Abstract

We prove that, for \( n \geq 4 \), the graphs \( K_n \) and \( K_n + K_{n-1} \) are Ramsey equivalent. That is, if \( G \) is such that any red-blue colouring of its edges creates a monochromatic \( K_n \) then it must also possess a monochromatic \( K_n + K_{n-1} \). This resolves a conjecture of Szabó, Zumstein, and Zürcher [10].

The result is tight in two directions. Firstly, it is known that \( K_n \) is not Ramsey equivalent to \( K_n + 2K_{n-1} \). Secondly, \( K_3 \) is not Ramsey equivalent to \( K_3 + K_2 \). We prove that any graph which witnesses this non-equivalence must contain \( K_6 \) as a subgraph.

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1 Introduction

A finite graph \( G \) is Ramsey for another finite graph \( H \), written \( G \rightarrow H \), if there is a monochromatic copy of \( H \) in every two-colouring of the edges of \( G \). We say that \( H_1 \) and \( H_2 \) are Ramsey equivalent, written \( H_1 \sim_R H_2 \), if, for any graph \( G \), we have \( G \rightarrow H_1 \) if and only if \( G \rightarrow H_2 \).

The concept of Ramsey equivalence was first introduced by Szabó, Zumstein, and Zürcher [10]. A fundamental question to ask is which graphs are Ramsey equivalent to the complete graph \( K_n \). It follows from a theorem of Folkman [5] that if a graph \( H \) is
Ramsey equivalent to $K_n$, then $\omega(H) = n$, where $\omega(H)$ denotes the size of the largest complete subgraph of $H$.

In a recent paper, Fox, Grinshpun, Person, Szabó, and the second author [6] showed that $K_n$ is not Ramsey equivalent to any connected graph containing $K_n$. Furthermore, it is easily seen that $K_n$ is not Ramsey equivalent to the vertex-disjoint union of two copies of $K_n$, see e.g. [10]. For two graphs $H_1$ and $H_2$ and an integer $t$, we denote by $H_1+tH_2$ the graph that consists of a copy of $H_1$ and $t$ pairwise vertex-disjoint copies of $H_2$. It follows that if $K_n \sim_R H$ then $H$ is of the form $K_n + H'$ where $\omega(H') < n$. In [10], it is proved that the graph $K_n + tK_k$ is Ramsey equivalent to $K_n$ for $k \leq n-2$ and $t \leq \frac{R(n,n-k+1)-2n}{2k}$, where $R(m_1,m_2)$ denotes the asymmetric Ramsey number. In particular, this implies that $K_n$ is Ramsey equivalent to $K_n + tK_{n-2}$ for some $t = \Omega(R(n,3)/n)$. However, the case of $k = n - 1$ was left open. It is easily checked that $K_3$ is not Ramsey equivalent to $K_3 + K_2$, since $K_6 \rightarrow K_3$ but $K_6 \rightarrow K_3 + K_2$. In [10] it is conjectured that this is an aberration, and that $K_n \sim_R K_n + K_{n-1}$ for large enough $n$.

The positive result on the Ramsey equivalence of $K_n$ and $K_n + tK_k$ is complemented by the following result, proved in [6]: For $n > k \geq 3$, the graph $K_n + tK_k$ is not Ramsey equivalent to $K_n$ if $t > \frac{R(n,n-k+1)-1}{k}$. In particular, $K_n$ is not Ramsey equivalent to $K_n + 2K_{n-1}$. In this paper, we prove the conjecture in [10], and thus, the latter statement is tight.

**Theorem 1.** For any $n \geq 4$, $K_n \sim_R K_n + K_{n-1}$.

Our methods are combinatorial and explicit, and the idea is the following: suppose for a contradiction that we have a graph $G$ which is Ramsey for $K_n$, yet has been coloured so as to avoid a monochromatic $K_n + K_{n-1}$. We will then attempt, by giving an explicit recolouring of some edges, to give a colouring which no longer possesses a monochromatic $K_n$, which contradicts the Ramsey property of $G$.

This is not quite possible directly, and instead we will build up our proof in stages: in a series of lemmas we will show that either a colouring of $G$ must have a monochromatic $K_n + K_{n-1}$, or if not we can deduce some further structural information about the colouring of $G$, which will help us in the lemmas to follow. Eventually, we will have accumulated enough information about our supposed counterexample that it collapses under the weight of contradiction into non-existence, which proves Theorem 1.

At several steps of the proof, we employ a similar argument: We remove a small number of vertices from a graph $G$ that is Ramsey for $K_n$ and show that the remaining graph $G'$ satisfies $G' \rightarrow K_{n-1}$. The following result is an instance of such a Ramsey stability result.

**Theorem 2.** Let $n \geq 3$ and $G \rightarrow K_n$. If $V \subset G$ has $|V| \leq 2n-2$ then $G \setminus V \rightarrow K_{n-1}$.

Note that this implies in particular that $R(n) \geq R(n-1) + 2n-2$, where $R(n)$ denotes the Ramsey number. It is known that $|R(n) - R(n-1)| \geq 4n + O(1)$ [4, 11]. Theorem 2 follows, however, from a more technical result, c.f. Lemma 4, which also allows us to retrieve information on the location of a monochromatic copy of $K_{n-1}$ in the remaining
graph. It is this additional information (which is crucial to prove our main result) that seems to yield the limitation on improving the lower bound on the difference of consecutive Ramsey numbers.

As mentioned above, the clique on six vertices is an unfortunate obstruction which prevents the Ramsey equivalence of \( K_3 \) and \( K_3 + K_2 \). Interestingly, Bodkin and Szabó (see [2]) have shown that, essentially, this is the only such obstruction.

**Theorem 3** ([2]). If \( G \to K_3 \) and \( G \not\to K_3 + K_2 \) then \( K_6 \subseteq G \).

In Section 4, we give an alternative proof of this theorem, using similar techniques to those developed for the proof of Theorem 1.

**Notation.** All graphs are simple and finite. As a convenient abuse of notation, we write \( G \) both for a graph and for its set of vertices. We write \( E(G) \) for the set of edges of \( G \).

**Structure of the paper.** In Section 2 we prove a Ramsey stability lemma, crucial for the proof of Theorem 1, but which also may be of independent interest. Theorem 2 is a direct corollary of that lemma. In Section 3 we give the proof of Theorem 1. In Section 4 we give the proof of Theorem 3. Finally, we conclude by giving a further discussion of Ramsey equivalence, including a discussion of some still-open conjectures in this field, and adding some more.

## 2 Ramsey stability

In this section we prove Theorem 2, a Ramsey stability result, showing that a small number of arbitrary vertices can be removed from a graph while still preserving much of its Ramsey properties.

The following lemma is the main stability result, from which Theorem 2 will be an easy corollary. The statement of the lemma is slightly technical, but this allows for additional flexibility which will be useful in the following section.

**Lemma 4.** Let \( n \geq 4 \) and \( G \to K_n \). Let \( V \subset G \) with \( 2 \leq |V| \leq 3n - 3 \) and let \( x \) and \( y \) be two distinct vertices from \( V \). Finally, let \( V_0 \subset V \setminus \{x, y\} \) be any set with \( |V_0| \leq 2n - 2 \).

Then, in any colouring of the edges of \( G \), there exists a monochromatic copy of \( K_{n-1} \) in \( G \setminus V_0 \), say with vertex set \( W \), such that either \( W \cap V = \{x\} \), or \( W \cap V = \{y\} \), or \( x, y \notin W \cap V \).

**Proof.** Without loss of generality, we may suppose that \( |V| = 3n - 3 \) and \( |V_0| = 2n - 2 \). We arbitrarily divide \( V_0 \) into two sets of \( n - 3 \) vertices each, say \( V_R \) and \( V_B \), and four single vertices, \( x_R, y_R, x_B, y_B \). For brevity, we let \( V' = V \setminus (V_0 \cup \{x, y\}) \).

Fix an arbitrary red/blue colouring of the edges of \( G \). The strategy is to recolour some of the edges incident to \( V \), and then show that the existence of a monochromatic copy of \( K_n \) in the recoloured graph (guaranteed by the Ramsey property of \( G \)) forces a monochromatic copy of \( K_{n-1} \) in the original colouring with the required properties.
This requires the recolouring to have some special features. The following explicit method of recolouring suffices, though we make no claims as to its uniqueness in this regard. Define auxiliary graphs $G_R$ and $G_B = G_R^c$, with vertex set

$$\{V_R, V_B, V', \{x\}, \{y\}, \{x_R\}, \{y_R\}, \{x_B\}, \{y_B\}\}.$$ 

Instead of giving an incomprehensible list of edges, we refer the reader to Figures 1 and 2 for the definition of $G_R$ and $G_B$, the complement of $G_R$. We now recolour the edges incident to $V$ as follows. If $u_1 \in U_1 \in G_R$ and $u_2 \in U_2 \in G_R$ such that $U_1 \neq U_2$, then colour the edge $u_1u_2$ red if $U_1U_2 \in E(G_R)$, and colour the edge $u_1u_2$ blue otherwise. Furthermore, colour all edges in $E(V_B)$ red, and all edges in $E(V_R)$ blue. The edges in $E(V')$ retain their original colouring. For all $u \in \{x_B, y_B\} \cup V_B$ and all $v \in G \setminus V$, colour the edge $uv$ blue. For all $u \in \{x_R, y_R\} \cup V_R$ and all $v \in G \setminus V$, colour the edge $uv$ red. It will be convenient to call the vertices in $\{x_B, y_B\} \cup V_B$ blue vertices, and to call the vertices in $\{x_R, y_R\} \cup V_R$ red vertices.
The crucial properties of this recolouring are the following, which are easy to verify from examining Figures 1 and 2:

1. Both $G_R$ and $G_B$ are $K_4$-free.

2. Every triangle in $G_R$ contains at least one of $\{x_B, y_B\}$, $V_B$, and every triangle in $G_B$ contains at least one of $\{x_R, y_R\}, V_R$.

3. The blue vertices $V_B \cup \{x_B, y_B\}$ are connected by only red edges in $G$, and the red vertices $V_R \cup \{x_R, y_R\}$ are connected by only blue edges in $G$.

Since $G$ is Ramsey for $K_n$ there must be a monochromatic copy of $K_n$ present in $G$ after this recolouring. We claim that, thanks to the fortuitous properties of our recolouring, this forces a monochromatic $K_{n-1}$ in the original colouring with the required properties.

Let $U$ be the vertex set of the monochromatic $K_n$ present in $G$ after this recolouring. If $|U \cap (V_0 \cup \{x, y\})| \leq 1$ then the lemma follows immediately, since discarding at most one vertex would leave a monochromatic $K_{n-1}$ in the original colouring (as the only edges which are recoloured are incident with $V_0 \cup \{x, y\}$), completely disjoint from $V_0 \cup \{x, y\}$ as required.
We may suppose, therefore, that $\left| U \cap (V_0 \cup \{x, y\}) \right| \geq 2$. The first case to consider is when $U \subset V$. By Property (1) and since $n \geq 4$, $U$ must contain at least two vertices from one of the classes $V_R$, $V_B$, or $V'$.

Suppose first that $\left| U \cap V_B \right| \geq 2$. Then $U$ must form a red $K_n$, and hence can contain at most one vertex from $V_R$ and no vertex from $V'$, since all edges between $V_B$ and $V'$ are blue. By similar reasoning, if $\left| U \cap V_R \right| \geq 2$, then $U$ must form a blue $K_n$, and hence cannot use any vertex from $V' \cup V_B$. Therefore, there exists at most one class $V'' \in \{V_R, V_B, V'\}$ such that $\left| U \cap V'' \right| \geq 2$. Since each such class contains at most $n - 3$ vertices, there is a monochromatic copy of $K_4$ within $V$, using at most one vertex from each of $V_R, V_B$, and $V'$. This gives a copy of $K_4$ in either $G_R$ or $G_B$, which contradicts Property (1).

Assume now that $U \not\subset V$, and suppose that $U$ hosts a red copy of $K_n$. Since all blue vertices are connected to $G \setminus V$ by blue edges, $U$ cannot contain any blue vertices. Therefore, by Property (2), $U$ uses vertices of at most two nodes in $G_R$. Furthermore, since the copy is red, $\left| U \cap V_R \right| \leq 1$.

If $V' \cap U \neq \emptyset$ then $U$ can use at most one vertex from $V \setminus V'$, and discarding this vertex leaves a monochromatic $K_{n-1}$ in the original colouring, disjoint from $V_0 \cup \{x, y\}$, as required.

If $V' \cap U = \emptyset$ then, by Property (2) again, it must use exactly two vertices from $V_R \cup \{x_R, y_R, x, y\}$. Since there are only blue edges between vertices in $V_R \cup \{x_R, y_R\}$, by Property (3), at least one of these two vertices in $U \cap V$ must be $x$ or $y$. Discarding the other vertex in $U \cap V$ leaves a monochromatic copy of $K_{n-1}$ in the original colouring which intersects $V$ in either $x$ or $y$, but no other vertices, as required.

The case when $U$ hosts a blue copy of $K_n$ is handled similarly, and the proof is complete.

\textbf{Proof of Theorem 2.} For $n \geq 4$ the theorem follows immediately from Lemma 4, after expanding $V$ by two arbitrary vertices from $G \setminus V$. For $n = 3$, it suffices to give an explicit colouring of $K_4$ in a similar fashion, as we do in Figure 3. Thus, if we recolour the edges adjacent to $V$ as indicated in Figure 3, then any monochromatic copy of $K_3$ in $G$ must have at least two vertices from $G \setminus V$, and hence $G \setminus V \rightarrow K_2$ as required.

Figure 3: Dashed lines indicate red edges, straight lines indicate blue edges.
3 Proof of the main result

We recall our goal: to show that $K_n$ is Ramsey equivalent to $K_n + K_{n-1}$ for $n \geq 4$. It is, of course, trivial that if $G \rightarrow K_n + K_{n-1}$ then $G \rightarrow K_n$. It remains to show that if $G \rightarrow K_n$ then $G \rightarrow K_n + K_{n-1}$. Our strategy will be to accumulate more and more information about the monochromatic structures present in a colouring of a graph which is Ramsey for $K_n$, without a monochromatic $K_n + K_{n-1}$, until we are eventually able to obtain a contradiction.

Lemma 5. Let $n \geq 4$. If $G \rightarrow K_n$ then, in every colouring of $G$, there is either a monochromatic $K_n + K_{n-1}$, or a red $K_n$ and a blue $K_n$.

Proof. Suppose, without loss of generality, that the edges of $G$ are coloured so that there is a red copy of $K_n$. Let $V_R$ be the vertex set of this red $K_n$. As in the proof of Lemma 4, we will recolour some edges of $G$ and use the assumption that $G \rightarrow K_n$ to prove the claim.

Suppose first that there is an edge $ab$ of $V_R$ which has the property that every red $K_n$ intersects $V_R$ in at least one vertex besides $a$ and $b$. In this case, we recolour every other edge of $V_R$ blue, and colour the edges between $V_R \{a, b\}$ and $G \backslash V_R$ red.

Since $G \rightarrow K_n$ there must be a monochromatic $K_n$ in this recoloured $G$. Suppose first that there is a red $K_n$. If it uses at least $n-1$ vertices from $G \backslash V_R$ then there is a red $K_{n-1}$ present in $G \backslash V_R$ in the original colouring, and hence a red $K_n + K_{n-1}$ in $G$. Otherwise, it must use a red edge from $V_R$. But the only red edge remaining in $V_R$ is $ab$, and the edges from $\{a, b\}$ to $G \backslash V_R$ remained their original colouring. Therefore, we must have a red $K_n$ in the original colouring that intersects $V_R$ in exactly $\{a, b\}$, which contradicts our choice of $ab$. Secondly, suppose that there is a blue $K_n$ in the recoloured $G$. If it uses any of the new blue edges inside $V_R$, then it must be contained entirely inside $V_R$, since the edges from $V_R \{a, b\}$ to $G \backslash V_R$ are all red. However, this is impossible, since $V_R$ has $ab$ still coloured red. Therefore we must have a blue $K_n$ that uses only edges which were originally blue, and so we have a red $K_n$ and a blue $K_n$ in $G$, as required.

We may now assume that, for every pair $\{a, b\} \subseteq V_R$, there is another red $K_n$ intersecting $V_R$ in only the edge $ab$. Let $W_R$ be the vertex set of another red $K_n$ such that
\[|V_R \cap W_R| = 2, \text{ say } V_R \cap W_R = \{a, b\}, \] and let \(c, d\) be any two vertices in \(W_R \setminus V_R\). We recolour (some of) the edges incident to \(W_R\) in the following way. An illustration of this colouring can be found in Figure 4.

- For all \(w \in W_R \setminus \{a, b, c, d\}\), all \(w' \in W_R (w' \neq w)\), and all \(v \in G \setminus W_R\), we colour the edge \(ww'\) blue and the edge \(ww\) red (if present in \(G\)).

- For all \(v \in V_R \setminus \{a, b\}\), we recolour the edges \(av\) and \(bv\) blue, and the edges \(cv\) and \(dv\) red (if present in \(G\)).

- For all \(x \in G \setminus (V_R \cup W_R)\), we colour the edges \(ax, bx\) and \(dx\) in red (the edge \(cx\) retains its original colour).

- Every edge in \(\{a, b, c, d\}\) is recoloured blue, except for \(ac\) which remains red.

Again, since \(G \to K_n\) there must be a monochromatic \(K_n\) in this recoloured \(G\).

Suppose first that there is a red \(K_n\), say on vertex set \(W\). If it uses at least \(n - 1\) vertices from \(G \setminus W_R\) then there is a red \(K_{n-1}\) present in \(G \setminus W_R\) in the original colouring, and hence a red \(K_n + K_{n-1}\) in \(G\). Otherwise, it must use a red edge from \(W_R\). But the only red edge remaining in \(W_R\) is \(ac\). Then \(W\) must be disjoint from \(V_R \setminus \{a\}\), since \(ax\) is blue for every \(x \in V_R \setminus \{a\}\). Hence, \(W \cap (V_R \cup W_R) = \{a, c\}\). But none of the edges inside \(W \setminus \{a\}\) were recoloured, and hence \(W \setminus \{a\}\) hosts a red \(K_{n-1}\) in the original colouring that is vertex disjoint from \(V_R\).

Secondly, suppose that there is a blue \(K_n\) in the recoloured \(G\), say on vertex set \(W\). If it uses any of the new blue edges inside \(V_R \cup W_R\), then it must be contained entirely inside \(V_R \cup W_R\), since the edges from \(W_R \setminus \{c\}\) to \(G \setminus (V_R \cup W_R)\) are all red. However, \(V_R \cup W_R\) does not host a blue \(K_n\) in this recolouring. Therefore we must have a blue \(K_n\) that uses only edges which were originally blue, and so we have a red \(K_n\) and a blue \(K_{n-1}\) as required.

**Lemma 6.** Let \(n \geq 4\). If \(G \to K_n\) then, in any colouring of \(G\), if there is a monochromatic \(K_{n+1}\) then there is a monochromatic \(K_n + K_{n-1}\).

**Proof.** Suppose that \(G\) has, say, a red \(K_{n+1}\), on vertex set \(V_R\). By Lemma 5, we may assume that there exists a blue \(K_n\), say on vertex set \(V_B\). Let \(V = V_R \cup V_B\), so that \(2 \leq |V| \leq 2n + 1\). We now apply Lemma 4, with \(V_0 \subset V\) being any set of \(2n - 2\) vertices containing \(V_B\), and \(x\) and \(y\) two arbitrary vertices from \(V_R \setminus V_B\). This yields a monochromatic \(K_{n-1}\) which intersects the red \(K_{n+1}\) in at most one vertex, as \(|V \setminus V_0| \leq 3\), and the blue \(K_n\) not at all, and hence we must have a monochromatic \(K_n + K_{n-1}\).

**Lemma 7.** Let \(n \geq 4\), and let \(G\) be a graph such that \(G \to K_n\). Assume that there is a colouring of the edges of \(G\) with no monochromatic copy of \(K_n + K_{n-1}\). Then, in this colouring, no two monochromatic copies of \(K_n\) intersect in exactly two vertices.

**Proof.** Suppose otherwise; without loss of generality, we have two red copies of \(K_n\), say on vertex sets \(V_R\) and \(V'_R\), such that \(|V_R \cap V'_R| = 2\). By Lemma 5 we may further assume that there is a blue \(K_n\), say on vertex set \(V_B\).
Assume first that \( V_B \cap V_R \neq \emptyset \). Let \( x \in V_R \setminus (V_B \cup V_R') \) and \( y \in V_B' \setminus (V_B \cup V_R) \) (which exist since \( n \geq 4 \) and since \( V_B \) intersects with \( V_R \) and \( V_B' \) with at most one vertex each). Further, set \( V := V_R \cup V_B \cup V_R' \) and \( V_0 := (V_R \cup V_R') \setminus \{x\} \subseteq V \). By assumption, \(|V| \leq 3n - 3\) and \(|V_0| \leq 2n - 2\). Therefore, by Lemma 4, there is a monochromatic copy of \( K_{n-1} \), say on set \( W \), such that either \( W \cap V = \{x\} \), or \( W \cap V \subseteq \{V_0 \cup \{x\}\} \). In the first case, when \( W \cap V = \{x\} \), then \( W \) is disjoint from both \( V_B \) and \( V_R' \), and hence there is a monochromatic copy of \( K_n + K_{n-1} \), a contradiction. Otherwise, \( W \) is disjoint from both \( V_B \) and \( V_R' \), and again, we find a monochromatic copy of \( K_n + K_{n-1} \), a contradiction.

We argue similarly if \( V_B \cap V_R' \neq \emptyset \), and therefore assume from now on that \( V_B \cap (V_R \cup V_R') = \emptyset \). Let \( x, y \in V_R \setminus V_R' \) and \( z \in V_R' \setminus V_R \) be some arbitrarily chosen vertices. We again apply Lemma 4, with \( V := V_B \cup V_R \cup W \), where \( W = V_R' \setminus (V_R \cup \{z\}) \), and \( V_0 := (V_R \cup V_R') \setminus \{x, y\} \). It is clear that \(|V| \leq 3n - 3\) and \(|V_0| = 2n - 2\), as required.

Suppose that there is a monochromatic copy of \( K_{n-1} \) which intersects \( V \) in only vertices of \( W \). In particular, it is vertex-disjoint from \( V_B \cup V_R' \), and hence it creates a monochromatic \( K_n + K_{n-1} \), which is a contradiction.

It follows that there exists a monochromatic copy of \( K_{n-1} \) which intersects \( V \) in either \( x \) or \( y \), but no other vertices. Since it is disjoint from \( V_B \), we may assume that it is red. If this red \( K_{n-1} \) does not use \( z \), however, then together with \( V_R' \) we have a red \( K_n + K_{n-1} \), which is a contradiction. Therefore, either \( xz \) or \( yz \) is red. Since \( x \) and \( y \) were an arbitrary choice of two vertices from \( V_R \setminus V_R' \), it follows that all but at most one vertex of \( V_R \) is connected to \( z \) by a red edge.

That is, \( V_R \cup \{z\} \) hosts two red copies of \( K_n \) that intersect in \( n - 1 \) vertices. Note that if \( V_R \cup \{z\} \) forms in fact a red copy of \( K_{n+1} \), then we are done by Lemma 6. Therefore, to finish the argument, let \( x \in V_R \setminus V_R' \) such that the edge \( xz \) is blue or not present in \( G \). As noted, there is at most one such \( x \). We apply Lemma 4 yet again to reach a contradiction. Let \( y \in V_R \setminus (V_R' \cup \{x\}) \), set \( V_0 := (V_R \cup V_B) \setminus \{y, z\} \) and \( V := (V_R \cup V_R' \cup V_B) \setminus \{x\} \). Then \(|V| = 3n - 3\) and \(|V_0| = 2n - 2\). By Lemma 4, there exists a monochromatic copy of \( K_{n-1} \), say on vertex set \( W \), such that either \( W \cap V = \{y\} \), \( W \cap V = \{z\} \), or \( W \cap V \subseteq V \setminus (V_0 \cup \{y, z\}) \). If \( W \cap V = \{y\} \), then \( W \) is disjoint from \( V_R' \cup V_R \) and hence forms a monochromatic copy of \( K_n + K_{n-1} \) in the original colouring, a contradiction. If \( W \cap V = \{z\} \), then \( W \) is disjoint from \( V_B \), and hence we may assume that it is red. But then, \( W \) is either disjoint from \( V_R \) and forms a red copy of \( K_n + K_{n-1} \), or \( x \in W \), and hence the edge \( xz \) is red, a contradiction. Finally, if \( W \cap V \subseteq V \setminus (V_0 \cup \{y, z\}) \), then \( W \) together with \( V_0 \cup \{y, z\} \) forms a monochromatic copy of \( K_n + K_{n-1} \). \( \square \)

We will now conclude the proof of the main result.

**Proof of Theorem 1.** Let \( n \geq 4 \), and let \( G \) be a graph such that \( G \to K_n \). Assume that there exists a colouring of the edges of \( G \) without a monochromatic copy of \( K_n + K_{n-1} \). By Lemma 5, we can assume that there are two (not necessarily disjoint) sets \( V_R \) and \( V_B \) of vertices such that \( G[V_R] \) and \( G[V_B] \) form a red and a blue copy of \( K_n \), respectively.

By assumption, any other red (blue) copy of \( K_n \) intersects \( V_R \cup V_B \) in at least two vertices; in fact, by Lemma 7, any other red (blue) copy of \( K_n \) intersects \( V_R \cup V_B \) in at least three vertices. That is, every set \( W_R \subseteq V_R \) of size \(|W_R| = n - 2\) meets every red
copy of $K_n$ in at least one vertex, and every set $W_B \subset V_B$ of size $|W_B| = n - 2$ meets every blue copy of $K_n$ in at least one vertex.

If $V_R \cap V_B = \emptyset$, fix two arbitrary subsets $W_R \subset V_R$ and $W_B \subset V_B$, both of size $|W_R| = |W_B| = n - 2$. If $V_R \cap V_B \neq \emptyset$, let $W_B \subseteq V_B$ be a set of size $n - 2$ such that $V_R \cap V_B \subseteq W_B$, and let $W_R \subseteq V_R$ be a subset of size $n - 3$ such that $W_R \cap V_B = \emptyset$ (note that $|V_R \cap V_B| = 1$). In both cases, the sets $W_R \subset V_R$ and $W_B \subset V_B$ are disjoint and, by the above discussion, any monochromatic copy of $K_n$ meets $W_R \cup W_B$ in at least one vertex.

We now recolour the graph and show that the resulting colouring does not contain a monochromatic copy of $K_n$. We may assume, without loss of generality, that all edges in $V_R \cup V_B$ are present, since losing edges will only help prevent a monochromatic $K_n$ occurring. Let $\{x_R, y_R\} = V_R \setminus (W_R \cup W_B)$ and $\{x_B, y_B\} = V_B \setminus (W_R \cup W_B)$.

- If $n = 4$ and $V_R \cap V_B \neq \emptyset$ (i.e. $|W_R| = 1$), colour one edge between $W_R$ and $W_B$ red, and the other one blue. Otherwise, colour the edges between $W_R$ and $W_B$ so that for every $v \in W_R$ there are $w_r, w_b \in W_B$ such that $vw_r$ is red and $vw_b$ is blue, and for every $v \in W_B$ there are $w_r, w_b \in W_R$ such that $vw_r$ is red and $vw_b$ is blue.\footnote{This is clearly possible if $|W_R|, |W_B| \geq 2$, i.e. if $V_R \cap V_B = \emptyset$ or $n \geq 5$.}

- For all $x \in W_R$, $y \in V_R$, and $z \not\in V_R \cup W_B$, colour the edge $xy$ blue and colour the edge $xz$ red.

- For all $x \in W_B$, $y \in V_B$, and $z \not\in W_R \cup V_B$, colour the edge $xy$ red and colour the edge $xz$ blue.

This recolouring is illustrated in Figure 5 (where we label as black those edges which retain their original colouring).

Note that we only recolour edges incident to $W_R \cup W_B$. Therefore, by our choice of $W_R \cup W_B$, any monochromatic copy of $K_n$ (after recolouring the edges) must meet $W_R \cup W_B$ in at least one vertex.

Suppose now that a red $K_n$ exists in the recoloured graph and uses vertices from $W_R$ but not $W_B$. Then it must use just one vertex from $W_R$ and $n - 1$ from $G \setminus (V_R \cup W_B)$, and hence we have a red $K_n + K_n-1$ in the original colouring. If a blue $K_n$ exists and uses vertices from $W_R$ but not $W_B$, then it cannot use any vertices from $\{x_B, y_B\}$ or $G \setminus (V_R \cup V_B)$, and can only use at most one vertex from $\{x_R, y_R\}$ (since the edge $x_R y_R$ remains red as in the original colouring). But this contradicts the fact that $|W_R| \leq n - 2$.

Similarly, we can rule out the case that a monochromatic copy of $K_n$ uses vertices from $W_B$ but not $W_R$. Therefore, if there is a monochromatic copy of $K_n$ after recolouring the edges, then it must use vertices from both $W_R$ and $W_B$. Assume first that this copy is red. Since all vertices in $W_B$ are connected to $G \setminus (V_R \cup V_B)$ via blue edges, the red copy of $K_n$ must lie entirely inside $V_R \cup V_B$. But then, it can use at most one vertex from $W_B$ and at most one of $\{x_B, y_B\}$. The remaining $n - 2$ vertices must come from $W_B$, so we must use all vertices from $W_B$. However, in the case $V_R \cap V_B = \emptyset$ or $n \geq 5$, every vertex in $W_R$ sees at least one vertex of $W_B$ in blue. In the case $n = 4$, $|W_R| = 1$ and $|W_B| = 2$, 

\footnote{This is clearly possible if $|W_R|, |W_B| \geq 2$, i.e. if $V_R \cap V_B = \emptyset$ or $n \geq 5$.}
the two edges between $W_R$ and $W_B$ are of opposite colour, and hence, at most one vertex of $W_B$ can contribute to a red $K_4$.

A similar argument shows that we do not find a blue copy of $K_n$ using vertices from both $W_R$ and $W_B$. We have therefore constructed a colouring of $G$ which has no monochromatic $K_n$, contradicting the original Ramsey property of $G$ and concluding the proof. 

4 Ramsey equivalence of $K_3$

In this section we give a proof of Theorem 3, a result of Szabó and Bodkin (see [2]). We need to show that, if $G \rightarrow K_3$ and $G \not\rightarrow K_3 + K_2$, then $K_6 \subset G$.

Proof of Theorem 3. Let $G$ be a graph which is Ramsey for $K_3$ and not Ramsey for $K_3 + K_2$, and fix some colouring of $G$ with no monochromatic copy of $K_3 + K_2$. We first show that $G$ must possess both a red copy of $K_3$ and a blue copy of $K_3$.

Without loss of generality, there is a red $K_3$, say on vertex set $V_R = \{x_R, y_R, z_R\}$. We now recolour the edges $x_Ry_R$ and $x_Rz_R$ blue, and colour all the edges from $x_R$ to $G \setminus V_R$ red. It is now straightforward that a blue copy of $K_3$ must be a blue copy in the original colouring, and that a red copy of $K_3$ forces either a monochromatic copy of $K_3 + K_2$ in the original colouring, or it uses the edge $y_Rz_R$ and single new vertex, say $v_R$. In this case, we recolour once again in the following way, as indicated in Figure 6.

We colour the three-edge path $(z_R, x_R, v_R, y_R)$ red, and the complement in $V_r \cup \{v_R\}$ blue. Furthermore, we colour all edges between $\{z_R, y_R\}$ and $G - (V_r \cup \{v_R\})$ red. As before, if there is now a blue $K_3$, then it cannot use either of the vertices $y_R$ or $z_R$, and

Figure 5: The colouring for the proof of Theorem 1. Edges inside $W_B$ are red, edges inside $W_R$ are blue. Dashed lines indicate red edges, straight lines indicate blue edges. Thin grey edges retain their original colouring.
hence it must have been already present in the original colouring of $G$. Otherwise, a red $K_3$ must use exactly two vertices from $\{x_R, y_R, z_R\}$. In particular, we have a red $K_2$ that is either disjoint from $\{x_R, y_R, z_R\}$ or $\{v_R, y_R, z_R\}$, and hence a red $K_3 + K_2$ in the original colouring.

We have shown that there must be, in our coloured graph $G$, a red $K_3$, say on $V_R$, and a blue $K_3$, say on $V_B$. We now show that we can assume that $V_R$ and $V_B$ are disjoint.

Suppose our original choices are not, so that $|V_R \cup V_B| = 5$. Suppose $V_R \cap V_B = \{y_R\}$ and $V_R = \{x, y_R, z_R\}$ and $V_B = \{x, y_B, z_B\}$. Clearly, any edges between $\{y_R, z_R\}$ and $G \setminus (V_R \cup V_B)$ must be red. If their neighbourhoods intersect in $G \setminus (V_R \cup V_B)$ we have found another red $K_3$, entirely disjoint from $V_B$, and we may proceed. Otherwise, we may assume that the neighbourhoods of $y_R$ and $z_R$ in $G \setminus (V_R \cup V_B)$ are disjoint. Similarly, we can assume that the neighbourhoods of $y_B$ and $z_B$ in $G \setminus (V_R \cup V_B)$ are disjoint. We now colour the edges incident to $V_R \cup V_B$ as indicated in Figure 7. Since $G \rightarrow K_3$, there must be a monochromatic copy of $K_3$ after recolouring. Furthermore, it must intersect $V_R \cup V_B$ in exactly two vertices, since the original colouring would contain a monochromatic $K_3 + K_2$ otherwise. If it is a red $K_3$, say, then it must therefore use $y_R, z_R$, and a single vertex from $G \setminus (V_R \cup V_B)$, which contradicts the fact that their neighbourhoods are disjoint as discussed above, and we argue similarly if we have found a blue $K_3$.

We may therefore assume that we have produced two disjoint sets, $V_R$ and $V_B$, each of which spans a red and blue $K_3$ respectively.

Suppose first that there are two vertex-disjoint edges missing from $V_R \cup V_B$. We then recolour the edges incident to $V_R \cup V_B$ as in Figure 8 (where, as usual, a red (blue) vertex represents the fact that the edges between that vertex and $G \setminus (V_R \cup V_B)$ are coloured red (blue)). It is easy to check that this colouring of $V_R \cup V_B$ contains no monochromatic $K_3$. Moreover, there are no blue edges between blue vertices, and, vice versa, no red edges between red vertices. It follows that a monochromatic copy of $K_3$ in this recoloured $G$ must use at least two vertices from $G \setminus (V_R \cup V_B)$, which would create a monochromatic $K_3 + K_2$ in the original colouring of $G$, a contradiction.

We may suppose, therefore, that there is a vertex, without loss of generality say $x_R \in V_R$, such that every missing edge in $V_R \cup V_B$ is adjacent to $x_R$. Furthermore, we
Figure 7: The recolouring when $V_R \cap V_B = \{x\}$. The edges between $\{y_R, z_R\}$ and $G - (V_R \cup V_B)$ are red, the edges between $\{y_B, z_B\}$ and $G - (V_R \cup V_B)$ are blue. Dashed edges are red, straight edges are blue.

may suppose that at least one edge is missing, or else we have a $K_6$ in $G$ as required. Let $x_RX_B$ be some missing edge, where $x_B \in V_B$.

Assume first that there is a vertex, say $w$, in $G \setminus (V_R \cup V_B)$ that has at least five neighbours in $V_R \cup V_B$. If it is adjacent to every vertex of $(V_R \cup V_B) \setminus \{x_R\}$ then this creates a $K_6$, as required. Hence, we can assume that $wx_R$ is an edge in $G$. Furthermore, all edges between $w$ and $V_R$ (if present in $G$) must be red, and all edges between $w$ and $V_B$ must be blue (as otherwise they create a monochromatic copy of $K_3 + K_2$ in the original colouring).

Suppose that $w$ is adjacent to every vertex of $V_R$ and to two vertices of $V_B$, say $a$ and $b$, and that the edge $wc$ is missing. If either of the edges $x_RA$ or $x_RB$ is missing, then by considering $\{w, x_R, y_R\} \cup V_B$ we have a similar situation as above – namely, disjoint vertex sets of a red and a blue copy of $K_3$ with two vertex disjoint edges missing, and we are done. Otherwise, we have a $K_6$ in $\{w, x_R, y_R, z_R, a, b\}$. Suppose now that $w$ is adjacent to every vertex of $V_B$ and $x_R$ and some other vertex of $V_R$, say $a$, and the edge $wb$ is missing, where $b \in V_R$. As above, we are now done by considering $V_R \cup \{w, x_B, y_B\}$, since $wb$ and $x_Bx_R$ are two independent edges missing.

For the remainder of the argument, we may therefore assume that every vertex of $G \setminus (V_R \cup V_B)$ has at most four neighbours in $V_R \cup V_B$. We now describe a recolouring of the edges incident to $V_R \cup V_B$ such that there is no monochromatic $K_3$ that uses at least two vertices from $V_R \cup V_B$. Recolour the interior edges of $V_R \cup V_B$ as in Figure 9.

Let now $w \in G \setminus (V_R \cup V_B)$ and let $N_w \subseteq V_R \cup V_B$ be any set of four vertices containing $N(w) \cap (V_R \cup V_B)$. Then, either (1), $\{x_R, x_B\} \not\subseteq N_w$ and we see a red copy of the three-edge
Figure 8: The two thin grey edges are absent (other edges may or may not be present). Dashed edges are red, straight edges are blue.

path $P_3$ and a blue copy of $P_3$ in $N_w$, or (2), \{x_R, x_B\} ⊆ $N_w$, say $N_w = \{a, b, x_R, x_B\}$, for some $a, b$, and we see a monochromatic copy of $C_4$ or a monochromatic star $K_{1,3}$ with $a$ being the centre of the star.

In case (1), say $(a, b, c, d)$ forms the red $P_3$ in $N_w$, then we colour the edges $wb$ and $wc$ blue, and the edges $wa$ and $wd$ red (if present in $G$). In case (2), we colour the edge $wa$, $wx_R$ and $wx_B$ the opposite colour of $ax_R$ and $wb$ the same colour as $ax_R$.

Note first that the colouring of $V_R ∪ V_B$ does not contain a monochromatic triangle. Furthermore, it is evident that we do not create a monochromatic triangle on vertices $w, x, y$ with $x, y ∈ V_R ∪ V_B$ and $w ∉ V_R ∪ V_B$, since no such $w$ sees both vertices of a red edge in red nor both vertices of a blue edge in blue.

However, since $G → K_3$, there must be an edge $vw$ with $v, w ∉ V_R ∪ V_B$, which creates a monochromatic $K_3 + K_2$ in the original colouring, a contradiction.

5 Further remarks

Theorem 2 states that the removal of any $2n - 2$ vertices of a graph $G$ that is Ramsey for $K_n$ leaves a graph that is Ramsey for $K_{n-1}$. We wonder whether $2n - 2$ can be replaced by $2n$ in that statement, since our main result would then follow immediately from Lemma 5. More generally, it is natural to ask the following.

**Question 8.** What is the maximum number $f(n)$ of vertices that can be removed from any graph Ramsey for $K_n$ such that the remainder is Ramsey for $K_{n-1}$?

Trivially, $f(n) ≤ R(n) - R(n - 1)$ which, together with our lower bound of $2n - 2$, implies that $f(3) = 4$ and that $6 ≤ f(4) ≤ 12$. To the best of our knowledge, nothing better is known.
We have shown that $K_n$ and $K_n + K_{n-1}$ are Ramsey equivalent for $n \geq 4$. Furthermore, we have seen that $K_6$ is the only obstruction to the Ramsey equivalence of $K_3$ and $K_3 + K_2$, i.e. any graph $G$ that satisfies $G ightarrow K_3$ and $G \nrightarrow K_3 + K_2$ must contain $K_6$ as a subgraph.

The only pairs of graphs $(H_1, H_2)$ known to be Ramsey equivalent are of the form $H_1 \cong K_n$ and $H_2 \cong K_n + H_3$, where $H_3$ is a graph of clique number less than $n$. Furthermore, it is known ([6] and [7]) that the only connected graph that is Ramsey equivalent to $K_n$ is the clique $K_n$ itself.

It is an open question, first posed in [6], whether there are two connected non-isomorphic graphs $H_1$ and $H_2$ that are Ramsey equivalent. It follows from [7] that, if such a pair exist, they must have the same clique number. In [1] it is shown that they must also have the same chromatic number, under the assumption that one of the two graphs satisfies an additional property, called clique-splittability.

To tackle problems on Ramsey equivalence, a weaker concept was proposed by Szabó [9]. We will first introduce some necessary notation. We say that $G$ is Ramsey minimal for $H$ if $G$ is Ramsey for $H$ and no proper subgraph of $G$ is Ramsey for $H$. Denote by $\mathcal{M}(H)$ the set of all graphs which are Ramsey minimal for $H$, and by $\mathcal{R}(H)$ the set of all graphs which are Ramsey for $H$. Finally, let $\mathcal{D}(H_1, H_2) := (\mathcal{M}(H_1) \setminus \mathcal{R}(H_2)) \cup (\mathcal{M}(H_2) \setminus \mathcal{R}(H_1))$ be the class of graphs $G$ that are Ramsey minimal for $H_1$, but which are not Ramsey for $H_2$, or vice versa. Equivalently, $\mathcal{D}(H_1, H_2)$ is the set of minimal obstructions to the Ramsey equivalence of $H_1$ and $H_2$.

In particular, $H_1$ and $H_2$ are Ramsey equivalent if and only if $\mathcal{D}(H_1, H_2) = \emptyset$. We say that $H_1$ and $H_2$ are Ramsey close, denoted by $H_1 \sim_c H_2$, if $\mathcal{D}(H_1, H_2)$ is finite. We stress that this is not an equivalence relation: reflexivity and symmetry are trivial, but transitivity does not hold, since every graph containing at least one edge is close to $K_2$.

Two graphs may be Ramsey close in a rather trivial sense if $\mathcal{M}(H_1)$ and $\mathcal{M}(H_2)$ are both finite, or if $H_2 \subset H_1$ and $\mathcal{M}(H_2)$ is finite. Graphs such that $\mathcal{M}(H)$ is finite are
known as Ramsey-finite graphs. The class of Ramsey-finite graphs has been studied quite intensively; see, for example, [3] for some results and further references. In particular, it has been shown that the only Ramsey-finite graphs are disjoint unions of stars.

If one wishes to prove that two graphs are Ramsey equivalent, a possible first step is to show that the two graphs are Ramsey close. Szabó [9] has posed the following weaker version of the open problem mentioned earlier.

**Question 9.** Is there a pair of non-isomorphic, Ramsey-infinite, connected graphs which are Ramsey close?

We suspect that the answer to Question 9 is negative, even with this weakening of the notion of Ramsey equivalence.

Nešetřil and Rödl [8] proved that if $\omega(H) \geq 3$ then there exist infinitely many Ramsey-minimal graphs $G \in \mathcal{M}(H)$ such that $\omega(H) = \omega(G)$. In particular, it follows that if $\omega(G_1) \geq 3$ and $\omega(G_2) \geq 3$, and $G_1 \sim_c G_2$, then $\omega(G_1) = \omega(G_2)$.

Theorem 3 states that, although $K_3$ and $K_3 + K_2$ are not Ramsey equivalent, they are Ramsey close. Indeed, the only graph $G$ that is Ramsey minimal for $K_3$ and satisfies $G \rightarrow K_3 + K_2$ is $K_6$ itself. This is the only example of a pair of Ramsey-infinite graphs which are Ramsey close but not Ramsey equivalent that we know of. In this case, $|\mathcal{D}(K_3, K_3 + K_2)| = 1$. We pose the following.

**Question 10.** For any integer $k \geq 2$, is there a pair of Ramsey-infinite graphs $H_1$ and $H_2$ such that $|\mathcal{D}(H_1, H_2)| = k$?

An affirmative answer, which we believe to exist, would in particular imply the following conjecture.

**Conjecture 11.** There are infinitely many pairs of Ramsey-infinite graphs which are Ramsey close but not Ramsey equivalent.

In an earlier draft of this paper, we posed the following question.

**Question 12.** Are $K_n$ and $K_n + K_n$ Ramsey close for $n \geq 3$?

Shagnik Das observed that the answer is no: any graph that is Ramsey-minimal for $K_n$ is not Ramsey for $K_n + K_n$ and if $n \geq 3$ then there are infinitely many Ramsey-minimal graphs for $K_n$.

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