and $\eta_2$ (magenta); note that $\eta_3$ is not shown in this picture, because it cannot be distinguished from $\gamma_w$ at this scale. The intersection curves $C_{\pm1}^-$ and $C_{\pm1}^+$ in $\Sigma_{-1}$ and $\Sigma_1$, respectively, give some idea of the exponential growth away from the fold curve in forward time (for $C^-$) and backward time (for $C^+$) of the solutions on the slow manifolds. Indeed, the maximal canards $\gamma_s$, $\gamma_w$ and $\eta_i$, for $i = 1, \ldots, \lfloor \frac{\mu - 1}{2} \rfloor$, are the only solutions of (1.2) that grow algebraically both in forward and backward time.
Figure 6 shows $C^\pm$ for much larger values of $\mu$, namely, for $\mu = 25.5$ and $\mu = 49.5$ in panels (a) and (b), respectively. Due to the exponential growth of the slow manifolds, we show here ribbons of $C^\pm$ only in between $\Sigma_{-0.5}$ and $\Sigma_{0.5}$. The figure shows the increased complexity of $C^\pm$ with many more intersection curves that form additional secondary canards. For $\mu = 25.5$ there are twelve secondary canards that wind around $\gamma_w$; only the first four are labeled in Fig. 6(a). For $\mu = 49.5$ there are 24 secondary canards, but again only $\eta_1$, $\eta_2$, $\eta_3$, and $\eta_4$ are labeled in Fig. 6(b). The increase in $\mu$ causes an increase of the spiralling amplitude of the intersection curves $C_{-0.5}^\pm$ and $C_{0.5}^\pm$ in $\Sigma_{-0.5}$ and $\Sigma_{0.5}$, respectively. By comparing Fig. 6(a) and (b), one can clearly see that the strong canard $\gamma_s$ and the labeled secondary canards $\eta_1$, $\eta_2$, $\eta_3$, and $\eta_4$ lie further away from the weak canard $\gamma_w$ and from each other; see also the accompanying animation dko_ezn_a03.gif.

Figure 7 illustrates the increasing complexity with $\mu$ of the two slow manifolds near the folded node by showing $C^\pm$ for $\mu = 8.5$, $\mu = 14.5$, $\mu = 25.5$ and $\mu = 49.5$. Here we rotated the slow manifolds $C^\pm$ about the $z$-axis with the visualization package GEOMVIEW [39] to generate common enlarged views centered around the weak canard $\gamma_w$. In this way, one obtains a good impression of the spiralling behavior of the secondary canards around $\gamma_w$ and how their positions and distances to $\gamma_w$ change with $\mu$. Figure 7(a) shows that for $\mu = 8.5$ $C^-$ and $C^+$ intersect in the two secondary canards $\eta_1$ and $\eta_2$. For $\mu = 14.5$ there are a total of six secondary canards, four of which, $\eta_1$ to $\eta_4$, are shown in Fig. 7(b). Note how the distance of $\eta_1$ and $\eta_2$ from the central weak canard $\gamma_w$ (black curve) is now much larger for $\mu = 14.5$; in a way, this creates space for $\eta_3$ and $\eta_4$ to spiral around $\gamma_w$ as well. For $\mu = 25.5$ there are twelve secondary canards in total, but only $\eta_1$ to $\eta_4$ are labeled in Fig. 7(c). Similarly, for $\mu = 49.5$ in Fig. 7(d) there are 24 secondary canards, of which the first nine are clearly visible while only $\eta_1$ to $\eta_4$ are labeled.

5. Geometric study of the secondary canards.

As we have seen, the secondary canards arise as intersections of the slow manifolds $C^-$ and $C^+$. We now find them directly as special orbits within the boundary value problem setup in Sec. 3. This allows us to visualize and discuss their spiralling behavior with respect to the weak canard $\gamma_w$. Furthermore, we continue the secondary canards in the parameter $\mu$ to reveal an overall bifurcation diagram.

5.1. Detection of secondary canards. During the continuation of (3.1)–(3.2) for fixed $\mu$ the end points $u(1) \in \Sigma_0$ of the computed orbit segments oscillate about the $z$-axis; see, for example, Fig. 5(a2). Due to symmetry, a secondary canard is characterized by the condition that the $z$-coordinate $u_z(1)$ satisfies $u_z(1) = 0$. Hence, secondary canards can be detected during the continuation by monitoring this condition (with a user-defined function in AUTO). Recall that the point $u(1)$ is a function of the $y$-coordinate $u_y(0)$ of its begin point $u(0)$, which varies along $L_{\Sigma_0}$.

Figure 8 shows the graph of $u_y(1)$ as a function of $u_y(0)$ for $\mu = 49.5$, where we show data for the run that starts from the strong canard $\gamma_s$ for which we have $u_y(0) = t_0 \approx -1.29$. Due to the spiralling nature of $C^- \cap \Sigma_0$ the graph oscillates with a rapidly decreasing amplitude; note that the continuation is in the direction of negative $u_y(0)$. The enlargement in the inset of Fig. 8 shows the oscillation of $u_z(1)$ in the region where $\eta_{12}$ to $\eta_{15}$ are detected and the oscillation amplitude has decreased to values of order $10^{-4}$. Numerically it becomes increasingly difficult to detect where $u_z(1)$ changes sign when the oscillation amplitude becomes very small. In other words, for large $\mu$ as in Fig. 8 for $\mu = 49.5$ it is quite a challenge to detect the secondary canards that lie very close to $\gamma_w$.

For all values of $\mu$ in this paper we start the continuation with the AUTO accuracy settings as shown in row (a) of Table 5.1. This is sufficient for the reliable detection of the $\eta_i$ even for $\mu = 25.5$, but for the case $\mu = 49.5$ shown in Fig. 8 the detection stops when extrema of $u_z(1)$ are less than $10^{-9}$ in modulus. At this stage the secondary canards $\eta_{17}$ to $\eta_{17}^{\pm}$ have
Fig. 8. Graph of the z-coordinate $u_z(1)$ of the end point $u(1)$ of the computed orbit segments on $C^-$ for $\mu = 49.5$ as a function of the y-coordinate $u_y(0)$ of its begin point $u(0) \in L^-$. The continuation starts from the weak canard $\gamma_w$ and secondary canards $\eta_i$ are detected when $u_z(1) = 0$.

Table 5.1

<table>
<thead>
<tr>
<th></th>
<th>NTST</th>
<th>NCOL</th>
<th>DS</th>
<th>DMIN</th>
<th>DMAX</th>
<th>EPSS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>200</td>
<td>4</td>
<td>0.001</td>
<td>$5 \times 10^{-4}$</td>
<td>0.01</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>(b)</td>
<td>400</td>
<td>6</td>
<td>0.001</td>
<td>$10^{-7}$</td>
<td>0.01</td>
<td>$10^{-7}$</td>
</tr>
</tbody>
</table>

Auto accuracy parameters as used during the detecting the secondary canards of (1.2). Row (a) is our regular accuracy setting, and row (b) is the increased accuracy as used for $\mu = 49.5$. Here NTST is the number of mesh points, NCOL the number of collocation points, DMIN and DMAX are the minimal and maximal stepsizes for the continuation, and EPSS is the relative arclength convergence criterion for the detection of special solutions; all other AUTO accuracy parameters are set to their default values.

been detected reliably. The next four secondary canards $\eta_{18}$ to $\eta_{21}$ are found in a second run, where we increase the accuracy parameters to the settings given in row (b) of Table 5.1. Nevertheless, the detection of $\eta_{22}$ to $\eta_{24}$ is very difficult even with the increased accuracy settings, because $u_z(1)$ is now consistently below $10^{-15}$ in modulus. Since we are reaching the limit of machine precision spurious roots of $u_z(1)$ are reported, out of which we need to select $\eta_{22}$ to $\eta_{24}$. This can be done by taking into consideration the distance between roots in $u_y(0)$, which leads to a selection that is consistent with the detected secondary canards $\eta_1$ to $\eta_{21}$; see already Fig. 11(d).

Once a secondary canard has been detected for a fixed value of $\mu$ as a zero of $u_z(1)$, it can be continued in the parameter $\mu$ by imposing $u_z(1) = 0$ as an additional boundary condition. In this way, we can compute the $\mu$-dependent families of all detected secondary canards. We remark here that the oscillations of $u_z(1)$ near a fixed canard increase as $\mu$ is increased. Hence, canards $\eta_i$ for large $i$ can be detected reliably for larger $\mu$ and then continued back into the range of lower values of $\mu$.  

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5.2. Spiralling behavior of the secondary canards. To explain the spiralling of the secondary canards around the weak canard we concentrate on the case $\mu = 8.5$, for which there are three secondary canards, $\eta_1$ to $\eta_3$. They are shown in Fig. 9 together with the primary canards $\gamma_s$ and $\gamma_w$ (black curves). Figure 9(a) is a three-dimensional view of the canards, where we also show for orientation the critical manifold $S$ (the grey parabolic cylinder) with the fold curve $L$ (thick grey line) and the singular canards $\hat{\gamma}_s$ and $\hat{\gamma}_w$ (grey) on $S$. The primary canards $\gamma_s$ and $\gamma_w$ (black) are perturbations (on the blown-up sphere) of $\hat{\gamma}_s$ and $\hat{\gamma}_w$. Furthermore, the secondary canards $\eta_i$ lie seemingly parallel to $\gamma_s$ for $|x|$ large, but follow $\gamma_w$ near the fold. With increasing $i$ the $\eta_i$ lie closer to $\gamma_w$ as they spiral increasingly around it. Figure 9(b) and (c) are projections of the primary and secondary canards onto the $(x, z)$- and $(y, z)$-planes, respectively. Figure 9(b) illustrates how each new secondary canard makes one more full rotation around the weak canard $\gamma_w$. Figure 9(c) focuses on the (slow) dynamics of
the secondary canards in the neighborhood of the fold. The grey regions indicate the funnel through which the weak canards move, which is delimited by the two primary canards $\gamma_w$ and $\gamma_s$; see also Sec. 2.

Figure 10 is a visualization with the package GEOMVIEW [39] in the spirit of a wire and cardboard model to bring out the spiralling of the secondary canards $\eta_1$ to $\eta_3$ around $\gamma_w$. Namely, shown are the plane $\Sigma_w = \{y = 2z\}$ that contains $\gamma_w$ and the plane $\Sigma_w^\perp = \{x = -10\}$ that is perpendicular to $\Sigma_w$ chosen so that all spiralling behavior is captured. The secondary canards $\eta_1$ to $\eta_3$ start on $\Sigma_w^\perp$ below $\Sigma_w$ and return to $\Sigma_w^\perp$ above $\Sigma_w$. Notice that $\eta_1$ has three intersection points (yellow dots), $\eta_2$ has five intersection points (magenta dots), and $\eta_3$ has seven intersection points (blue dots) with $\Sigma_w$. This illustrates the theoretical results that $\eta_i$ makes $\lfloor \frac{\mu - 1}{2} \rfloor$ rotations around $\gamma_w$. Figure 10 also illustrates that the secondary canards $\eta_i$ lie successively closer to $\gamma_w$ in the region of the fold.

The spiralling character of a secondary canard $\eta_i$ does not depend on the value of $\mu$. When a secondary canards is created in a transcritical bifurcation closest to $\gamma_w$ at an odd integer value of $\mu$, its rotating property is fixed. This is illustrated in Fig. 11 with projections onto the $(y, x)$-plane of all canards for $\mu = 8.5$, $\mu = 14.5$, $\mu = 25.5$, and $\mu = 49.5$, respectively. For each case we choose a region of the $(y, x)$-plane that allows for a comparison between the panels; specifically, the $x$-maximum of $\gamma_s$ is fixed and the $y$-range is adjusted so that the last secondary canard is seen to ‘leave’ $\gamma_w$. Figure 11(a) for $\mu = 8.5$ should be compared directly with Fig. 9. There are six secondary canards in Fig. 11(b), twelve in Fig. 11(c) and 24 in Fig. 11(d). Overall, the secondary canards run parallel to (have the same slope as) the strong canard $\gamma_s$ for large $|x|$, and then spiral around $\gamma_w$. Observe that the
maxima of the $\eta_i$ appear to line up along a curve that connects $\gamma_s$ to the maximum of $\gamma_s$; an initial investigation showed that this curve is not a straight line. It would be an interesting challenge to study limiting features of the canards in a suitably rescaled $(y, x)$-plane for $\mu$ tending to $\infty$.

5.3. Continuation of the secondary canards in $\mu$. It is a particular advantage of our boundary value problem setup that secondary canards can be continued in the parameter $\mu$. Figure 12 shows the result of the continuation of the secondary canards $\eta_1$ to $\eta_{12}$, where we plot the $x$-coordinate $u_x(1)$ of the end point in $\Sigma_0$. Also shown in this bifurcation diagram are the primary canards $\gamma_w$ and $\gamma_s$. They are determined from (2.12) as the straight lines $u_x(1) = \frac{1}{2}$ and $u_x(1) = \frac{\mu}{2}$, respectively, which intersect transversely at $\mu = 1$. The continuation was started by detecting all twelve secondary canards for $\mu = 25.5$, where we used $\xi = 1000$ to ensure sufficient accuracy for their continuation for $\mu > 25.5$.

Figure 12(a) shows how $\eta_1$ to $\eta_{12}$ bifurcate from $\gamma_w$ at odd integer values. Notice that the branches of $\eta_i$ for odd $i$ are in the region below $\gamma_w = \frac{1}{2}$, while those for even $i$ are in the region above $\gamma_w = \frac{1}{2}$. This agrees with [48, Fig. 17] of Wechselberger, who sketched the
branches of secondary canards $\eta_i$ as straight lines that bifurcate from $\gamma_w$ at an angle. As can be seen in Fig. 12(a), the branches of $\eta_i$ are actually tangent to $\gamma_w$ and the degree of tangency appears to increase with $i$. Figure 12(b) shows the branches of $\eta_i$ for the much larger $\mu$-range up to $\mu = 500$. This image suggests that in the limit of large $\mu$ the branches for odd $i$ appear to have slope $\frac{1}{2}$ as $\gamma_s$, while those for even $i$ appear to have slope $-\frac{1}{2}$. These slopes agree with the slopes suggested by Wechselberger’s sketch [48, Fig. 17].

6. **Beyond the normal form.** It is in the nature of a normal form that (1.2) has special properties. Specifically $C^-$ and $C^+$ are each other’s images under a symmetry operation.
From the computational point of view, this means that only $C^-$ needs to be computed. Furthermore, secondary canards can be detected and continued by considering the condition $u_z(1) = 0$. However, for a system that is not in normal form the symmetry of the normal form is typically lost. Hence, in general the attracting and repelling slow manifolds must be computed separately as the solution families of two different two-point boundary value problems. As a consequence, the primary and the secondary canards must be detected as solutions corresponding to intersection points of the curves $C^- \cap \Sigma_0$ and $C^+ \cap \Sigma_0$.

As an example, we show here what the slow manifolds look like in the perturbation of...
the normal form (1.2) that is given by (2.9) for small nonzero \( \rho = \sqrt{\varepsilon} \), that is,

\[
\begin{aligned}
\dot{x} &= \frac{1}{2} \mu y - (\mu + 1) z + \sqrt{\varepsilon}, \\
\dot{y} &= 1, \\
\dot{z} &= x + z^2 + \sqrt{\varepsilon}.
\end{aligned}
\]  

(6.1)

Note that system (6.1) has the same critical manifold as the normal form (1.2), so that its slow manifolds can be computed with the same boundary conditions as for (1.2). Namely, we require that the orbit segments start on the line \( L^- \) defined by (3.3). For the computation of the repelling slow manifold, we reverse time in (6.1) and consider

\[
L^+ := \{ (-\xi, s, \sqrt{\xi}) \mid s \in \mathbb{R} \},
\]

which lies on \( S^r \) instead of \( S^u \), again parallel to \( F \) at ‘distance’ \( \xi > 0 \) from \( F \). Furthermore, we require the orbit segments to end in a transverse section \( \Sigma_\alpha \) given by (3.4) with \( \alpha = 0 \) or \( \alpha = \pm 1 \).

We consider here the case \( \mu = 8.5 \) and \( \varepsilon = 0.1 \) and fix \( \xi = 100 \) as before. For \( \varepsilon = 0.1 \) the explicitly known solution of \( \gamma_s \) of the unperturbed system (1.2) does not work as a starting solution. (Newton’s method does not converge.) Therefore, starting from \( \gamma_s \) we first compute the \( \varepsilon \)-dependent family of orbit segments that solve (6.1) subject to the boundary conditions (3.2). Here we fix the parameter \( s \) that defines the position on \( L^- \) to the value (given by \( \gamma_s \)) of

\[
s = t_0 = -\sqrt{\frac{2\mu + 4\xi}{\mu^2}}
\]

as defined in (3.6). When \( \varepsilon = 0.1 \) is reached a first orbit segment on \( C^- \) has been found; a first orbit segment on \( C^+ \) is found similarly by starting a continuation in \( \varepsilon \) from the part of \( \gamma_s \) that connects \( L^+ \) to \( \Sigma_\alpha \). We now fix \( \varepsilon = 0.1 \) and continue in \( s \) to sweep out \( C^- \) and \( C^+ \), respectively.

Figure 13 shows the slow manifolds of (6.1) for \( \mu = 8.5 \) and \( \varepsilon = 0.1 \). This figure should be compared with Fig. 5 for \( \varepsilon = 0 \); for ease of comparison we use the same viewpoints in both figures. In Fig. 13 the slow manifolds \( C^- \) and \( C^+ \) have deformed and are no longer each other’s image under a symmetry operation. Nevertheless, the situation is topologically the same as that for \( \varepsilon = 0 \) in Fig. 5. Namely, \( C^- \) and \( C^+ \) intersect in the same way in the primary canards \( \gamma_s \) and \( \gamma_w \) and the secondary canards \( \eta_1 \) to \( \eta_3 \); see Fig. 13(a2) and (b). In particular, the rotating behavior of \( \eta_1 \) to \( \eta_3 \) around \( \gamma_w \) is preserved. The canards are found by detecting orbit segments on \( C^- \) and \( C^+ \) that end at the same point in \( \Sigma_0 \) (within the accuracy of the computation). Concatenation of the two respective orbit segments results in the representation of the secondary canard as a solution that starts on \( L^- \) and ends at \( L^+ \). After applying a Newton step to get an exact solution to this boundary value problem, detected canards can be continued in a system parameter.

7. Conclusion. We performed a study of slow manifolds and associated canard solutions in a three-dimensional normal form of a slow-fast system with a folded node. Specifically, we computed the two-dimensional attracting and repelling slow manifolds as one-parameter families of orbit segments that satisfy appropriately chosen boundary conditions. This approach also allows us to detect and continue the associated canard solutions in which the two slow manifolds intersect. The visualization of these geometric objects for different values of the normal form parameter \( \mu \) (the ratio of eigenvalues at the folded node) provided unprecedented insight into the geometry of the dynamics near a folded node. We discussed
in detail how the secondary canards spiral around the weak primary canard and presented the first computed bifurcation diagram showing branches of the secondary canards as a function of \( \mu \).

The numerical continuation of solution families of a well-posed boundary value problems can be performed extremely accurately. In our computations we use the continuation and boundary value solver routines of AUTO, which uses pseudo-arclength continuation and collocation with piecewise-polynomial approximations. Hence, the boundary value problems we define are solved subject to established error bounds. Therefore, the accuracy of our calculations of slow manifolds and canard solutions comes down to determining how the choice of boundary condition influences the distance of the approximation from the real object. In our setup we make use of the fact that far away from the fold line the slow manifolds are very close to the critical manifold. Hence, we restrict the approximating orbit segments to such a far away line on the critical manifold. We ensured by numerical checks that the pointwise distance to the true slow manifolds along selected orbit segments is sufficiently small. A more detailed error analysis is a challenging subject for further investigation, because the overall approximation error depends, in general, not only on the distance from the fold curve but also on system parameters.

While this paper concentrates on the normal form of a folded node, our boundary value problem approach to computing slow manifolds and canard solutions can be applied more widely. This was demonstrated with the example of a perturbation of the normal form that breaks the underlying symmetry. In [10] we computed slow manifolds and canard solutions in the self-coupled FitzHugh Nagumo model. This required the implementation of a homotopy approach to generating initial approximate orbits on the attracting and repelling slow manifolds. In this way, we were able to identify sectors between different secondary canards that correspond to mixed-mode oscillations with different numbers of small oscillations.

In the near future we plan to use our computational approach to investigate other slow-fast systems arising in applications, especially those showing mixed-mode oscillations. This is relatively straightforward for the case of three-dimensional vector field models with a clear splitting of the phase space into slow and fast variables, such as the self-coupled FitzHugh Nagumo model [10] or the forced Van der Pol system [6, 28]. However, we believe that the computation of invariant manifolds would also be a very helpful tool in situations where there is no obvious split of the system into slow and fast variables. The goal here would be to identify slow and fast components of the dynamics numerically, and to use this knowledge to unravel the geometry of slow manifolds and associated canard solutions.

REFERENCES

[8] M. Brøns, M. Krupa, and M. Wechselberger, Mixed mode oscillations due to the generalized canard


