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THE THREE GAP THEOREM AND THE SPACE OF LATTICES

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Abstract. The three gap theorem (or Steinhaus conjecture) asserts that there are at most three distinct gap lengths in the fractional parts of the sequence $\alpha, 2\alpha, \ldots, N\alpha$, for any integer $N$ and real number $\alpha$. This statement was proved in the 1950s independently by various authors. Here we present a different approach using the space of two-dimensional Euclidean lattices.

Imagine we divide a cake by cutting a first wedge at an angle $\alpha$, then an identical second, third, and so on as illustrated in Figure 1 (left), until the remaining piece is either of the same size as the previous, or smaller. We now have a cake comprising wedges of at most two distinct sizes: the size of the original and that of the left-over wedge. Suppose we continue cutting but insist that after each cut we rotate the knife by the same angle $\alpha$ as before, see Figure 1 (right). How many different sizes of cake wedges are there after $N$ cuts? The celebrated “three gap theorem” states that for each $N$ there will be at most three! This surprising fact was understood by number theorists in the late 1950s [6, 7, 8, 9]. Various new proofs have appeared since then, with connections to continued fractions [5, 10], Riemannian geometry [1], and elementary topology [4, App. A], as well as higher-dimensional generalisations [2, 3, 11]. Our aim here is to provide a simple proof of the three gap phenomenon by exploiting the geometry of the space of two-dimensional Euclidean lattices.

Figure 1. For each given $N$, there are at most three different wedge sizes.

The standard example of a Euclidean lattice in $\mathbb{R}^2$ is the square lattice $\mathbb{Z}^2$. We can generate any other Euclidean lattice $\mathcal{L}$ in $\mathbb{R}^2$ by applying a linear transformation to $\mathbb{Z}^2$. Writing points in $\mathbb{R}^2$ as row vectors $x = (x_1, x_2)$, we have explicitly

$$\mathcal{L} = \mathbb{Z}^2 M = \{(m, n)M \mid (m, n) \in \mathbb{Z}^2\},$$

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where $M$ is a $2 \times 2$ matrix with real coefficients. If
\begin{equation}
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det M = ad - bc \neq 0,
\end{equation}
then a basis of the lattice $\mathcal{L} = \mathbb{Z}^2 M$ is given by the linearly independent vectors
\begin{equation}
b_1 = e_1 M = (a, b), \quad b_2 = e_2 M = (c, d),
\end{equation}
where $e_1 = (1, 0)$, $e_2 = (0, 1)$ is the standard basis of $\mathbb{Z}^2$. All other bases of $\mathcal{L}$ with the same orientation can be obtained by replacing $M$ by $\gamma M$ provided $\gamma \in \Gamma = \text{SL}(2, \mathbb{Z})$, the group of matrices with integer coefficients and unit determinant. In the following we restrict our attention to lattices $\mathcal{L} = \mathbb{Z}^2 M$ whose basis vectors span a parallelogram of unit area. This means that $\det M = \pm 1$, and by reversing the orientation of a basis vector where necessary (this will not change the lattice), we can assume in fact that $\det M = 1$. Let us therefore denote by $G = \text{SL}(2, \mathbb{R})$ the group of real matrices with unit determinant. The “modular group” $\Gamma = \text{SL}(2, \mathbb{Z})$ is a discrete subgroup of $G$, and the space of lattices can in this way be identified with the coset space $\Gamma \backslash G = \{ \Gamma g \mid g \in G \}$.

In order to translate the three gap problem into the setting of lattices, let us measure all angles in units of $360^\circ$. That is, angles are parametrized by the coset space $\mathbb{R}/\mathbb{Z} = \{ x + \mathbb{Z} \mid x \in \mathbb{R} \}$ (the set of reals taken modulo one), which we can think of as the unit interval $[0, 1]$ with the endpoints 0 and 1 identified. Fix $\alpha \in \mathbb{R}/\mathbb{Z}$, and let $\xi_k = \{ ka \}$ be the fractional part of $ka$. The quantity $\xi_k$ represents the angular position of the $k$th cut. The angles of the resulting cake wedges after $N$ cuts are precisely the gaps between the elements of the sequence $(\xi_k)_{k=1}^N$ on $\mathbb{R}/\mathbb{Z}$. These gaps are, in other words, the lengths of the $N$ intervals that $\mathbb{R}/\mathbb{Z}$ is partitioned into by $(\xi_k)_{k=1}^N$.

The gap between $\xi_k$ and its next neighbor on $\mathbb{R}/\mathbb{Z}$ (this is not necessarily the nearest neighbor, as the gap to the element preceding $\xi_k$ may be the smaller one) is given by
\begin{equation}
s_{k,N} = \min\{ (\ell - k)\alpha + n \geq 0 \mid (\ell, n) \in \mathbb{Z}^2, \ 0 < \ell \leq N, \ \ell \neq k \}.
\end{equation}
The substitution $m = \ell - k$ yields
\begin{equation}
s_{k,N} = \min\{ m\alpha + n \geq 0 \mid (m, n) \in \mathbb{Z}^2, \ -k < m \leq N - k, \ m \neq 0 \}.
\end{equation}

We now claim that in fact
\begin{equation}
s_{k,N} = \min\{ m\alpha + n \geq 0 \mid (m, n) \in \mathbb{Z}^2 \setminus \{0\}, \ -k < m \leq N - k \}.
\end{equation}
To see this, we note that the minimum in (6) is taken over a larger set than that in (5), where the additional elements correspond to $m = 0$ and $n \neq 0$. For these values $\min\{ m\alpha + n \geq 0 \} = 1$, which means they do not contribute to the minimum in (6). We rewrite (6) as
\begin{equation}
s_{k,N} = \min\{ y \geq 0 \mid (x, y) \in \mathbb{Z}^2 A_1 \setminus \{0\}, \ -k < x \leq N - k \},
\end{equation}
with the matrix
\begin{equation}
A_1 = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}.
\end{equation}
The lattice $\mathbb{Z}^2 A_1$ and $s_{k,N}$ are illustrated in Figure 2.
Now take a general element $M \in G$ and $0 < t \leq 1$, and define the function $F$ by

(9) \[ F(M, t) = \min \left\{ y \geq 0 \mid (x, y) \in \mathbb{Z}^2 \setminus \{0\}, \, -t < x \leq 1 - t \right\}. \]

To see the connection of $F$ with the gap $s_{k,N}$, define

(10) \[ A_N = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N^{-1} & 0 \\ 0 & N \end{pmatrix} \in G, \]

and note that, by rescaling the set in (7), we have

(11) \[ s_{k,N} = \frac{1}{N} \min \left\{ y \geq 0 \mid (x, y) \in \mathbb{Z}^2 A_N \setminus \{0\}, \, -k < x \leq 1 - \frac{k}{N} \right\}. \]

Thus,

(12) \[ s_{k,N} = \frac{1}{N} F(A_N, \frac{k}{N}). \]

We first check $F$ is well-defined as a function on the space of lattices $\Gamma \setminus G$ (Proposition 1), and then establish that the function $t \mapsto F(M, t)$ only takes at most three values for every fixed $M \in G$ (Proposition 2). The latter implies the three gap theorem via (12).

**Proposition 1.** $F$ is well-defined as a function $\Gamma \setminus G \times (0, 1] \to \mathbb{R}_{\geq 0}$.

**Proof.** Let us begin by showing that

(13) \[ \left\{ y \geq 0 \mid (x, y) \in \mathbb{Z}^2 M \setminus \{0\}, \, -t < x \leq 1 - t \right\} \]

is nonempty for every $M \in G$, $t \in (0, 1]$. Let

(14) \[ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \]

Figure 2. Illustration of the the expression for $s_{k,N}$ in (7) (here $N = 4$, $k = 1$).
and assume first that $a = 0$. Then $c \neq 0$ and $b = -1/c$, and (13) becomes
\[
\begin{aligned}
\{ bm + dn \geq 0 \mid (m, n) \in \mathbb{Z}^2 \setminus \{0\}, -t < cn \leq 1 - t \} \supset |b|\mathbb{N},
\end{aligned}
\]
which is nonempty. If $a \neq 0$, we have
\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & ba^{-1} \\ 0 & 1 \end{pmatrix},
\]
and so (13) equals
\[
\begin{aligned}
\{ y + ba^{-1}x \geq 0 \mid (x, y) \in \mathbb{Z}^2 \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \setminus \{0\}, -t < x \leq 1 - t \}.
\end{aligned}
\]
Since $-t < x \leq 1 - t$ implies $|x| \leq 1$, the set in (17) contains the set
\[
\begin{aligned}
\{ y + ba^{-1}x \mid (x, y) \in \mathbb{Z}^2 \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \setminus \{0\}, -t < x \leq 1 - t, y \geq |ba^{-1}| \}
\end{aligned}
\]
(18) equals
\[
\begin{aligned}
\{ bm + dn \mid (m, n) \in \mathbb{Z}^2 \setminus \{0\}, -t < am + cn \leq 1 - t, n \geq |b| \}.
\end{aligned}
\]
If $c/a$ is rational, there exist $(m, n) \in \mathbb{Z}^2 \setminus \{0\}$ with $n \geq |b|$ such that $am + cn = 0$. If $c/a$ is irrational, then the set $\{ am + cn \mid (m, n) \in \mathbb{Z}^2 \setminus \{0\}, n \geq |b| \}$ is dense in $\mathbb{R}$. Therefore, in both cases, (18) is nonempty, and the minimum of (13) exists due to the discreteness of $\mathbb{Z}^2 M$.

Finally, we note that $F(\cdot, t)$ is well-defined on $\Gamma \setminus G$ since $F(M, t) = F(\gamma M, t)$ for all $M \in G, \gamma \in \Gamma$. □

The following assertion implies the classical three gap theorem; recall (12).

**Proposition 2.** For every given $M \in G$, the function $t \mapsto F(M, t)$ is piecewise constant and takes at most three distinct values. If there are three values, then the third is the sum of the first and second.

**Proof.** Among all points of the set $\mathcal{L} \setminus \{0\}$ with $\mathcal{L} = \mathbb{Z}^2 M$ in the region $A = (-1, 1) \times [0, \infty)$, let $r = (r_1, r_2)$ be a point with minimal second coordinate $r_2$. See
Let us assume \( r_2 > 0 \) (the case \( r_2 = 0 \) is treated at the end of the proof). Next let \( s = (s_1, s_2) \) be a point in \( A \cap \mathcal{L} \setminus \mathbb{Z}r \) with \( s_2 \) minimal. Then \( s_2 \geq r_2 > 0 \).

The parallelogram \( 0, r, s, r + s \) does not contain any other lattice points: if \( u \) were such a lattice point, then \( u \) or \( r + s - u \) would have second coordinate smaller than \( s_2 \), contradicting the assumed minimality of \( s_2 \). This implies that \( r, s \) form a basis of \( \mathcal{L} \).

Note that \( r_1 \) and \( s_1 \) must have opposite signs, i.e. \( r_1 s_1 < 0 \), since otherwise \( s - r \in A \) with a second coordinate that is smaller than \( s_2 \), contradicting the assumed minimality of \( s_2 \). It follows that, if we set \( \mathcal{J}_r = (0, 1] \cap (-r_1, 1 - r_1] \) and \( \mathcal{J}_s = (0, 1] \cap (-s_1, 1 - s_1] \), then one of these intervals is of the form \( (0, q] \) and the other is of the form \( (q', 1] \), for some \( q, q' \in (0, 1) \). Note that both intervals are nonempty since \( r, s \in A \) by construction, and thus \( |r_1|, |s_1| < 1 \). More explicitly,

\[
\mathcal{J}_r = \begin{cases} 
(-r_1, 1] & \text{if } -1 < r_1 \leq 0 \\
(0, 1 - r_1] & \text{if } 0 \leq r_1 < 1,
\end{cases}
\]

and similarly for \( \mathcal{J}_s \). Now in view of definition (9), we obtain

\[
F(M, t) = \begin{cases} 
\mathcal{J}_r & \text{if } t \in \mathcal{J}_r \\
\mathcal{J}_s \setminus \mathcal{J}_r & \text{if } t \in \mathcal{J}_s \setminus \mathcal{J}_r \\
r_2 + s_2 & \text{if } t \in (0, 1] \setminus (\mathcal{J}_r \cup \mathcal{J}_s).
\end{cases}
\]

(Here the set \((0, 1] \setminus (\mathcal{J}_r \cup \mathcal{J}_s)\) may be empty.) Thus, for any fixed \( M \), the function \( F(M, \cdot) \) can only take one of the three values \( r_2, s_2, r_2 + s_2 \).

Now consider the remaining case \( r_2 = 0 \). Let us then also require that \( r = (r_1, r_2) \) is a primitive lattice point (primitive means that there is no lattice point on the line segment between \( 0 \) and \( r \)) and again let \( s = (s_1, s_2) \) be a point in \( A \cap \mathcal{L} \setminus \mathbb{Z}r \) with \( s_2 \) minimal (then \( s_2 > 0 \)). If \( |r_1| \leq 1/2 \) then \( F(M, t) = 0 \) for all \( t \in (0, 1] \). On the other hand, if \( |r_1| > 1/2 \) then \( F(M, t) = s_2 \) for \( t \in (1 - |r_1|, |r_1|) \) and \( F(M, t) = 0 \) for all other \( t \) in \((0, 1]\).

\[ \square \]

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