STABLE CATEGORIES AND RECONSTRUCTION

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Dedicated to the memory of Sandy Green

1. Introduction

The Green correspondence is a fundamental construction in modular representation theory of finite groups. It is expected (Broué’s abelian defect group conjecture for example) to be the shadow of a more structural categorical correspondence, yet to be found. In an inductive approach to this, a key case is when the Green correspondence induces a stable equivalence between blocks. This work is an attempt towards a Morita theory for stable equivalences between self-injective algebras. More precisely, given two self-injective algebras $A$ and $B$ and an equivalence between their stable categories, consider the set $S$ of images of simple $B$-modules inside the stable category of $A$. That set satisfies some obvious properties of Hom-spaces and it generates the stable category of $A$. Keep now only $S$ and $A$. Can $B$ be reconstructed? We show how to reconstruct the graded algebra associated to the radical filtration of (an algebra Morita equivalent to) $B$. It would be interesting to develop further an obstruction theory for the existence of an algebra $B$ with that given filtration, starting only with $S$ (this might be studied in terms of localization of $A_\infty$-algebras). Note that a result of Linckelmann [Li] shows that, if we consider only stable equivalence of Morita type, then $B$ is characterized by $S$ — but this result does not provide a reconstruction of $B$ from $S$.

We also study a similar problem in the more general setting of a triangulated category $\mathcal{T}$. Given a finite set $S$ of objects satisfying Hom-properties analogous to those satisfied by the set of simple modules in the derived category of a ring and assuming that the set generates $\mathcal{T}$, we construct a $t$-structure on $\mathcal{T}$. In the case $\mathcal{T} = D^b(A)$ and $A$ is a symmetric algebra, the first author has shown [Ri] that there is a symmetric algebra $B$ with an equivalence $D^b(B) \cong D^b(A)$ sending the set of simple $B$-modules to $S$. The case of a self-injective algebra leads to a slightly more general situation: there is a finite dimensional differential graded algebra $B$ with $H^i(B) = 0$ for $i > 0$ and for $i \ll 0$ with the same property as above.

2. Notations

Let $\mathcal{C}$ be an additive category. Given $S$ a set of objects of $\mathcal{C}$, we denote by $\text{add} \ S$ the full subcategory of $\mathcal{C}$ of objects isomorphic to finite direct sums of objects of $S$.

Let $k$ be a field and $A$ a finite dimensional $k$-algebra. We say that $A$ is split if the endomorphism ring of every simple $A$-module is $k$. We denote by $A\text{-mod}$ the category of finitely generated left $A$-modules and by $D^b(A)$ its derived category. For $A$ self-injective, we denote by $A\text{-stab}$ the stable category, the quotient of $A\text{-mod}$ by projective modules. Given $M$ an $A$-module, we denote by $\Omega M$ the kernel of a projective cover of $M$ and by $\Omega^{-1} M$ the cokernel of an injective hull of $M$. 
3. Simple generators for triangulated categories

3.1. Category of filtered objects. Let $T$ be a triangulated category and $S$ a full subcategory of $T$.

We define a category $F$ as follows.

- Its objects are diagrams

$$M = (\cdots \to M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \xrightarrow{\epsilon_0} N_0)$$

where $M_i$ is an object of $T$, $M_i = 0$ for $i \gg 0$, such that

(i) $M_1 \xrightarrow{f_1} M_0 \xrightarrow{\epsilon_0} N_0$ is the beginning of a distinguished triangle

(ii) for all $i \geq 1$, the cone $N_{i-1}$ of $f_i$ is in add$S$

(iii) the canonical map $\text{Hom}(N_0, S) \to \text{Hom}(M_0, S)$ is surjective for all $S \in S$

(iv) the canonical map $\text{Hom}(N_i, S) \to \text{Hom}(M_i, S)$ is bijective for all $S \in S$ and $i \geq 1$.

Note that $\epsilon_i : M_i \to N_i = \text{cone}(f_{i+1})$ is well defined up to unique isomorphism for $i \geq 1$ thanks to property (iv). For $i \geq 0$, we define a new object $M_{i+1}$ of $F$ as

$$\cdots \to M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{\epsilon_i} N_i \to M_i \xrightarrow{\rho_i} N_{i-1}$$

- Given another diagram $M'$, we define $\text{Hom}_F(M, M'_0)$ as the subspace of $\text{Hom}(N_0, N'_0)$ consisting of those maps $g$ such that there is $h : M_0 \to M'_0$ with $\epsilon'_0 h = g \epsilon_0$.

We put $\text{Hom}_F(M, M'_{1}) = \text{Hom}_F(M, M'_{2})_0$ and $\text{Hom}_F(M, M') = \oplus_{i \geq 0} \text{Hom}_F(M, M'_i)$.

- Consider now $g_0 \in \text{Hom}_F(M, M')$. By (iv), there are maps $h_0, h_1, \ldots$ and $g_1, g_2, \ldots$ making the following diagrams commutative

$$\begin{array}{ccc}
N_i[-1] & \xrightarrow{\rho_i} & M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{\epsilon_i} N_i \\
| & | & | \\
g_i[-1] & \xrightarrow{h_{i+1}} & h_i \xrightarrow{g_i} N_i \\
\downarrow \rho'_i & \downarrow h_i & \downarrow g_i \\
N'_i[-1] & \xrightarrow{\rho'_i} & M'_{i+1} \xrightarrow{f'_{i+1}} M'_i \xrightarrow{\epsilon'_i} N'_i
\end{array}$$

Here, $\rho_i : N_i[-1] \to M_{i+1}$ and $\rho'_i : N'_i[-1] \to M'_{i+1}$ are the maps making the horizontal rows in the diagram above into distinguished triangles.

**Lemma 3.1.** The maps $g_i : N_i \to N'_i$ (for $i \geq 1$) depend only on $g_0$.

**Proof.** We proceed by induction on $i$. We assume $g_{i-1}$ has been shown to depend only on $g_0$. Let us consider the lack of unicity of $h_i$. Consider $h_i, \tilde{h}_i : M_i \to M'_i$ such that $h_i \rho_{i-1} = \rho'_{i-1} g_{i-1}[-1] = \tilde{h}_i \rho_{i-1}$. There is $p : M_{i-1} \to M'_i$ such that $h_i - \tilde{h}_i = p f_i$.

By (iii) and (iv), there exists $q : N_{i-1} \to N_i$ such that $q \epsilon_{i-1} = \epsilon'_i p$. We have $\epsilon'_i p f_i = q \epsilon_{i-1} f_i = 0$, hence $\epsilon'_i h_i = \epsilon'_i \tilde{h}_i$.

By (iv), we deduce that there is a unique map $g_i : N_i \to N'_i$ such that $g_i \epsilon_i = \epsilon'_i h_i$ and that map $g_i$ is the unique one such that $g_i \epsilon_i = \epsilon'_i \hat{h}_i$. \qed

Let $g_0 \in \text{Hom}_F(M, M')$, and $g'_0 \in \text{Hom}_F(M', M'')_j$. We define the product $g'_0 g_0$ as the composition $N_0 \xrightarrow{g_0} N'_i \xrightarrow{g'_i} N''_{i+j}$.

**Lemma 3.2.** Assume $\text{Hom}(S, T[n]) = 0$ for all $S, T \in S$ and $n < 0$. Let $M$ be an object of $F$. Then, the canonical map $\text{Hom}(N_0, S) \to \text{Hom}(M_0, S)$ is an isomorphism.
Proposition 3.4. \((T^{\leq 0}, T^{> 0})\) is a bounded t-structure on \(T\).

Proof. By induction, we see there is no non-zero map from an object of \(T^{\leq 0}\) to an object of \(T^{> 0}\). Furthermore, we have \(T^{\leq 0}[1] \subset T^{\leq 0}\) and \(T^{> 0} \subset T^{> 0}[1]\).
Let $N \in \mathcal{T}$. Pick a sequence as in Lemma 3.3. Take $s$ such that $d(s) > 0$ and $d(s + 1) \leq 0$. Let $L$ be the cone of $f_1 \cdots f_s : M_s \to N$. We have a distinguished triangle

$$M_s \to N \to L \rightsquigarrow$$

with $M_s \in \mathcal{T}^{\leq 0}$ and $L \in \mathcal{T}^{> 0}$.

We have a characterization of $\mathcal{T}^{\geq 0}$ and $\mathcal{T}^{\leq 0}$:

**Proposition 3.5.** Let $N \in \mathcal{T}$. Then, $N \in \mathcal{T}^{\leq 0}$ if and only if $\text{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$.

Similarly, $N \in \mathcal{T}^{\geq 0}$ if and only if $\text{Hom}(S[i], N) = 0$ for $S \in \mathcal{S}$ and $i > 0$.

**Proof.** We have $\text{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$, if $N \in \mathcal{S}[r]$ with $r \geq 0$. By induction, it follows that if $N \in \mathcal{T}^{\leq 0}$, then $\text{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$.

Assume now $\text{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$. Pick a filtration of $N$ as in Lemma 3.3. Then, $d(1) \leq 0$, hence $d(i) \leq 0$ for all $i$ and $N \in \mathcal{T}^{\leq 0}$.

The other case is similar.

Note that the heart $\mathcal{A}$ of the $t$-structure is artinian and noetherian. Its set of simple objects is $\mathcal{S}$.

**Remark 3.6.** Assume $\mathcal{T}$ can be generated by a finite set of objects. Then, there is a finite subcategory $\mathcal{S}'$ of $\mathcal{S}$ generating $\mathcal{T}$. It follows immediately from condition (i) that $\mathcal{S} = \mathcal{S}'$. So, $\mathcal{S}$ has only finitely many objects.

3.2.2. In §3.2.2, we assume $\mathcal{T} = D^b(A)$ where $A$ is a finite dimensional $k$-algebra. By Remark 3.6, $\mathcal{S}$ is finite (note that $\mathcal{T}$ is generated by the simple $A$-modules, up to isomorphism).

**Proposition 3.7.** Let $S \in \mathcal{S}$. There is a bounded complex of finitely generated injective $A$-modules $I_S(S) \in \mathcal{T}^{\geq 0}$ such that, given $T \in \mathcal{S}$ and $i \in \mathbb{Z}$, we have

$$\text{Hom}_{D^b(A)}(T, I_S(S)[i]) = \begin{cases} k & \text{for } i = 0 \text{ and } S = T \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, there is a bounded complex of finitely generated projective $A$-modules $P_S(S) \in \mathcal{T}^{\leq 0}$ such that, given $T \in \mathcal{S}$ and $i \in \mathbb{Z}$, we have

$$\text{Hom}_{D^b(A)}(P_S(S)[i], T) = \begin{cases} k & \text{for } i = 0 \text{ and } S = T \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** The construction of a complex $I_S(S)$ of $A$-modules with the Hom property is [Ri, §5] (note that the proof of [Ri, Lemma 5.4] is valid for non-symmetric algebras). It is in $\mathcal{T}^{\geq 0}$ by Proposition 3.5. Since $\bigoplus_{i \in \mathbb{Z}} \dim \text{Hom}_{D^b(A)}(V, I_S(S)[i]) = 0$ for all simple $A$-modules $V$, we deduce that $I_S(S)$ is isomorphic to a bounded complex of finitely generated injective $A$-modules.

The second case follows from the first one by passing to $A^{\text{opp}}$ and taking the $k$-duals of elements of $\mathcal{S}$.

We denote by $\tau^{> 0}$, etc... the truncation functors and $\mathcal{H}^0$ the $\mathcal{H}^0$-functor associated to the $t$-structure constructed in §3.2.1.

**Lemma 3.8.** The object $\mathcal{H}^0(I_S(S))$ of $\mathcal{A}$ is an injective hull of $S$ and $\mathcal{H}^0(P_S(S))$ is a projective cover of $S$. 
**Proof.** We have a distinguished triangle
\[ \tau^{<0}C \to C \to \tau^{>0}C \sim. \]

Let \( N \in \mathcal{A} \). We have \( \text{Hom}(N, \tau^{>0}C) = 0 \) and \( \text{Hom}(N, \tau^{<0}C) = 0 \), so we deduce that \( \text{Hom}(N, \tau^{<0}C) = 0 \). It follows that \( \text{Ext}^1_A(N, \tau^{<0}C) = 0 \), hence \( \tau^{<0}C \) is injective. Since \( \text{Hom}(T, (\tau^{<0}C)[-1]) = 0 \), we have \( \text{Hom}(T, \tau^{<0}C) \sim \text{Hom}(T, C) = k^{\text{stab}} \) for \( T \in \mathcal{S} \). So \( \tau^{<0}C \) is an injective hull of \( S \). The projective case is similar. \( \square \)

Let us consider the finite dimensional differential graded algebra
\[ B = \text{End}_A\left( \bigoplus_S P_S(S) \right) = \bigoplus_S \text{End}_A\left( \bigoplus_P P_S(S) \right). \]

Denote by \( D^b(B) \) the derived category of finite dimensional differential graded \( B \)-modules.

**Theorem 3.9.** We have \( H^i(B) = 0 \) for \( i > 0 \) and for \( i \ll 0 \). We have \( H^0(B) \text{-mod} \simeq \mathcal{A} \) and \( D^b(B) \simeq D^b(A) \).

**Proof.** Let \( N \in \mathcal{T} \) and consider a filtration of \( N \) as in Lemma 3.3. Take \( S \in \mathcal{S} \) such that \( S[i] \) is isomorphic to the cone of \( M_i \to M_{i-1} \). Then, \( \text{Hom}(P_S(S)[i], N) \neq 0 \). It follows that the right orthogonal category of \( \{P_S(S)[i]\}_{S \in \mathcal{S}, i \in \mathbb{Z}} \) is zero. Since the \( P_S(S) \) are perfect, it follows that \( \bigoplus_S P_S(S) \) generates the category of perfect complexes of \( A \)-modules as a triangulated category closed under taking direct summands [Nee, Lemma 2.2]. The functor \( \text{Hom}_A^\bullet\left( \bigoplus_S P_S(S), - \right) \) gives an equivalence \( D^b(A) \sim D^b(B) \) [Ke, Theorem 4.3].

Let \( C = \bigoplus_{S \in \mathcal{S}} P_S(S) \) and \( N = \tau^{<0}C \). We have a distinguished triangle \( \tau^{<0}C \to C \to N \sim. \)

We have \( \text{Hom}(\tau^{<0}C, N[i]) = 0 \) for \( i \leq 0 \). We deduce that the canonical morphism \( \text{Hom}(N, N) \to \text{Hom}(C, N) \) is an isomorphism. We have \( \text{Hom}(C, (\tau^{<0}C)[i]) = 0 \) for \( i \geq 0 \) since \( \tau^{<0}C \) is filtered by objects in \( S[d], d > 0 \) (cf Proposition 3.7). It follows that the canonical morphism \( \text{Hom}(C, C) \to \text{Hom}(C, N) \) is an isomorphism.

This shows that the canonical morphism \( \text{End}(C) \to \text{End}(\tau^{<0}C) \) is an isomorphism. By Lemma 3.8, \( \tau^{<0}C \) is a progenerator for \( \mathcal{A} \). So \( H^0(B) \text{-mod} \simeq \mathcal{A} \).

Note that \( H^i(B) = 0 \) for \( i \ll 0 \) because \( \bigoplus_S P_S(S) \) is bounded. Since \( P_S(S) \) is filtered by objects in \( S[d] \) with \( d \geq 0 \), it follows from Proposition 3.7 that \( \text{Hom}(P_S(T), P_S(S)[i]) = 0 \) for \( i > 0 \). So, \( H^i(B) = 0 \) for \( i > 0 \). \( \square \)

The following proposition is clear.

**Proposition 3.10.** Let \( B \) be a dg-algebra with \( H^i(B) = 0 \) for \( i > 0 \) and for \( i \ll 0 \). Let \( C \) be the sub-dg-algebra of \( B \) given by \( C^i = B^i \) for \( i < 0 \), \( C^0 = \ker d^0 \) and \( C^i = 0 \) for \( i > 0 \). Then the restriction \( D(B) \to D(C) \) is an equivalence.

Let \( \mathcal{S} \) be a complete set of representatives of isomorphism classes of simple \( H^0(B) \text{-modules} \) (viewed as dg-\( C \)-modules). Then \( \mathcal{S} \) satisfies Hypothesis 1. Furthermore, \( \mathcal{A} \simeq H^0(B) \text{-mod} \).

So we have a bijection between

- the sets \( \mathcal{S} \) (up to isomorphism) satisfying Hypothesis 1
- the equivalences \( D^b(B) \to D^b(A) \) where \( B \) is a dg-algebra with \( H^i(B) = 0 \) for \( i > 0 \) and for \( i \ll 0 \) and where \( B \) is well-defined up to quasi-isomorphism and the equivalence is taken modulo self-equivalences of \( D^b(B) \) that fix the isomorphism classes of simple \( H^0(B) \text{-modules} \).
We recover a result of Al-Nofayee [Al, Theorem 4]:

**Proposition 3.11.** Assume $A$ is self-injective with Nakayama functor $\nu$. The following are equivalent

- $H^i(B) = 0$ for $i \neq 0$
- $\nu(S) = S$ (up to isomorphism).

**Proof.** Note that $S$ is stable under $\nu$ if and only if $\{P_S(S)\}_{S \in S}$ is stable under $\nu$ (up to isomorphism). Given $S, T \in S$ and $i \in \mathbb{Z}$, we have

$$\text{Hom}_{D^b(A)}(P_S(S), P_S(T)[i])^* \simeq \text{Hom}_{D^b(A)}(P_S(T), \nu(P_S(S))[-i]).$$

If $S$ is stable under $\nu$, then $\text{Hom}_{D^b(A)}(P_S(T), \nu(P_S(S))[-i]) = 0$ for $i > 0$, hence $H^{<0}(B) = 0$.

Assume now $H^{<0}(B) = 0$. Then, viewed as an object of $D^b(B)$, $\nu(P_S(S))$ is concentrated in degree 0. Since it is perfect, it is isomorphic to a projective indecomposable module, hence to $P_S(S')$ for some $S' \in S$. So, $S$ is stable under $\nu$. $\square$

We recover now the main result of [AlRi]:

**Corollary 3.12.** Let $A$ be a self-injective algebra and $B$ an algebra derived equivalent to $A$. Then $B$ is self-injective.

From Proposition 3.11, we recover [Ri, Theorem 5.1]:

**Theorem 3.13.** If $A$ is symmetric then $H^i(B) = 0$ for $i \neq 0$, i.e., there is an equivalence $D^b(A) \simeq D^b(A)$ where $S$ is the set of images of the simple objects of $A$.

**Remark 3.14.** Theorem 3.13 does not hold in general for a self-injective algebra. Take $A = k[\varepsilon]/(\varepsilon^2) \rtimes \mu_2$, where $\mu_2 = \{\pm 1\}$ acts on $k[\varepsilon]/(\varepsilon^2)$ by multiplication on $\varepsilon$. Assume $k$ does not have characteristic 2. This is a self-injective algebra which is not symmetric. The Nakayama functor swaps the two simple $A$-modules $U$ and $V$.

Let $P_U$ (resp. $P_V$) be a projective cover of $U$ (resp. $V$). Take $S = U$ and $T = P_U[1]$. Then, the set $S = \{S, T\}$ satisfies Hypothesis 1. We have $I_S(T) \simeq T$ and $I_S(S) \simeq 0 \to P_U \to P_V \to 0$, a complex with homology $V$ in degree 0 and $-1$.

The dg-algebra $B$ has homology $H^0(B)$ isomorphic to the path algebra of the quiver $\bullet \longrightarrow \bullet$, $H^{-1}(B) = k$ and $H^i(B) = 0$ for $i \neq 0, -1$.

The derived category of the hereditary algebra $H^0(B)$ is not equivalent to $D^b(A)$.

### 3.3. Graded of an abelian category.

Let $A$ be an abelian $k$-linear artinian and noetherian category with finitely many simple objects up to isomorphism and $S$ a complete set of representatives of isomorphism classes of simple objects. We assume $A$ is split, i.e., endomorphism rings of simple objects are isomorphic to $k$. Let $T = D^b(A)$.

Let $\text{gr}A$ be the category with objects the objects of $A$ and where $\text{Hom}_{\text{gr}A}(M, N)$ is the graded vector space associated to the filtration of $\text{Hom}_A(M, N)$ given by $\text{Hom}_A(M, N)^i = \{f | \text{im } f \subseteq \text{rad}^i N\}$.

Given $M$ in $A$, let $M_i = \text{rad}^i M$, $f_i : M_i \to M_{i-1}$ the inclusion, $N_0 = M/M_1$ and $\varepsilon_0 : M \to M/M_1$ the projection. This defines an object of $\mathcal{F}$.

We obtain a functor $\text{gr}A \to \mathcal{F}$.

**Proposition 3.15.** The canonical functor $\text{gr}A \to \mathcal{F}$ is an equivalence.
4. Simple generators for stable categories

4.1. From equivalences. Let \( k \) be a field and \( A \) a split self-injective \( k \)-algebra with no projective simple module.

Let \( B \) be another split self-injective \( k \)-algebra with no projective simple module, and let \( F : B\text{-}\text{stab} \xrightarrow{\sim} A\text{-}\text{stab} \) be an equivalence of triangulated categories. Let \( S' \) be a complete set of representatives of isomorphism classes of simple \( B \)-modules. For \( L \in S' \), let \( L' \) be an indecomposable \( A \)-module isomorphic to \( F(L) \) in \( A\text{-}\text{stab} \). Let \( S = \{L'\}_{L \in S'} \). Then,

(i) \( \text{Hom}_{A\text{-}\text{stab}}(S,T) = k^{d_{S,T}} \) for \( S,T \in S \)

(ii) Every object \( M \) of \( A\text{-}\text{stab} \) has a filtration \( 0 = M_r \to M_{r-1} \to \cdots \to M_1 \to M_0 = M \) such that the cone of \( M_i \to M_{i-1} \) is isomorphic to an object of \( S \).

Note that (ii) is equivalent to

(ii') Given \( M \) in \( A\text{-}\text{mod} \), there is a projective module \( P \) such that \( M \oplus P \) has a filtration \( 0 = N_r \subset N_{r-1} \subset \cdots \subset N_1 \subset N_0 = M \oplus P \) with the property that \( N_i/N_{i-1} \) is isomorphic (in \( A\text{-}\text{mod} \)) to an object of \( S \).

Linckelmann has shown the following [Li, Theorem 2.1 (iii)]:

**Proposition 4.1.** Assume that \( F \) is induced by an exact functor \( B\text{-}\text{mod} \to A\text{-}\text{mod} \). If \( S \) consists of simple modules, then there is a direct summand of \( F \) that is an equivalence \( B\text{-}\text{mod} \xrightarrow{\sim} A\text{-}\text{mod} \).

We deduce :

**Corollary 4.2.** Let \( B_1, B_2 \) be split self-injective algebras with no projective simple modules and \( G_t : B_t\text{-}\text{mod} \to A\text{-}\text{mod} \) exact functors inducing stable equivalences. Assume \( S_1 = S_2 \) (up to isomorphism). Then, \( B_1 \) and \( B_2 \) are Morita equivalent.

So, if we assume in addition that \( F \) comes from an exact functor \( G \) between module categories, then \( B \) is determined by \( S \), up to Morita equivalence.

The functor \( G \) is isomorphic to \( X \otimes_B - \) where \( X \) is an \( (A,B) \)-bimodule. We can (and will) choose \( G \) so that \( X \) has no non-zero projective direct summand. Then, \( G(L) \) is indecomposable for \( L \) simple [Li, Theorem 2.1 (ii)], so \( S = \{G(L)\}_{L \in S'} \), up to isomorphism.

**Proposition 4.3.** An \( A \)-module \( M \) is in the image of \( G \) if and only if there is a filtration \( 0 = M_r \subset M_{r-1} \subset \cdots \subset M_1 \subset M_0 = M \) such that \( M_i/M_{i-1} \) is isomorphic to an object of \( S \).
We have $M$ projective and a morphism between $L$ and $N$, which implies the existence of an isomorphism $\zeta \in \text{Ext}^1_A(G(L), G(N))$. We have an isomorphism $\text{Ext}^1_B(L, N) \cong \text{Ext}^1_A(G(L), G(N))$ and we take $\zeta'$ to be the inverse image of $\zeta$ under this isomorphism. This gives an exact sequence $0 \rightarrow N \rightarrow M' \rightarrow L \rightarrow 0$, and hence an exact sequence $0 \rightarrow G(N) \rightarrow G(M') \rightarrow G(L) \rightarrow 0$ with class $\zeta$. It follows that $M \cong G(M')$ and we are done. \hfill \Box

4.2. Filtrable objects.

4.2.1. Given two $A$-modules $M$ and $N$, we write $M \sim N$ to denote the existence of an isomorphism between $M$ and $N$ in $A$-stab. Given $f, g \in \text{Hom}_A(M, N)$, we write $f \sim g$ if $f - g$ is a projective map.

Lemma 4.4. Let $f, f' : M \rightarrow N$ be two surjective maps with $f \sim g$. Then there is $\sigma \in \text{Aut}_A(M)$ with $f' = f \sigma$ and $\sigma \sim \text{id}_M$.

Similarly, let $f, f' : N \rightarrow M$ be two injective maps with $f \sim g$. Then there is $\sigma \in \text{Aut}_A(M)$ with $f' = f \sigma$ and $\sigma \sim \text{id}_M$.

Proof. Let $L = \ker f$ and $L' = \ker f'$. Let $L = L_0 \oplus P$ and $L' = L_0' \oplus P'$ with $P, P'$ projective and $L_0, L_0'$ without non-zero projective direct summands. We have an isomorphism $\tilde{\alpha}_0 \in \text{Hom}_{A, \text{stab}}(L_0, L_0')$ in $A$-stab giving rise to an isomorphism of distinguished triangles in $A$-stab

$$
\begin{array}{cccc}
L_0 & \rightarrow & M & \rightarrow & N & \rightarrow & \Omega^{-1}L_0 \\
\downarrow \tilde{\alpha}_0 & \sim & \sim & \sim & \Omega^{-1}(\tilde{\alpha}_0) & \sim \\
L_0' & \rightarrow & M & \rightarrow & N & \rightarrow & \Omega^{-1}L_0'
\end{array}
$$

Let $\alpha_0 \in \text{Hom}_A(L_0, L_0')$ lifting $\tilde{\alpha}_0$. This is an isomorphism. There is now a commutative diagram of $A$-modules, where the exact rows come from the elements of $\text{Ext}^1_A(N, L_0)$ and $\text{Ext}^1_A(N, L_0')$ defined above:

$$
\begin{array}{cccc}
0 & \rightarrow & L_0 & \rightarrow & M_0 & \rightarrow & N & \rightarrow & 0 \\
\alpha_0 & \sim & \sigma_0 & \sim & \sim & \sim & \sim & \sim \\
0 & \rightarrow & L_0' & \rightarrow & M_0' & \rightarrow & N & \rightarrow & 0
\end{array}
$$

We have $M \simeq M_0 \oplus P \simeq M_0' \oplus P'$, hence $P \simeq P'$. Let $\alpha : L \rightarrow L'$ extending $\alpha_0$. Then there is $\sigma : M \rightarrow M$ making the following diagram commute:

$$
\begin{array}{cccc}
0 & \rightarrow & L & \rightarrow & M & \rightarrow & N & \rightarrow & 0 \\
\alpha & \sim & \sigma & \sim & \sim & \sim & \sim & \sim \\
0 & \rightarrow & L & \rightarrow & M & \rightarrow & N & \rightarrow & 0
\end{array}
$$

and we are done.

The second part of the lemma has a similar proof — it can also be deduced from the first part by duality. \hfill \Box
4.2.2.

**Hypothesis 2.** Let $\mathcal{S}$ be a finite set of indecomposable finitely generated $A$-modules such that $\text{Hom}_{A\text{-stab}}(S, T) = k^{\mathcal{S} \times \mathcal{T}}$ for $S, T \in \mathcal{S}$.

An $\mathcal{S}$-filtration for an $A$-module $M$ is a filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ such that $M_i = M_i/M_{i+1}$ is in add($\mathcal{S}$) for $0 \leq i \leq r - 1$.

We say that $M$ is filtrable if it admits an $\mathcal{S}$-filtration.

**Lemma 4.5.** Let $M$ be a non-projective filtrable $A$-module. Then there is $S \in \mathcal{S}$ such that $\text{Hom}_{A\text{-stab}}(M, S) \neq 0$ (resp. such that $\text{Hom}_{A\text{-stab}}(S, M) \neq 0$).

**Proof.** Assume $\text{Hom}_{A\text{-stab}}(M, S) = 0$ for all $S \in \mathcal{S}$. Since $M$ is filtrable, it follows that $\text{End}_{A\text{-stab}}(M) = 0$, and hence $M$ is projective, which is not true. The second case is similar. \qed

**Lemma 4.6.** Let $M$ be a filtrable module and $S \in \mathcal{S}$. Given $f : M \to S$ non-projective, there is $g : M \to S$ surjective with filtrable kernel such that $f \sim g$. Similarly, given $f : S \to M$ non-projective, there is $g : S \to M$ injective with filtrable cokernel such that $f \sim g$.

**Proof.** We proceed by induction on the number of terms in a filtration of $M$. The result is clear if $M \in \mathcal{S}$.

Let $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} T \to 0$ be an exact sequence with $T \in \mathcal{S}$ and $N$ filtrable.

Assume first $f \alpha : N \to S$ is projective. Then there is $p : M \to S$ projective and $g : T \to S$ with $f - p = g \beta$. Since $g$ is not projective, it is an isomorphism. Consequently, $f - p$ is surjective and its kernel is isomorphic to $N$ by Lemma 4.4, so we are done.

Assume now $f \alpha : N \to S$ is not projective. By induction, there is $g : N \to S$ projective such that $f \alpha + g$ is surjective with filtrable kernel $N'$. Since $\alpha : N \to M$ is injective, there is a projective map $p : M \to S$ with $q = p \alpha$. Now, we have an exact sequence $0 \to N/N' \xrightarrow{\alpha\bar{\alpha}} M/\alpha(N') \to T \to 0$ and a non-projective surjection $f + p : M/\alpha(N') \to S$. Since $(f + p)\bar{\alpha} : N/N' \to S$ is an isomorphism, it follows that the kernel of the map $M/\alpha(N') \to S$ is isomorphic to $T$. Since $N'$ is filtrable, it follows that ker$(f + p)$ is filtrable and we are done. The second assertion follows by duality. \qed

From Lemmas 4.4 and 4.6, we deduce:

**Lemma 4.7.** Let $S \in \mathcal{S}$ and let $M$ be a filtrable module.

If $f : M \to S$ is a surjective and non-projective map, then ker$f$ is filtrable.

Similarly, if $g : S \to M$ is injective and non-projective, then coker$g$ is filtrable.

From Lemmas 4.5 and 4.6, we deduce:

**Lemma 4.8.** Let $M$ be filtrable non-projective. Then there is a submodule $S$ of $M$, with $S \in \mathcal{S}$, such that $M/S$ is filtrable and the inclusion $S \to M$ is not projective. Similarly, there is a filtrable submodule $N$ of $M$ such that $M/N \in \mathcal{S}$ and $M \to M/N$ is not projective.

**Proposition 4.9.** Let $M$ be an $A$-module with a decomposition $M \sim M'_1 \oplus M'_2$ in the stable category. If $M$ is filtrable then there is a decomposition $M = M_1 \oplus M_2$ such that $M_i$ is filtrable and $M_i \sim M'_i$.
Proof. We can assume $M$ is not projective, for otherwise the proposition is trivial. We prove the proposition by induction on the dimension of $M$.

Let $M = T_1 \oplus T_2 \oplus P$ with $P$ projective, $T_i$ without non-zero projective direct summand and $T_i \sim M'_i$. Denote by $\pi : M \to T_1$ the projection.

By Lemma 4.5, there is $S \in \mathcal{S}$ such that $\text{Hom}_{A_{\text{stab}}}(M, S) \neq 0$. Hence, $\text{Hom}_{A_{\text{stab}}}(T_i, S) \neq 0$ for $i = 1$ or $i = 2$. Assume for instance $i = 1$. Pick a non-projective map $\alpha : T_1 \to S$. So, $\alpha \pi : M \to S$ is not projective. By Lemma 4.6, there is a surjective map $\beta : M \to S$ with $\beta \sim \alpha \pi$ and $N = \ker \beta$ filtrable. Then $N \sim L \oplus T_2$, where $L$ is the kernel of $\alpha + p : T_1 \oplus P \to S$ and $p : P \to S$ is a projective cover of $S$. By induction, we have $N = N_1 \oplus N_2$ with $N_i$ filtrable and $N_1 \sim L$, $N_2 \sim T_2$. Now, the map $S \to L[1]$ gives a map $S \to N_1[1]$ (in $A$-stab). Let $M_1$ be the extension of $S$ by $N_1$ corresponding to that map. Then $M \simeq M_1 \oplus N_2$, the modules $M_1$ and $N_2$ are filtrable, $M_1 \sim M'_1$, and $N_2 \sim M'_2$. \hfill $\Box$

Let $M$ be a filtrable module. We say that $M$ has no projective remainder if there is no direct sum decomposition $M = N \oplus P$ with $P \neq 0$ projective and $N$ filtrable.

Lemma 4.10. Let $M$ be a filtrable module with no projective remainder and let $S \in \mathcal{S}$.

For $f : M \to S$ surjective, $\ker f$ is filtrable if and only if $f$ is non-projective.

For $f : S \to M$ injective, $\text{coker } f$ is filtrable if and only if $f$ is non-projective.

Proof. Assume $f$ is projective. Then there is a decomposition $M = N \oplus P$ and $f = (0, g)$ with $P$ projective. Now, $\ker f = N \oplus \ker g$. If $\ker f$ is filtrable, then it follows from Lemma 4.9 that $M$ has a non-zero projective submodule whose quotient is filtrable.

The converse is given by Lemma 4.7. The second part of the Lemma has a similar proof. \hfill $\Box$

Lemma 4.11. Let $M = M_0 \oplus M_1$ with $M$ and $M_0$ filtrable and such that $M_0$ has no projective remainder. Then $M_1$ is filtrable.

Proof. We proceed by induction on $\text{dim } M_0$ — the result is clear for $M_0 = 0$. Assume $M_0 \neq 0$. Let $f : M_0 \to S$ be a surjection with $S \in \mathcal{S}$ and $\ker f$ filtrable. By Lemma 4.10, $f$ is not projective. Then $f' : M \to M_0 \to \to S$ is a non-projective surjection. By Lemma 4.7, $\ker f'$ is filtrable. We have $\ker f' = \ker f \oplus M_1$ and we are done. \hfill $\Box$

4.2.3. We now turn to filtrations by objects in $\text{add}(\mathcal{S})$.

Lemma 4.12. Let $M$ be a filtrable module and $N$ a filtrable submodule of $M$ such that $M/N \in \text{add } \mathcal{S}$. Then, $N$ is minimal with these properties if and only if $N$ has no projective remainder and the canonical map $\text{Hom}_{A_{\text{stab}}}(M/N, S) \to \text{Hom}_{A_{\text{stab}}}(M, S)$ is surjective for every $S \in \mathcal{S}$.

Proof. Let $N$ be a minimal filtrable submodule of $M$ such that $M/N \in \text{add } \mathcal{S}$. Denote by $i : N \to M$ the injection and $p : M \to M/N$ the quotient map.

Let $S \in \mathcal{S}$. Fix $f_1, \ldots, f_r : M/N \to S$ such that $\sum_i f_i : M/N \to S^r$ is surjective and ker $\sum_i f_i$ has no direct summand isomorphic to $S$. Let $T$ be the subspace of $\text{Hom}_{A_{\text{stab}}}(M, S)$ generated by $f_1p, \ldots, f_rp$. Assume this is a proper subspace, so there is $f' : M \to S$ whose image in $\text{Hom}_{A_{\text{stab}}}(M, S)$ is not in $T$. Then $f'i : N \to S$ is not projective, hence there is a projective map $q : N \to S$ such that $f'i + q$ is surjective and has filtrable kernel $N'$ (Lemma 4.6). There is $q' : M \to S$ projective such that $q = q'i$. Now, $M/N' \simeq M/N \oplus S$ and this contradicts the minimality of $N$. It follows that the canonical map $\text{Hom}_{A_{\text{stab}}}(M/N, S) \to \text{Hom}_{A_{\text{stab}}}(M, S)$ is surjective. Assume $N = N' \oplus P$ with $N'$ filtrable with no projective remainder and $P$ projective.
By Lemma 4.11, $P$ is filtrable. We have $M/N' \simeq M/N \oplus P$. Since $M/N$ is a maximal quotient of $M$ in $\text{add}(S)$ and $P$ is filtrable, it follows that $P = 0$.

Conversely, take $f : N \to S$ surjective with filtrable kernel such that the extension of $M/N$ by $S$ splits. Then $f$ lifts to $M \to S$ and it is not projective by Lemma 4.10. This contradicts the surjectivity of $\text{Hom}_{A-\text{stab}}(M/N, S) \to \text{Hom}_{A-\text{stab}}(M, S)$. Consequently, $N$ is minimal. \hfill $\Box$

**Lemma 4.13.** Let $M$ be a filtrable $A$-module with no projective remainder.

Let $f : M \to L$ be a surjection with $L \in \text{add}S$. Then $\ker f$ is filtrable if and only if the canonical map $\text{Hom}_{A-\text{stab}}(L, S) \to \text{Hom}_{A-\text{stab}}(M, S)$ is injective for all $S \in S$.

**Proof.** Note that the canonical map $\text{Hom}_{A-\text{stab}}(L, S) \to \text{Hom}_{A-\text{stab}}(M, S)$ is injective if and only if, given $p : L \to S$ surjective with $S \in S$, $pf$ is not projective.

Assume $\ker f$ is filtrable. Let $p : L \to S$ be a surjective map with $S \in S$. Then ker $pf$ is filtrable, hence $pf$ is not projective (Lemma 4.10).

Let us now prove the converse by induction on the dimension of $M$. Assume that given $p : L \to S$ surjective with $S \in S$, then $pf$ is not projective. Pick $p : L \to S$ surjective and let $L' = \ker p$. Let $M' = \ker pf$. Then $f$ induces a surjection $f' : M' \to L'$ and we have $L' \in \text{add}S$ (since $p$ is split). Let $p' : L' \to T$ be a surjective map with $T \in S$. Fix a left inverse $\sigma : L \to L'$ to the inclusion $L' \to L$.

If $S \neq T$, then $\text{Hom}_{A-\text{stab}}(S, T) = 0$, and hence $p'\sigma f$ doesn’t factor through $S$ in the stable category. On the other hand, if $S = T$ then $pf$ and $p'\sigma f$ define linearly independent elements of $\text{Hom}_{A-\text{stab}}(M, S)$. Consequently, $p'\sigma f$ doesn’t factor through $S$ in the stable category. It follows that $p'f'$ is not projective. By Lemma 4.7, $M'$ is filtrable. By induction, it follows that ker $f'$ is filtrable and we are done. \hfill $\Box$

**Proposition 4.14.** Let $M$ be a filtrable $A$-module with no projective remainder.

Let $N$ be a minimal filtrable submodule of $M$ such that $M/N \in \text{add}S$. Then there is an isomorphism

$$M/N \simeq \bigoplus_{S \in S} S \otimes \text{Hom}_{A-\text{stab}}(M, S)$$

that induces the canonical map $M \to \bigoplus_{S \in S} S \otimes \text{Hom}_{A-\text{stab}}(M, S)$ in the stable category.

Given $\tau \in \text{Aut}(N)$ such that $\tau \sim \text{id}_N$, there is $\sigma \in \text{Aut}(M)$ with $\sigma \sim \text{id}_M$ and $\sigma|_N = \tau$.

Let $N'$ be a minimal filtrable submodule of $M$ such that $M/N' \in \text{add}S$. Then there is $\sigma \in \text{Aut}(M)$ such that $N' = \sigma(N)$ and $\sigma \sim \text{id}_M$. 
Proposition 4.15. Let $M$ be a filtrable projective module. We prove this lemma by induction on the dimension of $M$.

Proof. The first part of the proposition follows from Lemmas 4.12 and 4.13.

Let $\tau \in \text{Aut}(N)$ such that $\tau = \text{id}_N + p$ with $p : N \to N$ projective. Then there is a projective map $q : M \to N$ with $p = qi$. Let $\sigma = \text{id}_M + q$. Then $\sigma|_N = \tau$. Now, we have a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0 \\
& & \tau \sim & \sigma \downarrow & \text{id} \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0
\end{array}
$$

and hence $\sigma$ is an automorphism of $M$.

Let $N'$ be a minimal filtrable submodule of $M$ such that $M/N' \in \text{add} \mathcal{S}$. Then we have shown that $M/N \xrightarrow{\sim} M/N'$ and that via such an isomorphism, the maps $M \to M/N$ and $M \to M/N'$ are stably equal. Now, Lemma 4.4 shows there is $\sigma \in \text{Aut}(M)$ with $N' = \sigma(N)$ and $\sigma \sim \text{id}_M$. \hfill $\Box$

Let $M$ be filtrable. An $\mathcal{S}$-radical filtration of $M$ is a filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ such that $M_i$ is a minimal filtrable submodule of $M_{i-1}$ with $M_{i-1}/M_i \in \text{add} \mathcal{S}$.

Proposition 4.15. Let $M$ be a filtrable $A$-module with no projective remainder. Let $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ and $0 = M'_r \subseteq M'_{r-1} \subseteq \cdots \subseteq M'_0 = M$ be two $\mathcal{S}$-radical filtrations of $M$. Then, $r = r'$ and there is an automorphism of $M$ that swaps the two filtrations and that is stably the identity.

Proof. We prove this lemma by induction on the dimension of $M$. By Proposition 4.14, there is $\sigma \in \text{Aut}(M)$ such that $\sigma(M'_i) = M_i$ and $\sigma \sim \text{id}_M$. Now, by induction, we have $r = r'$ and there is $\tau \in \text{Aut}(M_1)$ such that $\tau \sigma(M'_i) = M_i$ for $i > 0$ and $\tau \sim \text{id}_{M_1}$. By Proposition 4.14, there is $\tau' \in \text{Aut}(M)$ such that $\tau'|_{M_1} = \tau$ and $\tau' \sim \text{id}_{M_1}$. Now, $\tau' \sigma$ sends $M'_i$ onto $M_i$. \hfill $\Box$

Remark 4.16. A filtrable projective module can have two $\mathcal{S}$-radical filtrations with non-isomorphic layers.

Consider $A = k\mathfrak{A}_4$, the group algebra of the alternating group of degree 4 and assume $k$ has characteristic 2 and contains a cubic root of 1. Let $B$ be the principal block of $k\mathfrak{A}_5$. Then, the restriction functor is a stable equivalence between $B$ and $A$. Let $\mathcal{S}$ be the set of images of the simple $B$-modules. Denote by $k$ the trivial $A$-module and by $k_+, k_-$ the non-trivial simple $A$-modules. Then $\mathcal{S} = \{k, S_+, S_-\}$ where $S_+$ is a non-trivial extension of $k_-$ by $k_+$. Let $P$ and $P'$ be the two projective indecomposable $B$-modules that don’t have $k$ as a quotient. Then $\text{Res}_{A_4} P \simeq \text{Res}_{A_4} P'$. This projective module has two $\mathcal{S}$-radical filtrations with non-isomorphic layers : one coming from the radical filtration of $P$ and one coming from the radical filtration of $P'$.

While $\mathcal{S}$-radical filtrations are not unique in general for filtrable modules with a projective remainder, there are some cases where uniqueness still holds :

Proposition 4.17. Assume $A$ is a symmetric algebra. Let $0 \to S \to M \to T \to 0$ and $0 \to S' \to M \to T' \to 0$ be two exact sequences with $S, S', T, T' \in \mathcal{S}$. Assume that the sequences don’t both split. Then there is an automorphism of $M$ swapping the two exact sequences.

Proof. If $M$ is non-projective, then this is a consequence of Proposition 4.14.
Assume $M$ is projective. Since $A$ is symmetric, we have a non-projective map $T \simeq \Omega^{-1}S \to S$. It follows that $S = T$. Similarly, $T' = S'$. We have exact sequences
\[ 0 \to \text{Hom}(S', S) \to \text{Hom}(S', M) \to \text{Hom}(S', S) \to \text{Ext}^1(S', S) \to 0 \]
\[ 0 \to \text{Hom}(S', S') \to \text{Hom}(S', M) \to \text{Hom}(S', S') \to \text{Ext}^1(S', S') \to 0 \]
We have $\Omega^{-1}S' \simeq S'$, and hence $\dim \text{Ext}^1(S', S') = 1$. Consequently, $\dim \text{Hom}(S', M)$ is an odd integer. It follows that $\text{Ext}^1(S', S) \neq 0$, hence $\text{Hom}_{A\text{-stab}}(S', S) \neq 0$, so $S' = S$ and we are done by Lemma 4.4.

Lemma 4.18. Let $0 = M_r \subset M_{r-1} \subset \cdots \subset M_0 = M$ be a filtration of $M$ with $M_{i-1}/M_i \in \text{add} S$.

(i) If $M$ has no projective remainder, then $M_i$ has no projective remainder, for all $i$.

(ii) If the filtration is an $S$-radical filtration, then $M_i$ has no projective remainder for $i \geq 1$.

Proof. Consider an exact sequence $0 \to N \oplus P \to M \to L \to 0$ of filtrable modules with $P$ projective and $N$ filtrable. Then there is an extension $M'$ of $L$ by $N$ such that $M = M' \oplus P$ and $M'$ is filtrable. The first part of the lemma follows.

Assume now the filtration is an $S$-radical filtration. Assume for some $i \geq 1$, we have $M_i = N \oplus P$ with $N$ filtrable with no projective remainder and $P$ projective and filtrable (Lemma 4.11). Then, $M = M' \oplus P$ with $P$ filtrable by (i). There is an exact sequence $0 \to L \to P \to S \to 0$ with $S \in S$ and $L$ filtrable. Now, the canonical surjection $M' \oplus P \to M/M_1 \oplus S$ has filtrable kernel and this contradicts the minimality of $M_1$.

Proposition 4.19. Let $M_1$ and $M_2$ be two filtrable $A$-modules with no projective remainder. If $M_1 \sim M_2$, then $M_1 \simeq M_2$.

Proof. We prove the proposition by induction on $\min(\dim M_1, \dim M_2)$. Fix an isomorphism $\phi$ from $M_2$ to $M_1$ in the stable category. Let $X = \bigoplus_{S \in S} S \otimes \text{Hom}_{A\text{-stab}}(M_1, S)$ and $g_1 \in \text{Hom}_{A\text{-stab}}(M_1, X)$ be the canonical map. Let $g_2 = g_1 \phi$. By Propositions 4.14 and 4.15, there are exact sequences
\[ 0 \to N_1 \to M_1 \xrightarrow{f_1} X \to 0 \quad \text{and} \quad 0 \to N_2 \to M_2 \xrightarrow{f_2} X \to 0 \]
with the image of $f_i$ in the stable category equal to $g_i$. So, there is an isomorphism from $N_2$ to $N_1$ in the stable category compatible with $\phi$. By Lemma 4.18, $N_1$ and $N_2$ have no projective remainder. By induction, we deduce that there is an isomorphism $N_2 \simeq N_1$ lifting the stable isomorphism. So, $M_1$ and $M_2$ are extensions of isomorphic modules, with the same class in $\text{Ext}^1$, hence are isomorphic.

4.3. Generators and reconstruction.

4.3.1. We assume from now on that

Hypothesis 3. $S$ satisfies Hypothesis 2 and given $M \in A\text{-mod}$, there is a projective $A$-module $P$ such that $M \oplus P$ is filtrable.

Proposition 4.20. Let $S \in S$. Let $P_S \to S$ be a projective cover of $S$ and $P$ minimal projective such that $\Omega S \oplus P$ is filtrable. Let $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = \Omega S \oplus P$ be an $S$-radical filtration.

Then $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 \subseteq P_S \oplus P$ is an $S$-radical filtration.

If $A$ is symmetric, then $M_{r-1} \simeq S$. 
Proof. Let \( f_1 : P_S \to S \) be a surjective map and \( f = (f_1, 0) : P_S \oplus P \to S \). Let \( T \in \mathcal{S} \) and \( g : P_S \oplus P \to T \) such that we have an exact sequence \( 0 \to L \to P_S \oplus P \xrightarrow{f+g} S \oplus T \to 0 \) with \( L \) filtrable.

We have a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & L & \longrightarrow & P_S \oplus P & \xrightarrow{f+g} & S \oplus T & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & \Omega S \oplus P & \xrightarrow{(0,\text{id})} & T & \longrightarrow & 0
\end{array}
\]

The surjection \( \Omega S \oplus P \to T \) is projective and has filtrable kernel. From Lemma 4.10, we get a contradiction to the minimality of \( P \). It follows that \( \Omega S \oplus P \) is a minimal submodule of \( P_S \oplus P \) such that the quotient is in \( \text{add} \mathcal{S} \).

We have \( \text{Hom}_{A\text{-stab}}(T, \Omega S) \cong \text{Hom}_{A\text{-stab}}(S, T)^* \), since \( A \) is symmetric. Now, \( \text{Hom}_{A\text{-stab}}(M_{i-1}, \Omega S \oplus P) \neq 0 \) by Lemma 4.10. The second part of the proposition follows. \( \square \)

Let \( M \) and \( N \) be two \( A \)-modules with filtrations \( 0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M \), \( 0 = N_s \subseteq N_{s-1} \subseteq \cdots \subseteq N_0 = N \). Let \( \text{Hom}_A^f(M, N) \) be the subspace of \( \text{Hom}_A(M, N) \) of filtered maps (i.e., those \( g \) such that \( g(M_i) \subseteq N_i \)). We put \( \bar{M}_i = M_i/M_{i+1} \). We denote by \( \phi_i \) the composition of canonical maps \( \phi_i : \text{Hom}_A^f(M, N) \to \text{Hom}_A(\bar{M}_i, \bar{N}_i) \to \text{Hom}_{A\text{-stab}}(\bar{M}_i, \bar{N}_i) \).

We view \( N' = N_i \) as a filtered module with the induced filtration \( 0 = N'_{s-i} \subseteq N'_{s-i-1} = N_{s-i} \subseteq \cdots \subseteq N_i = N_{i+1} \subseteq N'_0 = N' \).

**Lemma 4.21.** Let \( M \) be a filtrable \( A \)-module with an \( \mathcal{S} \)-radical filtration and \( N \) be a filtrable \( A \)-module with an \( \mathcal{S} \)-filtration. Let \( f \in \text{Hom}_A^f(M, N) \) with \( \phi_0(f) = 0 \). Then \( \phi_i(f) = 0 \) for all \( i \).

**Proof.** The map \( \bar{f}_0 : \bar{M}_0 \to \bar{N}_0 \) induced by \( f \) is projective. So there is a projective module \( P \) and a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow & & \downarrow \\
\bar{M}_0 & \xrightarrow{\bar{f}_0} & \bar{N}_0
\end{array}
\]

Let \( p \) be the composition \( p : M \to \bar{M}_0 \to P \to N \). Then \( f - p \sim f \), \( f - p \) and \( f \) have the same restriction to \( M_1 \), and \( (f - p)_0 = 0 \). Consequently it is enough to prove the lemma in the case where \( \bar{f}_0 = 0 \).

From now on, we assume \( \bar{f}_0 = 0 \). Assume the map \( \bar{f}_1 : \bar{M}_1 \to \bar{N}_1 \) is not projective. So there is \( S \in \mathcal{S} \) and a (split) surjection \( g : \bar{N}_1 \to S \) such that \( g\bar{f}_1 : M_1 \to S \) is not projective. Let \( s : S \to M_1 \) be a right inverse to \( g \), and let \( L \) be the kernel of \( g\bar{f}_1 \).

We have an exact sequence \( 0 \to L \to M/M_2 \xrightarrow{(\text{can}, g)} M_0 \oplus S \to 0 \). So the inverse image of \( L \) in \( M_1 \) is a filtrable submodule of \( M \) with quotient isomorphic to \( \bar{M}_0 \oplus S \). This contradicts the fact that \( M_1 \) is a minimal filtrable submodule of \( M \) such that \( M/M_1 \in \text{add} \mathcal{S} \). So \( \bar{f}_1 \) is projective; i.e., \( \phi_1(f) = 0 \).
We now prove by induction that \( \phi_i(f) = 0 \) for all \( i \). Assume \( \phi_d(f) = 0 \). Then, we apply the result above to the filtered modules \( M_d \) and \( N_d \) to get \( \phi_{d+1}(f) = 0 \). \( \square \)

4.3.2. We define a category \( \mathcal{G} \) as follows.
- Its objects are \( \mathcal{A} \)-modules together with a fixed \( \mathcal{S} \)-radical filtration.
- We define \( \text{Hom}_\mathcal{G}(M, N)_i \) as the image of \( \text{Hom}_\mathcal{A}(M, N)_i \) in \( \text{Hom}_{\mathcal{A} \text{-stab}}(\bar{M}_0, \bar{N}_i) \). We put \( \text{Hom}_\mathcal{G}(M, N) = \oplus_i \text{Hom}_\mathcal{G}(M, N)_i \).
- Let \( f \in \text{Hom}_\mathcal{G}(M, N)_i \) and \( g \in \text{Hom}_\mathcal{G}(L, M)_j \). Let \( \tilde{f} : M \to N_i \) be a filtered map lifting \( f \). It induces a map \( \phi_j(\tilde{f}) \in \text{Hom}_{\mathcal{A} \text{-stab}}(M_j, \bar{N}_{i+j}) \) independent of the choice of \( \tilde{f} \) (Lemma 4.21). We define the product \( fg \) to be \( \phi_j(\tilde{f}) \circ \phi_0(g) \).

Given \( \mathcal{S} \in \mathcal{S} \), let \( P_S \to S \) be a projective cover of \( \mathcal{S} \) and \( Q_S \) projective minimal such that \( \Omega S \oplus Q_S \) is filtrable. Fix a radical filtration of \( P_S \oplus Q_S \) with first term \( \Omega S \oplus Q_S \).

Let \( \mathcal{M} = \oplus_{\mathcal{S} \in \mathcal{S}} (P_S \oplus Q_S) \). This comes with an \( \mathcal{S} \)-radical filtration. We have constructed a \( \mathbb{Z}_{\geq 0} \)-graded \( k \)-algebra \( \text{End}_\mathcal{G}(M) \).

The following Lemma is clear.

**Lemma 4.22.** Let \( \mathcal{S} \) be a complete set of representatives of isomorphism classes of simple \( \mathcal{A} \)-modules. Then we have an equivalence \( \text{gr}(\mathcal{A} \text{-mod}) \xrightarrow{\sim} \mathcal{G} \). If \( \mathcal{A} \) is basic, then \( \text{End}_\mathcal{G}(M) \) is isomorphic to the graded algebra associated with the radical filtration of \( \mathcal{A} \).

We have now obtained our partial reconstruction result:

**Theorem 4.23.** Let \( \mathcal{B} \) be a selfinjective algebra with no simple projective module. Let \( \mathcal{M} \) be an \( (\mathcal{A}, \mathcal{B}) \)-bimodule inducing a stable equivalence and having no projective direct summand. Let \( \mathcal{S} = \{ M \otimes_B L \} \) where \( L \) runs over a complete set of representatives of isomorphism classes of simple \( \mathcal{B} \)-modules.

Then, there is an equivalence \( \text{gr}(\mathcal{B} \text{-mod}) \xrightarrow{\sim} \mathcal{G} \). If \( \mathcal{B} \) is basic, there is an isomorphism between the graded algebra associated with the radical filtration of \( \mathcal{B} \) and \( \text{End}_\mathcal{G}(M) \).

4.3.3. The category \( \mathcal{G} \) can be constructed directly as in §3.1, using only the stable category with its triangulated structure.

**Proposition 4.24.** Let \( \mathcal{M} \) be a module with an \( \mathcal{S} \)-filtration \( 0 = \mathcal{M}_r \subseteq \mathcal{M}_{r-1} \subseteq \cdots \subseteq \mathcal{M}_0 = \mathcal{M} \). This is an \( \mathcal{S} \)-radical filtration if and only if
- \( \text{Hom}_{\mathcal{A} \text{-stab}}(\mathcal{M}_i/\mathcal{M}_{i+1}, S) \to \text{Hom}_{\mathcal{A} \text{-stab}}(\mathcal{M}_i, S) \) is an isomorphism for all \( S \in \mathcal{S} \) and \( i > 0 \),
- \( \text{Hom}_{\mathcal{A} \text{-stab}}(\mathcal{M}_0/\mathcal{M}_1, S) \to \text{Hom}_{\mathcal{A} \text{-stab}}(\mathcal{M}_0, S) \) is surjective for all \( S \in \mathcal{S} \), and
- \( \mathcal{M}_i \) has no projective remainder for \( i > 0 \).

Assume the filtration is an \( \mathcal{S} \)-radical filtration. Then \( \mathcal{M} \) has no projective remainder if and only if \( \text{Hom}_{\mathcal{A} \text{-stab}}(\mathcal{M}_0/\mathcal{M}_1, S) \to \text{Hom}_{\mathcal{A} \text{-stab}}(\mathcal{M}_0, S) \) is an isomorphism.

**Proof.** Let \( \mathcal{M} \) be a module with an \( \mathcal{S} \)-radical filtration \( 0 = \mathcal{M}_r \subseteq \mathcal{M}_{r-1} \subseteq \cdots \subseteq \mathcal{M}_0 = \mathcal{M} \). The canonical map \( \text{Hom}_{\mathcal{A} \text{-stab}}(\mathcal{M}_i/\mathcal{M}_{i+1}, S) \to \text{Hom}_{\mathcal{A} \text{-stab}}(\mathcal{M}_i, S) \) is surjective for all \( S \in \mathcal{S} \), by Lemma 4.12. Note that \( \mathcal{M}_i \) has no projective remainder for \( i > 0 \), by Lemma 4.18. It follows that the canonical map \( \text{Hom}_{\mathcal{A} \text{-stab}}(\mathcal{M}_i/\mathcal{M}_{i+1}, S) \to \text{Hom}_{\mathcal{A} \text{-stab}}(\mathcal{M}_i, S) \) is an isomorphism for all \( S \in \mathcal{S} \) (Lemma 4.13).

Let us now prove the other implication. Since \( \mathcal{M}_i \) has no projective remainder for \( i > 0 \), it follows from Lemma 4.12 that \( 0 = \mathcal{M}_r \subseteq \mathcal{M}_{r-1} \subseteq \cdots \subseteq \mathcal{M}_1 \) is an \( \mathcal{S} \)-radical filtration of \( \mathcal{M}_1 \).
Assume the filtration is an $\mathcal{S}$-radical filtration. If $M$ has no projective remainder, then $\text{Hom}_{A\text{-stab}}(M_0/M_1, S) \to \text{Hom}_{A\text{-stab}}(M_0, S)$ is injective by Lemma 4.13.

Assume now that $\text{Hom}_{A\text{-stab}}(M/M_1, S) \to \text{Hom}_{A\text{-stab}}(M, S)$ is bijective. Assume $M = M' \oplus P$ with $M'$ filtrable and $P$ projective. We have $\text{Hom}_{A\text{-stab}}(M'/M_1, S) \cong \text{Hom}_{A\text{-stab}}(M, S) \cong \text{Hom}_{A\text{-stab}}(M', S)$. There is a surjective map $g : M' \to M/M_1$ with filtrable kernel such that the composition $M \longrightarrow M' \twoheadrightarrow M/M_1$ is equal to the canonical map $M \to M/M_1$ in the stable category, by Proposition 4.14. By Lemma 4.4, we have $M_1 \simeq \ker g \oplus P$. Since $M_1$ has no projective remainder by the first part of the proposition, we get $P = 0$, hence $M$ has no projective remainder.

Let $\mathcal{T} = A\text{-stab}$. Note that $\mathcal{S}$ is determined by its image in $\mathcal{T}$ and it satisfies Hypothesis 3 if and only if $\text{Hom}_\mathcal{T}(S, T) = k^{\delta_{st}}$ for all $S, T \in \mathcal{S}$ and every object of $\mathcal{T}$ is an iterated extension of objects of $\mathcal{S}$.

We have a functor $\mathcal{G} \to \mathcal{F} :$ it sends a module $M$ with an $\mathcal{S}$-radical filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ to $\cdots \to 0 \to M_{r-1} \to \cdots \to M_1 \to M \to M/M_1$ (cf Proposition 4.24).

**Proposition 4.25.** The canonical functor $\mathcal{G} \cong \mathcal{F}$ is an equivalence.

**Proof.** The functor is clearly fully faithful.

Start with $0 = N_r \xrightarrow{f_r} N_{r-1} \rightarrow \cdots \rightarrow N_1 \xrightarrow{f_1} N_0 \xrightarrow{\varepsilon_0} M_0$. Adding a projective direct summand to the $N_i$'s, we can lift the maps $f_i$ to maps that are injective in the module category and such that the successive quotients have no projective direct summands. So we have a filtration $0 = M'_r \subseteq M'_{r-1} \subseteq \cdots \subseteq M'_1 \subseteq M'_0$ such that $M'_r/M'_{r+1}$ is stably isomorphic to a direct sum of objects of $\mathcal{S}$. Since it has no projective summand, it is actually isomorphic to a sum of objects of $\mathcal{S}$; i.e., we have an $\mathcal{S}$-filtration. Consider $i$ maximal such that $M'_i$ has a projective remainder. Then $0 = M'_r \subseteq M'_{r-1} \subseteq \cdots \subseteq M'_1$ is an $\mathcal{S}$-radical filtration by Proposition 4.24 (first part). The second part of Proposition 4.24 shows now that $M'_1$ has no projective remainder, a contradiction. So the filtration is an $\mathcal{S}$-filtration.

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**References**


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