Appendix: Proofs of Results 2 and 3

Proof of Result 2

Result 2: Lottery tickets are not purchased outside the interval \( x_1 \in \left[ x_2^* - \frac{1-q}{q}, x_2^* + 1 \right] \).

Proof:

The value functions for not buying a lottery ticket is given by:

\[
V^{t=0}_1 = \begin{cases} 
  u(x_i) & \text{if } x_1 < x_2^* \\
  u(x_i - p) + \eta & \text{if } x_1 \geq x_2^* 
\end{cases}
\]

The expected value function for buying a ticket is given by:

\[
E[V^{t=1}_1] = q \left( u \left( x_i + \frac{1-q}{q} \right) \right) \begin{cases} 
  u \left( x_i + \frac{1-q}{q} \right) & \text{if } x_1 < x_2^* - \frac{1-q}{q} \\
  u(x_i + \frac{1-q}{q} - p) + \eta & \text{if } x_1 \geq x_2^* - \frac{1-q}{q} 
\end{cases} + (1-q) \left( u(x_i - 1 - p) + \ln \eta \right) \begin{cases} 
  u(x_i - 1) & \text{if } x_1 < x_2^* + 1 \\
  u(x_i - 1 - p) + \eta & \text{if } x_1 \geq x_2^* + 1 
\end{cases}
\]

Now consider separately the incentive to buy a lottery ticket when cash-on-hand is below the interval and above the interval.

1) When \( x_1 < x_2^* - (1-q)/q \), cash-on-hand in period 2 will be sufficiently low that even if the lottery is won, \( x_2 < x_2^* \), and so the household does not buy the indivisible good, regardless of the lottery outcome. Thus, the expected value of buying a lottery ticket becomes:

\[
E[V^{t=1}_1] = q u \left( x_i + \frac{1-q}{q} \right) + (1-q)u(x_i - 1)
\]

The value of not buying becomes: \( V^{t=0}_1 = u(x_i) \). Since the gamble is actuarially fair and utility, u, is concave, the value of not buying a lottery ticket is always greater than the expected value of buying the lottery ticket:
\[ V_{i=0}^{t=1} = u(x_t) \]
\[ \geq qu \left( x_t + \frac{1-q}{q} \right) + (1-q)u(x_t - 1) = E[V_{i=1}^{t=1}] \]

2) When \( x_t > x_2^* + 1 \), cash-on-hand in period 2 will be sufficiently high that even if the lottery is lost, \( x_2 > x_2^* \), and so the household buys the indivisible good regardless of the lottery outcome. Thus, the expected value of buying a lottery ticket becomes:
\[ E[V_{i=1}^{t=1}] = qu \left( x_t + \frac{1-q}{q} - p \right) + (1-q)u(x_t - 1 - p) + \eta \]
And the value of not buying becomes:
\[ V_{i=0}^{t=1} = u(x_t - p) + \eta \]
Since the gamble is actuarially fair and utility, \( u \), is concave, the value of not buying the lottery ticket is always greater than the value of buying the ticket.

\[ V_{i=0}^{t=1} = u(x_t - p) + \eta \]
\[ \geq qu \left( x_t - p + \frac{1-q}{q} \right) + (1-q)u(x_t - p - 1) + \eta = E[V_{i=1}^{t=1}] \]

**Proof of Result 3**

**Result 3:** There exists a region, \( x_t \in [x_1, x_1^+] \), which contains \( x_2^* \left( x_1 < x_2^* < x_1^+ \right) \), in which the agent will purchase a lottery ticket.

**Proof:**
We consider the incentive to buy a lottery ticket in the region of \( x_2^* \). Define the difference in utility from purchasing the indivisible good and not purchasing it as
\[ \delta = u(x_2 - p) + \eta - u(x_2) \]

We consider separately the incentive \( \varepsilon \) above and \( \varepsilon \) below \( x_2^* \).

1) **Below** \( x_2^* \): When

\[ x_2 = x_2^* - \varepsilon \]

and so \( \delta < 0 \)

we can write the expected value of buying a lottery ticket as:

\[ E[V_{1l}^{i=1}] = q \left( u \left( x_2^* - \varepsilon + \frac{1-q}{q} - p \right) + \eta \right) + (1-q) u \left( x_2^* - \varepsilon - 1 \right) \]

And the value of not buying a ticket as:

\[ V_{1l}^{i=0} = u \left( x_2^* - \varepsilon \right) \]

\[ = q \left( u \left( x_2^* - \varepsilon - p \right) + \eta \right) - q\delta + (1-q) u \left( x_2^* - \varepsilon \right) \]

\[ E[V_{1l}^{i=1} - V_{1l}^{i=0}] = q \left( u \left( x_2^* - \varepsilon + \frac{1-q}{q} - p \right) + \eta - u \left( x_2^* - \varepsilon - p \right) - \eta \right) \]

\[ + q\delta \]

\[ + (1-q) \left( u \left( x_2^* - \varepsilon - 1 \right) - u \left( x_2^* - \varepsilon \right) \right) \]

This is approximately equal to:

\[ E[V_{1l}^{i=1} - V_{1l}^{i=0}] = q \left( u'(x_2 - \varepsilon - p) \left( \frac{1-q}{q} \right) \right) \]

\[ + q\delta \]

\[ + (1-q) \left( -u'(x_2 - \varepsilon) \right) \]
\[ E[V^{i=1}_1 - V^{i=0}_1] = (1-q)(u'(x_2 - \varepsilon - p) - u'(x_2 - \varepsilon)) + q\delta \]
\[ = -p(1-q)u^*(x_2 - \varepsilon) + q\delta \]

As
\[ \varepsilon \to 0, x_2 \uparrow x_2^*, \delta \uparrow 0 \]

and
\[ E[V^{i=1}_1 - V^{i=0}_1] > 0 \]

2) Above \( x_2^* \): When
\[ x_2 = x_2^* + \varepsilon \]

and so \( \delta > 0 \)

\[ E[V^{i=1}_1] = q\left(u\left(x_2^* + \varepsilon + \frac{1-q}{q} - p\right) + \eta\right) + (1-q)u(x_2^* + \varepsilon - 1) \]

The value of not buying a ticket is:

\[ V^{i=0}_1 = u\left(x_2^* + \varepsilon - p\right) + \eta \]
\[ = q\left(u\left(x_2^* + \varepsilon - p\right) + \eta\right) + (1-q)\left(u\left(x_2^* + \varepsilon - p\right) + \eta\right) \]
\[ = q\left(u\left(x_2^* + \varepsilon - p\right) + \eta\right) + (1-q)u(x_2^* + \varepsilon) + (1-q)\delta \]

\[ E[V^{i=1}_1 - V^{i=0}_1] = q\left(u\left(x_2^* + \varepsilon + \frac{1-q}{q} - p\right) + \eta - u\left(x_2^* + \varepsilon - p\right) - \eta\right) \]
\[ + (1-q)\left(u\left(x_2^* + \varepsilon - 1\right) - u(x_2^* + \varepsilon)\right) \]
\[ - (1-q)\delta \]
\[ E\left[ V_{i=1}^{l} - V_{i=0}^{l} \right] = q \left( u'(x_2 + \varepsilon - p) \left( \frac{1-q}{q} \right) \right) \]
\[ + (1-q) \left( -u'(x_2 + \varepsilon) \right) \]
\[ - (1-q) \delta \]

\[ E\left[ V_{i=1}^{l} - V_{i=0}^{l} \right] = (1-q) \left( u'(x_2 + \varepsilon - p) - u'(x_2 + \varepsilon) \right) - (1-q) \delta \]
\[ = -p(1-q)u''(x_2 + \varepsilon) - (1-q) \delta \]

As
\[ \varepsilon \to 0, \quad x_2 \downarrow x_2^*, \quad \delta \downarrow 0 \]

and
\[ E\left[ V_{i=1}^{l} - V_{i=0}^{l} \right] > 0 \]