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Detailed Analysis for Remark 5.1 in the Manuscript

The solution of the optimal control algorithm in each time interval \([t_k, t_{k+1}]\) can be transformed into a boundary value problem by applying Pontryagin’s minimum principle [1]. We start by constructing the Hamiltonian as follows

\[
H(X, u, \lambda) = \frac{1}{2} \theta \sigma (\dot{x} - r \sigma)^2 + \frac{1}{2} \eta_m u^2 + \lambda^T \begin{pmatrix} y \\ - (\alpha x^2 + \beta y^2 - \gamma) y - \omega^2 x + u \end{pmatrix}
\]

where \(X = [x, \dot{x}]^T = [x, y]^T\) and \(\lambda = [\lambda_1, \lambda_2]^T\). Using the minimum principle gives optimal open loop control

\[
u^* = \text{argmin}_{u \in \mathbb{R}} H(X^*, u, \lambda) = -\eta_m^{-1} \lambda^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\eta_m^{-1} \lambda_2
\]

and optimal state equation

\[
\dot{X}^* = \nabla_{\lambda} H = \begin{pmatrix} y^* \\ - (\alpha x^2 + \beta y^2 - \gamma) y^* - \omega^2 x^* - \eta_m^{-1} \lambda_2 \end{pmatrix}
\]

with initial condition \(X(t_k) = [x(t_k), \dot{x}(t_k)]^T\) and optimal costate equation

\[
\dot{\lambda} = -\nabla_{X} H = \begin{pmatrix} \frac{\lambda_2 (2 \alpha x^* y^* + \omega^2)}{\lambda_2 (2 \alpha x^* y^* + \omega^2) + 3 \beta y^2 - \gamma - \lambda_1 - \theta \sigma (y^* - r \sigma)} \\ \lambda_2 (2 \alpha x^* y^* + \omega^2) + 3 \beta y^2 - \gamma - \lambda_1 - \theta \sigma (y^* - r \sigma) \end{pmatrix}
\]

with the terminal condition

\[
\lambda(t_{k+1}) = \begin{pmatrix} \theta_p (x^*(t_{k+1}) - \hat{r}_p(t_{k+1})) \\ 0 \end{pmatrix}
\]

Let \(\hat{x}\) denote the approximation of the optimal solution \(x^*\), then it is feasible to estimate the position error between the VP and the HP based on the collocation method as.

\[
|x^* - \hat{r}_p| = |x^* - \hat{x} + \hat{x} - \hat{r}_p| \leq |x^* - \hat{x}| + |\hat{x} - \hat{r}_p|
\]

Notice that \(|x^* - \hat{x}|\) is negligible due to the high approximation accuracy of numerical methods [2]. In particular, considering that normally the optimal solution \(x^*\) is not available, the approximate
solution \( \tilde{x} \) exactly corresponds to the position of the VP in the simulation. Thus, we mainly focus on the estimation of \( |\tilde{x} - \hat{r}_p| \). For simplicity, we define \( \tilde{x}(t) = a_0 + a_1(t - t_k) + a_2(t - t_k)^2 \), \( \lambda_1(t) = b_0 + b_1(t - t_k) + b_2(t - t_k)^2 \) and \( \lambda_2(t) = c_o + c_1(t - t_k) + c_2(t - t_k)^2 \), where \( a_i, b_i \) and \( c_i, i \in \{0, 1, 2\} \) are unknown constants and \( t \in [t_k, t_{k+1}] \). Substituting \( \tilde{x}(t) \), \( \lambda_1(t) \) and \( \lambda_2(t) \) into the above optimal state equation and costate equation at the boundary points yields the linear matrix equation

\[
A_k X_k = B_k \tag{1}
\]

where

\[
A_k = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\theta_p & \theta_p T & \theta_p T^2 & -1 & -T & -T^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & T & T^2 \\
0 & 0 & 2 & 0 & 0 & 0 & \eta_m^{-1} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -(2\alpha x(t_k)y(t_k) + \omega^2) & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -(ax(t_k)^2 + 3\beta y(t_k)^2 - \gamma) & 1 & 0 \\
\theta_p & T\theta_p + \theta_\sigma & T(T\theta_p + 2\theta_\sigma) & 0 & 0 & 0 & 0 & 1 & 2T \\
0 & 0 & 0 & 0 & 1 & 2T & 0 & 0 & 0 
\end{pmatrix}
\]

and

\[
B_k = \begin{pmatrix}
x(t_k) \\
y(t_k) \\
\theta_p \hat{r}_p \\
0 \\
-(\alpha x(t_k)^2 + \beta y(t_k)^2 - \gamma)y(t_k) - \omega^2 x(t_k) \\
0 \\
-\theta_\sigma(y(t_k) - r_\sigma(t_k)) \\
\theta_p \hat{r}_p + \theta_\sigma r_\sigma(t_{k+1}) \\
0
\end{pmatrix}
\]

\[
X_k = \begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
b_0 \\
b_1 \\
b_2 \\
c_0 \\
c_1 \\
c_2
\end{pmatrix}
\]

Solving equation (1) determines the vector of unknown constants \( X_k = A_k^{-1}B_k \).

Thus, we obtain the approximate solution

\[
\tilde{x}(t) = x(t_k) + y(t_k)(t - t_k) + \frac{N}{D}(t - t_k)^2, \quad t \in [t_k, t_{k+1}]
\]
where

\[ N = 2T \left[ \frac{r_\sigma(t_k) + r_\sigma(t_{k+1})}{2} - y(t_k) + (\hat{r}_p - x(t_k) - Ty(t_k))\theta_p \right] \]

\[ - \eta_m \left( \frac{T^2 \omega^2}{2} + \alpha T^2 x(t_k)y(t_k) + \alpha T x(t_k)^2 + 3\beta T y(t_k)^2 - \gamma T + 2 \right) \]

\[ \left[ (\alpha x(t_k)^2 + \beta y(t_k)^2 - \gamma)y(t_k) + \omega^2 x(t_k) \right] \]

and

\[ D = 2T^2 (\theta_p T + \theta_\sigma) + 2\eta_m \left( \frac{T^2 \omega^2}{2} + \alpha T^2 x(t_k)y(t_k) + \alpha T x(t_k)^2 + 3\beta T y(t_k)^2 - \gamma T + 2 \right). \]

Then we can compute

\[ |\hat{x}(t_{k+1}) - \hat{r}_p(t_{k+1})| = \lim_{t \to t_{k+1}} |\hat{x}(t) - \hat{r}_p(t_{k+1})| \]

\[ = |x(t_k) + Ty(t_k) + \frac{N}{D} T^2 - \hat{r}_p(t_{k+1})| \]

\[ \leq T^2 (1 - \theta_p) \left| \frac{2|x(t_k) - \hat{r}_p(t_{k+1})| + T(r_\sigma(t_k) + r_\sigma(t_{k+1}))}{|D|} \right| + \eta_m \left| \frac{\mathcal{L} \cdot \mathcal{M}}{|D|} \right| \]

where

\[ \mathcal{L} = \frac{T^2 \omega^2}{2} + \alpha T^2 x(t_k)y(t_k) + \alpha T x(t_k)^2 + 3\beta T y(t_k)^2 - \gamma T + 2 \]

and

\[ \mathcal{M} = 2(x(t_k) + Ty(t_k) - \hat{r}_p(t_{k+1}) - T^2 y(t_k)(\alpha x(t_k)^2 + \beta y(t_k)^2 - \gamma) + \omega^2 x(t_k)) \]

Since \( \hat{r}_p, r_\sigma, D, \mathcal{L} \) and \( \mathcal{M} \) are all bounded, it follows from inequality (2) that the bound on the tracking error \( |\hat{x}(t_{k+1}) - \hat{r}_p(t_{k+1})| \) converges to 0 as \( \theta_p \to 1 \) and \( \eta_m \to 0 \). Similarly, we can estimate the velocity error between the VP and the reference signal encoding the desired signature as follows

\[ |\hat{x}(t_{k+1}) - r_\sigma(t_{k+1})| = \lim_{t \to t_{k+1}} |\hat{x}(t) - r_\sigma(t)| \]

\[ = |y(t_k) + \frac{2N}{D} T - r_\sigma(t_{k+1})| \]

\[ \leq (1 - \theta_\sigma) 2T^2 \left| \frac{T(y(t_k) - r_\sigma(t_{k+1})) + 2(\hat{r}_p(t_{k+1}) - x(t_k) - Ty(t_k))}{|D|} \right| + \theta_\sigma \left| \frac{2T^2 |r_\sigma(t_k) - y(t_k)|}{|D|} \right| + 2\eta_m \left| \frac{\mathcal{L} \cdot \mathcal{P}}{|D|} \right| \]

where

\[ \mathcal{P} = y(t_k) - r_\sigma(t_{k+1}) - T[(\alpha x(t_k)^2 + \beta y(t_k)^2 - \gamma)y(t_k) + \omega^2 x(t_k)] \]

According to inequality (3), the bound of the velocity error goes to 0 if \( \theta_\sigma \to 1, \eta_m \to 0 \) and \( r_\sigma(t_k) = y(t_k) \).
References
