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Distributed Graph Clustering by Load Balancing

He Sun  
University of Bristol  
Bristol, UK  
h.sun@bristol.ac.uk

Luca Zanetti  
University of Bristol  
Bristol, UK  
luca.zanetti@bristol.ac.uk

ABSTRACT
Graph clustering is a fundamental computational problem with a number of applications in algorithm design, machine learning, data mining, and analysis of social networks. Over the past decades, researchers have proposed a number of algorithmic design methods for graph clustering. However, most of these methods are based on complicated spectral techniques or convex optimisation, and cannot be applied directly for clustering many networks that occur in practice, whose information is often collected on different sites. Designing a simple and distributed clustering algorithm is of great interest, and has wide applications for processing big datasets.

In this paper we present a simple and distributed algorithm for graph clustering: for a wide class of graphs that are characterised by a strong cluster-structure, our algorithm finishes in a poly-logarithmic number of rounds, and recovers a partition of the graph close to an optimal partition. The main component of our algorithm is an application of the random matching model of load balancing, which is a fundamental protocol in distributed computing and has been extensively studied in the past 20 years. Hence, our result highlights an intrinsic and interesting connection between graph clustering and load balancing.

At a technical level, we present a purely algebraic result characterising the early behaviours of load balancing processes for graphs exhibiting a cluster-structure. We believe that this result can be further applied to analyse other gossip processes, such as rumour spreading and averaging processes.

CCS CONCEPTS
• Theory of computation → Graph algorithms analysis; Distributed algorithms; Random walks and Markov chains;

KEYWORDS
graph clustering; load balancing; spectral graph theory

1 Introduction
Analysis of large-scale networks has brought significant advances to our understanding of complex systems. One of the most relevant features possessed by networks occurring in practice is a strong cluster-structure, i.e., an organisation of nodes into clusters such that nodes within the same cluster are highly connected in contrast to nodes from different clusters. Recovering this cluster-structure is the aim of graph clustering, which is an important research topic in many disciplines, including computer science, physics, biology, and sociology. For instance, graph clustering is widely used in finding communities in social networks, webpages dealing with similar topics, and proteins having the same specific function within the cell in protein-protein interaction networks [13]. Most algorithms for graph clustering, however, require advanced algorithm design techniques such as spectral methods, or convex optimisation, which may make the algorithms difficult to be implemented in the setting of big data, where graphs may be allocated in sites that are physically remote. For this reason, designing a simple and distributed algorithm is of great interest in practice, and has received considerable attention in recent years [6, 20, 31].

In this work we propose a simple and distributed graph clustering algorithm that is mainly based on the following classical load balancing process (random matching model): assume that there is an application running on a parallel network with n processors. Every processor has initially a certain amount of loads (jobs) and the processors are connected by an arbitrary graph G. A load balancing process in the random matching model consists of synchronous rounds: in each round a random matching of G is generated in a distributed way, and every two matched nodes average their loads evenly. This process continues until every node has almost the same amount of load. Despite its low communication cost (at most [n/2] edges are involved in each round for load distribution) and highly distributed properties (every node only contacts its neighbors in the entire process), load balancing has been proven to be very efficient [27], and has been widely used in various domains, including scheduling [30], hashing [23], routing [10], and numerical computation such as solving partial differential equations [32].

1.1 Structure of Clusters
Let $G = (V, E)$ be an undirected graph with n nodes. For any set $S$, let the conductance of $S$ be

$$\phi_G(S) \triangleq \frac{|E(S, V \setminus S)|}{\text{vol}(S)},$$

where $E(S, V \setminus S)$ is the set of edges between $S$ and $V \setminus S$, and $\text{vol}(S)$ is the number of edges with at least one endpoint in $S$. Intuitively, nodes in $S$ form a cluster if $\phi_G(S)$ is small, i.e., there are few edges connecting the nodes of $S$ to the nodes in $V \setminus S$. We call subsets of nodes (i.e. clusters) $A_1, \ldots, A_k$ a k-way partition of $G$ if $A_i \cap A_j = \emptyset$ for different $i$ and $j$, and $\bigcup_{j=1}^k A_i = V$. Moreover, we define the $k$-way expansion constant by

$$\rho(k) \triangleq \min_{\text{partition } A_1, \ldots, A_k} \max_{1 \leq i \leq k} \phi_G(A_i).$$
We investigate the power of random matching model of load balancing: any well-clustered graph best of our knowledge. Our main result is summarised as follows: for graph clustering. Our algorithm is decentralised, and very easy to implement. Moreover, our approach corresponds to a natural centralised algorithm for graph clustering, which is also new to the best of our knowledge. Our main result is summarised as follows:

**Theorem 1.1.** There exists a distributed algorithm such that, for any well-clustered graph \( G = (V, E) \) with \( n \) nodes and \( k \) clusters \( S_1, \ldots, S_k \) that satisfies (2), finishes in \( T \leq \Theta \left( \frac{n}{\lambda_{k+1}^2} \right) \) rounds and, with constant probability, at the end of the algorithm the following statements hold:

1. Each node \( v \) receives a label \( \ell_v \) such that the total number of misclassified nodes is \( o(n) \), i.e., there exists a permutation of the labels \( \sigma \) such that
   \[
   | \{ i \leq k \mid \ell_i \neq \sigma(i) \} | = o(n);
   \]
   (2) The total information exchanged among these \( n \) nodes, i.e., the message complexity, is \( O(T \cdot n \cdot k \log k) \) words.

An important scenario for graph clustering is the case where \( G \) consists of \( k = \Theta(1) \) clusters \( S_1, \ldots, S_k \), where \( |S_i| = \Theta(n/k) \), every \( G[S_i] \) is a spectral expander, and has conductance \( \phi_G(S_i) = O(1/\text{poly log } n) \) for \( i = 1, \ldots, k \). It is easy to verify that for such graph \( G \) our gap assumption (2) on \( \gamma \) holds, and our algorithm finishes in \( O(n \log n) \) rounds with message complexity \( O(n \log n) \). Moreover, the non-distributed version of our algorithm runs in \( O(n \log n) \) time once we have an oracle which outputs a random neighbourhood of any node. That is, when the input graph is \( \Delta \)-regular with \( \Delta = o(\log n) \), our algorithm runs in sub-linear time. This example shows a clear distinction between our algorithm and most other graph clustering algorithms, which usually require at least linear-time. Hence, the techniques presented in our paper might be of interest for designing algorithms for other models of computation as well, e.g., local algorithms, and algorithms for property testing.

### 1.3 Related Work

There is a large amount of literature on graph clustering, and our work is most closely related to efficient algorithms for graph clustering under different formulations of clusters. Oveis Gharan and Trevisan [24] formulate the notion of clusters with respect to the **inner** and **outer** conductance: a cluster \( S \) should have low outer conductance, and the conductance of the induced subgraph by \( S \) should be high. Under a assumption between \( \lambda_{k+1} \) and \( \lambda_k \), they present a polynomial-time algorithm which finds a \( k \)-way partition \( \{A_i\}_{i=1}^k \) that satisfies the inner- and outer-conductance condition. To ensure that every \( A_i \) has high inner conductance, they assume that \( \lambda_{k+1} \geq \text{poly}(k)\lambda_k^{1/4} \), which has a stronger polynomial dependency on \( \lambda_k \).

Another line of research closely related to our result is the design of local algorithms for finding a subset of nodes of low conductance, e.g., [16, 29]. In particular, Allen-Zhu et al. [1] studies a cluster structure with a gap assumption similar to ours, and presents a local algorithm with better approximation guarantee than previously known algorithms under that gap assumption. However, there is substantial difference between our algorithm and most local algorithms [1, 16, 29] for the following reasons: (1) We need to run a local algorithm \( k \) times in order to find \( k \) clusters. However, as the output of each execution of a local algorithm only returns an approximate cluster, the approximation ratio of the final output cluster might not be guaranteed when the value of \( k \) is large. (2) For many instances, our algorithm requires only a poly-logarithmic number of rounds, while local algorithms run in time proportional to the volume of the output set. It is unclear how these algorithms could finish in a poly-logarithmic number of rounds, even if we were able to implement them in the distributed setting.

Recently, Becchetti et al. [3] studies a distributed process to partition an almost-regular graph into clusters, and their analysis focuses mostly on graphs generated randomly from stochastic block models. In contrast to ours, their algorithm requires each node to exchange information with all of its neighbours in each round, and has significantly higher communication cost for a dense graph.

We also notice that the distributed algorithm presented in Kempe et al. [21] for computing the top \( k \) eigenvectors of the adjacency matrix of a graph can be applied for graph clustering. Their algorithm is, however, much more involved than ours. Moreover, for an
input graph $G$ of $n$ nodes, the number of rounds required in their algorithm is proportional to the mixing time of a random walk in $G$. For a graph consisting of multiple expanders connected by a few edges, their algorithm requires a polynomial number of rounds, while ours only requires a poly-logarithmic number of rounds.

Finally, we point out that our work is closely related to multiple random walks [2, 9, 12], other variants of load balancing processes [14], and runtime analysis of gossip algorithms [7, 8, 18, 19].

1.4 Organisation
The remaining part of the paper is organised as follows: Section 2 lists the notations used in the paper, and discusses necessary knowledge of load balancing. Section 3 presents our algorithm, and Section 4 gives a detailed analysis of our algorithm.

2 Preliminaries

2.1 Notations
Let $G = (V, E)$ be an undirected graph with $n$ nodes and $m$ edges. For any set $S, T \subseteq V$, we define $E(S, T)$ to be the set of edges between $S$ and $T$, i.e., $E(S, T) \triangleq \{ (u, v) \mid u \in S \text{ and } v \in T \}$. For two sets $X$ and $Y$, the symmetric difference of $X$ and $Y$ is defined as $X \triangle Y \triangleq (X \setminus Y) \cup (Y \setminus X)$.

For any $d$-regular graph $G$, the random walk matrix of $G$ is defined by $P = (1/d) \cdot A$, where $A$ is the adjacency matrix of $G$ defined by $A_{u, v} = 1$ if $(u, v) \in E(G)$, and $A_{u, v} = 0$ otherwise. For this matrix, we will denote its $i$th eigenvalue with $\lambda_1 \geq \cdots \geq \lambda_n$, with their corresponding orthonormal eigenvectors $f_1, \ldots, f_n$.

For any set $S$ of nodes, let $\chi_S \in \mathbb{R}^n$ be the normalised indicator vector of $S$, where $\chi_S(v) = 1/|S|$ if $v \in S$, and $\chi_S(v) = 0$ otherwise. In particular, we simply write $\chi_v$ instead of $\chi_{\{v\}}$ when the set $S = \{v\}$. The Euclidean norm of any vector $x \in \mathbb{R}^n$ is defined by $\|x\| \triangleq \sqrt{\sum_{i=1}^n x_i^2}$, and the spectral norm of any matrix $M \in \mathbb{R}^{n \times n}$ is defined as $\|M\| \triangleq \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Mx\|}{\|x\|}$.

Given two symmetric $n \times n$ matrices $A, B$, we write $A \preceq B$ if $x^T A x \leq x^T B x$ holds for any $x \in \mathbb{R}^n$.

2.2 The Matching Model for Load Balancing
One of the key concepts used in our algorithm is the random matching model for load balancing [5, 26, 27], in which one generates a random matching in each round and every two matched nodes balance their loads evenly. There are several simple and distributed randomised protocols to generate such matching, and in the present paper we use the following protocol [5]: (1) every node is either active or non-active with probability $1/2$; (2) every active node chooses one of its neighbours uniformly at random; (3) every non-active node $v$ chosen by exactly one of its neighbours is included in the matching together with its neighbour $u$.

We use a matching matrix $M^{(t)} \in \mathbb{R}^{n \times n}$ to express the matching used in round $t$: for every matched nodes $u$ and $v$, we have $M^{(t)}_{u, v} \triangleq 1/2$, $M^{(t)}_{v, u} \triangleq 1/2$, and $M^{(t)}_{u, u} \triangleq 1/2$, if $u$ is not matched, then $M^{(t)}_{u, u} \triangleq 1$ and $M^{(t)}_{u, v} \triangleq 0$ if $u \neq v$. With slight abuse of notation, we also use $M^{(t)}_x$ to express the set of edges included in the matching in round $t$.

The following lemma describes the properties of matrix $M^{(t)}$, and is originally proven in [5].

**Lemma 2.1** ([5]). Let $\bar{d} = \left(1 - \frac{1}{2d}\right)^{d-1}$. Then, the following statements hold for any $t \geq 1$:

1. $E[M^{(t)}] = \left(1 - \frac{\bar{d}}{2}\right)I + \frac{\bar{d}}{4} \cdot P$.
2. $M^{(t)}$ is a projection matrix, i.e., $(M^{(t)})^2 = M^{(t)}$.

**Proof.** We start with the first statement. By definition, it holds for any edge $(u, v), u \neq v$, that

$$
\mathbb{P}[\{u, v\} \text{ is included in a matching in round } t] = 2 \cdot \mathbb{P}[u \text{ is active}] \cdot \mathbb{P}[v \text{ is non-active}]
$$

$$
\mathbb{P}[\{u, v\} \text{ is included in a matching in round } t] = 2 \cdot \frac{1}{4} \cdot \frac{\bar{d}}{4} \left(1 - \frac{1}{2d}\right)^{d-1} = \frac{1}{2} \cdot \frac{\bar{d}}{4}.
$$

Hence, we have for any edge $(u, v), u \neq v$, that

$$
\mathbb{E} \left[ M_{u, v}^{(t)} \right] = \frac{1}{2} \cdot \mathbb{P}[\{u, v\} \text{ is included in a matching in round } t] = \frac{1}{2} \cdot \frac{\bar{d}}{4} = \frac{\bar{d}}{4} P_{u, v}.
$$

Similarly, we have for any vertex $u$ that

$$
\mathbb{E} \left[ M_{u, u}^{(t)} \right] = \frac{1}{2} \cdot \mathbb{P}[u \text{ is included in a matching in round } t] + \mathbb{P}[u \text{ is not included in a matching}] = \frac{1}{2} \cdot \frac{\bar{d}}{4} + \left(1 - \frac{\bar{d}}{2}\right) = 1 - \frac{\bar{d}}{4}.
$$

Combining these two equations gives us the first statement.

The second statement follows from the fact that, for any $x \in \mathbb{R}^n$, $M^{(t)x}$ is the projection of $x$ on the subspace

$$
\{y \in \mathbb{R}^n \mid g(u) = g(v) \text{ for any } (u, v) \in M^{(t)}\}.
$$

3 Algorithm
Now we present the distributed algorithm for graph clustering. For completeness, in Section 3.1 we will first present the detailed implementation of our algorithm in the distributed setting. In Section 3.2 we will discuss our algorithm in a more abstract way, and show its connection to load balancing processes.

3.1 Formal Description
At the initialisation step, every node $v$ picks a random number from $1$ to $n^3$, which is used as the identification of node $v$. It is easy to show that, with high probability, all the nodes pick different numbers. We assume that this holds in the remaining part of the paper, and use $\text{ID}(v)$ to represent the ID of node $v$. Our algorithm consists of three procedures:
The Seeding Procedure: Every node v repeats the following experiment for 
\[ \tau \triangleq \frac{3}{\beta} \ln \frac{1}{\beta} \]
trials, where in each trial node v becomes active with probability 1/n. For every node v that has become active at least once, node v sets its initial state as Statev(0) = ([ID(v), 0]). Every non-active node v sets Statev(0) = 0. For simplicity, we call ID(v) and x the prefix and suffix of vector (ID(v), x).

The Averaging Procedure: The averaging procedure proceeds for T rounds, where in each round t each node v computes its state Statev(t) through the following operations: (1) nodes apply the distributed algorithm described in Section 2.2 to generate a matching; (2) each node v computes the vector Statev(t) in round t as follows: If node v is not involved in any matching, then node v sets its state in round t as Statev(t) = Statev(t − 1). Otherwise node v is matched to node u in round t, and their states are computed based on the following rule, where both Stateu(t) and Statev(t) are set to be empty initially:

- If there is (ID(w), x) ∈ Stateu(t − 1) and (ID(w), y) ∈ Statev(t − 1) with the same prefix, then both of u and v adds the vector (ID(w), (x + y)/2) to Stateu(t) and Statev(t).
- For any vector (ID(w), x) ∈ Stateu(t − 1) that does not share a common prefix with any vector in Statev(t − 1), both of u and v adds the vector (ID(w), x/2) to Stateu(t) and Statev(t) respectively.
- For any vector (ID(w), y) ∈ Statev(t − 1) that does not share a common prefix with any vector in Stateu(t − 1), both of u and v adds the vector (ID(w), y/2) to Stateu(t) and Statev(t) respectively.

The Query Procedure: The query procedure assigns every node v to a label \( \ell_v \), and any two nodes u, v belong to the same cluster if and only if \( \ell_u = \ell_v \). Formally, based on Stateu(T) (Node v) uses 
\[ \ell_v = \min \left\{ \text{ID}(w) \mid \text{ID}(w), x \in \text{State}_v(T) \land x \geq \frac{1}{\sqrt{2\beta n}} \right\} \]
as the label of the cluster it belongs to, and \( \ell_v \) is set to be an arbitrary ID if there is no vector \((\text{ID}(w), x) \in \text{State}_v(T) \) satisfying \( x \geq 1/\sqrt{2\beta n} \).

3.2 Connection to Multi-Dimensional Load Balancing

From the formal description above, it is easy to see that the prefix of any vector is only used to identify from which node the corresponding unit load is generated, and loads from the vectors with different prefix will not be balanced during the execution of the algorithm. Therefore, we can view our algorithm as a multi-dimensional load balancing process, which is described as follows.

The seeding procedure consists of \( \tau \triangleq (3/\beta) \ln(1/\beta) \) trials, where in each trial every node becomes active with probability 1/n. For simplicity, we use s to denote the number of active nodes at the end of these \( \tau \) trials, and use \( v_1, \ldots, v_s \) to denote these active nodes. Moreover, we introduce s vectors \( x^{(0,1)}, \ldots, x^{(0,s)} \in \mathbb{R}^n \), where \( x^{(0,i)} = \chi_{v_i} \) for any \( 1 \leq i \leq s \).

After that, the averaging procedure proceeds for T rounds, where in each round t the nodes apply the distributed algorithm described in Section 2.2 to generate a matching \( M(t) \), and update the vectors \( x^{(t,i)} \) as follows: if nodes u and v are matched in round t, then they simply average their load evenly, i.e., 
\[ x^{(t,i)}(u) = \frac{x^{(t-1,i)}(u) + x^{(t-1,i)}(v)}{2}, \quad i = 1, \ldots, s; \]
otherwise, for every unmatched node u, node u simply sets 
\[ x^{(t,i)}(u) = x^{(t-1,i)}(u), \quad i = 1, \ldots, s. \]

Notice that the evolution of these s load vectors can be described by \( x^{(t,i)} = M(t)x^{(t-1,i)} \) for any \( i = 1, \ldots, s \).

Finally, at the query procedure every node v checks its coordinates \( x^{(T,1)}(v), \ldots, x^{(T,s)}(v) \), and uses 
\[ \ell_v = \min \left\{ i \mid x^{(T,i)}(v) \geq \frac{1}{\sqrt{2\beta n}} \right\} \]
as the label of the cluster it belongs to. If no such index i exists, the algorithm assigns node v an arbitrary label \( \ell_v \in \{1, \ldots, s\} \).

As a side remark, notice that, compared with the standard load balancing process in which the configuration for any round is expressed by an n-dimensional vector, in our algorithm there are s vectors of dimension n in each round. However, in each round the same matching matrix is applied to update these s vectors. Notice that, since \( \mathbb{E}[|s|] = \tau \), the expected communication cost is low, i.e., \( O(T\cdot s\cdot n) \). Secondly, as an interesting feature, our algorithm does not need to know the exact number of clusters k, and a lower bound of \( \beta \) suffices for our algorithm. Thirdly, the number of rounds T required by our algorithm relates to an upper bound of the local mixing time of a cluster, i.e., the time required for a random walk to become mixed inside a cluster. In particular, a value of \( T = \Theta \left( \frac{\log n}{\sqrt{\beta}} \right) \) suffices and this value is typically a poly-logarithmic function of n for most graphs exhibiting a strong cluster-structure.

4 Analysis

In this section we analyse the algorithm, and prove Theorem 1.1. Remember that the configuration of our algorithm is expressed by s vectors \( x^{(1)}, \ldots, x^{(s)} \), and these vectors are updated with respect to the same matching matrix in each round. To elaborate the intuitions behind our analysis, we first look at the standard load balancing process (the 1-dimensional case), and use the symbols \( y^{(0)} \in \mathbb{R}^n \) to express the load distribution in round t for the 1-dimensional load balancing process, where \( y^{(0)} = f_u \) for some node u, and the load distribution in round t + 1 is defined by 
\[ y^{(t+1)} = M(t)y^{(t)}. \]

It is well-known that the sequence \( \{y^{(t)}\}_{t=1}^{\infty} \) converges to the stationary distribution of a random walk in \( \mathbb{G} \), i.e., the first eigenvector \( f_1 \) of \( \mathbb{P} \) [5, 28], and \( y^{(t)} \) is close to \( f_1 \) when t is the mixing time of a random walk in \( \mathbb{G} \) [27]. Studying the early behaviour of load balancing processes, however, is more complicated, and we will show that the early behaviour of this process depends on the cluster-structure of \( \mathbb{G} \).

Our starting point is to study the load distribution \( y^{(T)} \). Informally, our choice of T corresponds to the time when a random walk
gets well mixed and the resulting distribution becomes stable in $S_i$, as long as the random walk always stays in $S_i$. This happens if a random walk starts from a good node in $S_i$ so that it won’t leave $S_i$ quickly. We will prove that there are enough good nodes so that, if the load balancing process above starts with $\chi_u$ for a good node $u$, then $y^{(T)}$ is close to a linear combination of $\chi_{S_1}, \ldots, \chi_{S_k}$. This implies that $y^{(T)}(u)$ and $y^{(T)}(v)$ are approximately the same if $u$ and $v$ belong to the same cluster.

Generalising this argument, we study the multi-dimensional load balancing process and prove the following fact: if the load balancing process above starts with $\chi_u$, then

$$\|\tilde{y}_t - f_i\| \leq \Theta \left( k \sqrt{\frac{k}{T}} \right)$$

Moreover, $\{\tilde{y}_j\}_{j=1}^k$ form an orthonormal set.

**Lemma 4.2.** For any $1 \leq i \leq k$ there exists $\tilde{y}_i$ in the span of $\{\chi_{S_1}, \ldots, \chi_{S_k}\}$, such that

$$\|\tilde{y}_i - f_i\| \leq \Theta \left( k \sqrt{\frac{k}{T}} \right)$$

**Proof of Theorem 1.1.** The seeding procedure consists of $\tilde{s}$ trials, where in each trial a node is active with probability $1/n$. Hence, the total number of active nodes $s$ satisfies $\mathbb{E}[s] = \tilde{s}$ and, by Markov inequality, $s = O(\tilde{s})$ with probability at least $1 - e$ for an arbitrary small constant $e > 0$. We assume this holds in the remaining part of the proof.

For any fixed cluster $S_i$, the probability that no node in $S_i$ is active in any one of the $\tilde{s}$ trials is at most

$$\prod_{v \in S_i} \left( 1 - \frac{1}{n} \right)^{\tilde{s}} \leq \prod_{v \in S_i} e^{-\tilde{s}/n} = e^{-\tilde{s} \sum_{v \in S_i} 1/n} \leq e^{-\tilde{s}\beta I} \leq e^{-3\ln\beta^{-1}} \leq e^{-3/k}$$

where we use the fact $1 - x \leq e^{-x}$ for $x \leq 1$ in the first inequality, and the assumption that $\|S_i\| \geq \beta n$, $\beta \leq 1/k$. Applying a union bound, with probability at least $1 - e^{-3}$ there is at least one active node in each cluster.

Let $I = \{v_1, \ldots, v_k\}$ be the set of active nodes, and denote by $\mathcal{S}(v)$ the cluster to which node $v$ belongs to. By the definition of
and the fact $\sum_{\nu} \alpha_{\nu}^2 = k\mathcal{E}^2$, the number of bad nodes is at most

$$k\mathcal{E}^2 \cdot \left( k\mathcal{E} \cdot \sqrt{\frac{\log n \log (1/\beta)}{\beta n}} \right)^2 = \frac{\beta n}{C \cdot k \log n \log(1/\beta)}.$$ 

by the averaging argument. Hence, the probability that in any given trial a bad node is active is at most

$$\frac{1}{n} \cdot \frac{\beta n}{C \cdot k \log n \log(1/\beta)} = \frac{\beta}{C \cdot k \log n \log(1/\beta)},$$

and with constant probability all the active nodes are good. From now on we assume that this event occurs.

Now we apply Lemma 4.3 on each coordinate of the multidimensional load vector, and obtain

$$\mathbb{E} \left[ \left\| x^{(T,i)} - \chi S_{(v_j)} \right\|^2 \right] = O \left( k \cdot \mathcal{E} \cdot \sqrt{\frac{\log n \cdot \log(1/\beta)}{\beta \cdot n}} \right).$$

for $i = 1, 2, \ldots, s$. By Markov inequality and the union bound, with constant probability it holds for all $i = 1, \ldots, s$ that

$$\left\| x^{(T,i)} - \chi S_{(v_j)} \right\|^2 = O \left( \frac{\log n \cdot \log(1/\beta)}{\beta \cdot n} \right).$$

To analyse the performance of the query procedure, notice that node $v$ can be misclassified only if there is $i \in \{1, \ldots, s\}$ such that

$$\left| x^{(T,i)}(v) - \chi S_{(v_j)}(v) \right|^2 \geq \frac{1}{2\beta n^2}.$$ 

By a simple averaging argument and assuming (5) holds, the number of misclassified nodes is at most

$$\sum_{i=1}^{s} \sum_{v \in V} \mathbb{I} \left\{ \left| x^{(T,i)}(v) - \chi S_{(v_j)}(v) \right|^2 \geq \frac{1}{2\beta n^2} \right\} \leq \sum_{i=1}^{s} \left( \frac{\beta}{\beta \cdot n} \right)^2 \cdot 2\beta n^2$$

$$= O \left( \frac{\beta^2 \cdot \mathcal{E}^2 \cdot n \cdot \log(1/\beta)}{\beta \cdot n} \right) \cdot n$$

$$= O \left( \frac{k^2 \cdot \mathcal{E}^2 \cdot n \cdot \log(1/\beta)}{\beta} \cdot \log(1/\beta) \right).$$

Combining this with the definition of $\mathcal{E}$ gives us that

$$\sum_{i=1}^{s} \sum_{v \in V} \mathbb{I} \left\{ \left| x^{(T,i)}(v) - \chi S_{(v_j)}(v) \right|^2 \geq \frac{1}{2\beta n^2} \right\} \leq O \left( \frac{k^5 \cdot n \cdot \log(1/\beta)}{\beta} \cdot \log(1/\beta) \right) = o(n),$$

where the last equality holds by the assumption on $Y$.

The total information exchanged follows from the fact that the algorithm finishes in $T$ rounds, and in each round only matched nodes exchange the information of $O(k \log k)$ words.

### 4.2 Proof of Lemma 4.1

**Proof of Lemma 4.1.** Without loss of generality, we denote by $Q^t \triangleq I - \mathcal{Q}$ the projection on the subspace spanned by the eigenvectors $f_k, \ldots, f_n$. Since $Q^t y^{(t)}$ and $Q^t y^{(t)}$ are orthogonal to each other, it holds that

$$\mathbb{E} \left[ \| Q^t y^{(0)} - y^{(t)} \|^2 \right] = \mathbb{E} \left[ \| Q^t y^{(0)} - (Q + Q^t) y^{(t)} \|^2 \right] = \mathbb{E} \left[ \| Q^t y^{(0)} - y^{(t)} \|^2 \right] + \mathbb{E} \left[ \| Q^t y^{(t)} \|^2 \right].$$

(6)

Proving that the first term in (6) is small corresponds to show that after $t = T$ rounds the contribution of the top $k$ eigenvectors $f_1, \ldots, f_k$ to $y^{(t)}$ is dominant, while proving that the second term is small means that the contribution of the bottom $k$ eigenvectors $f_{k+1}, \ldots, f_n$ to $y^{(t)}$ becomes negligible. This is what we would expect if at each round we were able to apply directly the expected matrix $\mathbb{E} \left[ \mathbf{M}^{(t)} \right]$. We prove that in expectation these facts hold, although different matching matrices $\mathbf{M}^{(t)}$ are applied in different rounds.

Formally, we analyse the first term in (6) and have that

$$\mathbb{E} \left[ \| Q^t (y^{(0)} - y^{(t)}) \|^2 \right] = \mathbb{E} \left[ \sum_{i=1}^{k} \left( \langle y^{(0)} - y^{(t)}, f_i \rangle \right)^2 \right]$$

$$= \sum_{i=1}^{k} \mathbb{E} \left[ \left( \langle y^{(0)} - (Q + Q^t) y^{(t)}, f_i \rangle \right)^2 \right]$$

$$= \mathbb{E} \left[ \sum_{i=1}^{k} \left( \langle y^{(0)}, f_i \rangle - \langle y^{(t)}, f_i \rangle \right)^2 \right]$$

$$= \mathbb{E} \left[ \sum_{i=1}^{k} \left( \langle y^{(0)} - Q y^{(0)}, f_i \rangle \right)^2 \right] - 2 \langle y^{(0)}, f_i \rangle \mathbb{E} \left[ \langle y^{(t)}, f_i \rangle \right]$$

$$\leq \sum_{i=1}^{k} \left( 2 \langle y^{(0)} - Q y^{(0)}, f_i \rangle^2 - 2 \langle y^{(0)}, f_i \rangle \mathbb{E} \left[ \langle y^{(t)}, f_i \rangle \right] \right),$$

(7)

where the last inequality uses the fact that, for every $t$, $\mathbf{M}^{(t)}$ is a projection matrix with norm at most one, and therefore

$$\mathbb{E} \left[ \left( \langle y^{(t)}, f_i \rangle \right)^2 \right] \leq \langle y^{(0)}, f_i \rangle^2.$$

Also, since at every round $t$ the picked matrix $\mathbf{M}^{(t)}$ is independent from previous matchings, it holds that

$$\mathbb{E} \left[ \langle y^{(t)}, f_i \rangle \right] = y^{(0)} \mathbb{E} \left[ \mathbf{M}^{(t)} \cdots \mathbf{M}^{(0)} \right] f_i$$

$$= y^{(0)} \mathbb{E} \left[ \left( \mathbf{M}^{(0)} \right)^t \right] f_i$$

$$= \left( 1 - \frac{d - d \cdot \lambda_i}{4} \right)^t \langle y^{(0)}, f_i \rangle \cdot$$

(8)

Therefore, it holds that

$$\mathbb{E} \left[ \| Q^t y^{(0)} - Q y^{(t)} \|^2 \right]$$

$$\leq \sum_{i=1}^{k} \left( 2 \langle y^{(0)} - Q y^{(0)}, f_i \rangle^2 - 2 \left( 1 - \frac{d - d \cdot \lambda_i}{4} \right)^t \langle y^{(0)}, f_i \rangle^2 \right)$$

$$\leq 2 \beta (1 - \lambda_k) \| Q y^{(0)} \|^2.$$

(9)
To bound the second term in (6), we study the total expected norm of $y^{(t)}$, and prove that, for any $t, k \geq 1$, it holds that

$$
E \left[ \|M^{(t)}P^e M^{(t)}\| \right] \leq \left( 1 - \frac{\bar{d}}{8} \right) P^e + \frac{\bar{d}}{8} P^{e+1}. \quad (10)
$$

To see this, we fix two nodes $u, v$. Then, the value of $M^{(t)}P^eM^{(t)}_{u,v}$ depends on how nodes $u$ and $v$ are matched in round $t$:

Case 1: If both of $u$ and $v$ are not involved in the matching in round $t$, then $M^{(t)}P^eM^{(t)}_{u,v} = P^e_{u,v}$.

Case 2: If $u$ is not involved in the matching but $v$ is matched to a node $\sigma(v) \neq u$, then $M^{(t)}P^eM^{(t)}_{u,v} = (1/2) \cdot P^e_{u,v} + (1/2) \cdot P^e_{u,\sigma(v)}$.

Case 3: Similarly, if $u$ is matched to $\sigma(u) \neq u$ but $v$ is not involved in the matching in round $t$, then $M^{(t)}P^eM^{(t)}_{u,v} = (1/2) \cdot P^e_{u,v} + (1/2) \cdot P^e_{\sigma(u),v}$.

Case 4: If $u$ is matched to $\sigma(u) \neq u$ and $v$ to $\sigma(v) \neq v$, then $M^{(t)}P^eM^{(t)}_{u,v} = (1/4) \cdot \left( P^e_{u,v} + P^e_{\sigma(u),v} + P^e_{\sigma(u),\sigma(v)} + P^e_{\sigma(u),\sigma(v)} \right)$.

Notice that the exact value of $E \left[ M^{(t)}P^eM^{(t)}_{u,v} \right]$ depends on how node $v$ can be reached from node $u$ from one or two matching edges in round $t$, as well as a walk of length $\ell$. Hence, we can write

$$
E \left[ \|M^{(t)}P^eM^{(t)}_{u,v}\| \right] = \alpha_1 P^e_{u,v} + \alpha_2 P^{e+1}_{u,v} + \alpha_3 P^{e+2}_{u,v},
$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 1$. In particular, since the first case occurs with probability at most $1 - \bar{d}/4$, it holds that $\alpha_1 \leq 1 - \frac{\bar{d}}{8}$. Then, (10) follows from the fact that $P^{e+2} \leq P^{e+1} \leq P^e$, and we have that

$$
E \left[ \|y^{(t)}\|^2 \right] = y^{(0)^T} E \left[ M^{(t)}M^{(t-1)} \cdots M^{(1)}M^{(0)} \cdots M^{(t-1)}M^{(t)} \right] y^{(0)}
\leq y^{(0)^T} \left( \left( \frac{1 - \frac{\bar{d}}{8}}{1 + \frac{\bar{d}}{8}} \right) P \right)^t y^{(0)}. \quad (11)
$$

To bound $E \left[ \|Qy^{(t)}\|^2 \right]$, we use (8) and obtain that

$$
E \left[ \|Qy^{(t)}\|^2 \right] = \sum_{i=1}^k \left( y^{(i)^T}, f_i \right)^2
\geq (1 - 2t(1 - \lambda_k)) \sum_{i=1}^k \left( y^{(i)^T}, f_i \right)^2
= (1 - 2t(1 - \lambda_k)) \|Qy^{(0)}\|^2, \quad (12)
$$

where the first inequality follows from the Johnson’s inequality. Combining (11), (12) and the fact that $\left( y^{(t)^T}, Q^e y^{(t)} \right) = 0$, we obtain that

$$
E \left[ \|Q^e y^{(t)}\|^2 \right] = E \left[ \|y^{(t)}\|^2 \right] - E \left[ \|Qy^{(t)}\|^2 \right]
\leq y^{(0)^T} \left( \left( \frac{1 - \frac{\bar{d}}{8}}{1 + \frac{\bar{d}}{8}} \right) P \right)^T y^{(0)} - (1 - 2t(1 - \lambda_k)) \|Qy^{(0)}\|^2
\leq 2t(1 - \lambda_k) \|Qy^{(0)}\|^2 \left( \frac{1 - \frac{\bar{d}}{8} + \frac{\bar{d}}{8} \lambda_{k+1}}{1 - \frac{\bar{d}}{8} + \frac{\bar{d}}{8} \lambda_k} \right)^t
\leq 2t(1 - \lambda_k) \|Qy^{(0)}\|^2 + o(n^{-c}), \quad (13)
$$

where (13) holds for a large constant $c > 0$ due to our choice of $t \gg T$.

Finally, combining (9) with (13) gives us that

$$
E \left[ \|\bar{Q}y^{(0)} - y^{(t)}\|^2 \right] \leq \lambda (1 - \lambda_k) \|\bar{Q}y^{(0)}\|^2 + o(n^{-c}),
$$

and Lemma 4.1 holds by applying the Jensen’s inequality.

4.3 Proof of Lemma 4.2

To prove Lemma 4.2, we need the following lemma:

**Lemma 4.4** (25). Let $(S_t)_{t=1}^\infty$ be a $k$-way partition of $G$ achieving $p(k)$, and let $\Gamma = \Omega \left( k^2 \right)$. Assume that $\bar{X}_t$ is the projection of $f_t$ in the span of $\{\bar{X}_1, \ldots, \bar{X}_k\}$. Then, it holds for any $1 \leq i \leq k$ that

$$
\|\bar{X}_t - f_t\| = O \left( \sqrt{\frac{k}{\Gamma}} \right).
$$

**Proof of Lemma 4.2.** Since $\{f_t\}_{t=1}^\infty$ is an orthonormal set, it holds by Lemma 4.4 that $\{\bar{X}_t\}_{t=1}^\infty$ are almost orthonormal. Hence, our task is to construct an orthonormal set $\{\bar{X}_t\}_{t=1}^\infty$ based on $\{\bar{X}_t\}_{t=1}^\infty$, which can be achieved by applying the Gram-Schmidt orthonormalisation procedure. The error bound follows from the fact that

$$
\langle \bar{X}_t, \bar{X}_i \rangle = O \left( \sqrt{\frac{k}{\Gamma}} \right)
$$

holds for $i \neq j$.

4.4 Proof of Lemma 4.3

**Proof.** We first show that $\bar{X}_S$ is the projection of the initial load vector $y^{(0)} = \chi_0$ in the span of $\{\bar{X}_1, \ldots, \bar{X}_k\}$. Since every $\bar{X}_i$ (1 $\leq i \leq k$) is a linear combination of vectors in $\{\bar{X}_1, \ldots, \bar{X}_k\}$, and $\bar{X}_1, \ldots, \bar{X}_k$ are orthonormal by Lemma 4.2, we have that $\text{span} \{\bar{X}_1, \ldots, \bar{X}_k\} = \text{span} \{\bar{X}_1, \ldots, \bar{X}_S\}$. Hence,

$$
\sum_{i=1}^k \langle \chi_0, \bar{X}_i \rangle \bar{X}_i = \sum_{i=1}^k \langle \chi_0, \bar{X}_i \rangle \bar{X}_i \|\bar{X}_S\| \|\bar{X}_S\| = \langle \chi_0, \bar{X}_S \rangle \|\bar{X}_S\|^2 = \bar{X}_S, \quad (14)
$$

where the first equality holds by the fact that span $\{\bar{X}_1, \ldots, \bar{X}_k\} = \text{span} \{\bar{X}_1, \ldots, \bar{X}_S\}$, the second equality holds since $\bar{X}_0$ is orthogonal to every $\bar{X}_S$ with $\ell \neq j$, and the third equality holds by the fact that $\langle \chi_0, \bar{X}_S \rangle = 1/\|\bar{X}_S\|^2$.

Based on this, we bound the expected distance between $y^{(T)}$ and $\bar{X}_S$. By the triangle inequality, it holds that

$$
E \left[ \|y^{(T)} - \bar{X}_S\|^2 \right] \leq E \left[ \|\bar{Q}x - y^{(T)}\|^2 \right] + \|\bar{Q}x - \bar{X}_S\|^2, \quad (15)
$$

where the expectation is over all possible random matchings generated within the first $T$ rounds. By Lemma 4.1, we have that

$$
E \left[ \|\bar{Q}x - y^{(T)}\|^2 \right] \leq 2 \sqrt{T} \cdot (1 - \lambda_k) \|\bar{Q}x\|^2 + o(n^{-c}). \quad (16)
$$
For the second term in the right hand side of (15), by the triangle inequality we have that
\[
\|QX_v - \chi S_j\| \leq \sum_{i=1}^k |\langle \chi_v, f_i \rangle| |f_i - \chi_{i}|, \\
\sum_{i=1}^k \langle \chi_v, f_i \rangle f_i - \sum_{i=1}^k \langle \chi_v, f_i \rangle \chi_{i} \leq \sum_{i=1}^k \langle \chi_v, f_i \rangle f_i - \sum_{i=1}^k \langle \chi_v, \chi_{i} \rangle \chi_{i}
\]
To bound the first term in (17), we have that
\[
\left\| \sum_{i=1}^k \langle \chi_v, f_i \rangle f_i - \sum_{i=1}^k \langle \chi_v, f_i \rangle \chi_{i} \right\| \leq \sum_{i=1}^k |\langle \chi_v, f_i \rangle| |f_i - \chi_{i}| \\
\leq E \sum_{i=1}^k |\langle \chi_v, f_i \rangle| \leq kE \|QX_v\| 
\]
where the first line follows from the triangle inequality, the second follows by Lemma 4.2, and the last follows by the Cauchy-Schwarz inequality. To bound the second term in (17), we have that
\[
\left\| \sum_{i=1}^k \langle \chi_v, f_i \rangle - \langle \chi_v, \chi_{i} \rangle \right\| \chi_{i} \leq \sqrt{\sum_{i=1}^k (f_i(v) - \chi_{i}(v))^2} \chi_{i} \\
= \sqrt{\sum_{i=1}^k (f_i(v) - \chi_{i}(v))^2} \chi_{i} \\
= \alpha_v
\]
where the second inequality follows from the orthonormality of \( \{\chi_{i}\} \), and the third equality from the definition of \( \alpha_v \), and, again, the orthonormality of \( \{\chi_{i}\} \). Thus, we rewrite (17) as
\[
\|QX_v - \chi S_j\| \leq k \cdot E \cdot \|QX_v\| + \alpha_v. 
\]
Combining (15), (16) with (19), we have that
\[
\mathbb{E} \left[ \|g(T) - \chi S_j\| \right] \leq \sqrt{T \cdot (1 - \lambda_k) + k \cdot E} \cdot \|QX_v\| + \alpha_v \\
= O \left( k \cdot E \cdot \sqrt{\log n} \right) \cdot \|QX_v\| + \alpha_v, 
\]
where the last equality follows by (1) and the fact that
\[
\sqrt{T \cdot (1 - \lambda_k)} = O \left( \frac{(1 - \lambda_k) \log n}{1 - \lambda_{k+1}} \right) = O \left( k \cdot E \cdot \sqrt{\log n} \right).
\]
Hence, it suffices to bound \( \|QX_v\|^2 \). Direct calculation shows that
\[
\|QX_v\|^2 = \sum_{i=1}^k (\chi_v f_i)^2 \\
= \sum_{i=1}^k (\chi_v \tilde{x}_i - (\tilde{x}_i - f_i))^2 \\
\leq \sum_{i=1}^k 2 (\chi_v \tilde{x}_i)^2 + (\chi_v \tilde{x}_i - f_i)^2 \\
= 2 \left( \|X_S\|^2 + (\chi_v \tilde{x}_i - f_i)^2 \right) \\
\leq 2 \left( \|X_S\|^2 + 2 \|f_i\|^2 \right) 
\]
where (21) follows from the inequality
\[
(a - b)^2 \leq 2(a^2 + b^2), 
\]
(22) follows from (14), and (23) follows from the definition of \( \alpha_v \). Hence, it holds that \( \|QX_v\| = O \left( \|X_S\| + \alpha_v \right) \), and we can rewrite (20) as
\[
\mathbb{E} \left[ \|g(T) - \chi S_j\| \right] = O \left( k \cdot E \cdot \sqrt{\log n} \cdot \|X_S\| + \alpha_v \right) \\
= O \left( k \cdot E \cdot \sqrt{\log n} \cdot \|X_S\| + \alpha_v \right). 
\]

4.5 Analysis for Almost-Regular Graphs

Finally, we show that our algorithm and analysis can be easily modified to work for almost-regular graphs, i.e., the graphs for which the ratio between maximum degree \( \Delta = \max_{v \in V} |d_v| \) and the minimum degree \( \delta = \min_{v \in V} |d_v| \) is upper bounded by some constant. We also assume each node knows an upper bound \( D \geq \Delta \) of the maximum degree such that \( D / \delta = \Theta(\Delta / \delta) \). With these assumptions, we only need to slightly modify the seeding procedure, in which every node \( v \) sets to be active with probability \( \frac{1}{2} + \frac{D - d_v}{2D} \), instead of \( 1/2 \) for the case of regular graphs. The Averaging and Query procedures remain the same.

To show our algorithm and analysis holds for almost-regular graphs, we view the underlying almost-regular graph \( G \) as a \( D \)-regular graph \( G' \), which is obtained from \( G \) by adding \( D - d_v \) self-loops to each node \( v \). Then, the conductance of any set \( S \) is almost the same in \( G \) and \( G' \), i.e., the graphs for the case of regular graphs. The Averaging and Query procedures remain the same.

It is also easy to see that the \( (k+1) \)th eigenvalues of the random walk matrix of \( G \) and \( G' \) differ by at most a constant factor, and therefore \( G' \) is well-clustered. Hence, Theorem 1.1 holds for almost-regular graphs as well.