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Pinning Controllability of Complex Network Systems With Noise

Daniel A. Burbano-L. *Member, IEEE*, Giovanni Russo *Member, IEEE* and Mario di Bernardo, *Fellow, IEEE*

Abstract—The ability of a network of nonlinear systems to synchronize onto the desired reference trajectory in the presence of one or more leader nodes is known as the pinning controllability problem. This paper studies the pinning controllability of multi-agent networks subject to three different types of noise diffusion processes; namely, noise affecting the node dynamics, the communication links, and the pinning control action itself. By using appropriate stochastic Lyapunov functions, sufficient pinning controllability conditions are derived depending on the node dynamics, network structure, noise intensity, and control parameters. Counterintuitively, it is found that under some specific conditions noise may enhance the pinning controllability of the network making it easier to drive all agents towards the desired collective behavior. The effectiveness of the theoretical results is illustrated via two application examples arising in the context of gene regulatory networks and synchronization of chaotic systems.

Index Terms—Pinning Control, Nonlinear Systems, Stochastic/Uncertain Systems, Stability, Networks of Autonomous Agents

I. INTRODUCTION

PINNING control has been shown to be an effective approach for controlling network systems towards a desired collective behavior. This strategy consists of injecting control signals only into a fraction of agents (or nodes) so that the whole network converges to a desired target state. This makes pinning control applicable for large networks and even in those cases where only some of the nodes are accessible for control [1]. Pinning control strategies and their extensions have been widely used to steer networks toward a synchronous state. For instance in [2], the control gains self-tune their value for guaranteeing synchronization in networks of circuits, while in [3], an impulsive control law is used for synchronizing firing neurons. For a comprehensive review of pinning control strategies the reader is referred to e.g. [4], [5] and references therein.

When designing a pinning control strategy, it is crucial to decide the number and location of nodes to be controlled and find the value of the control parameters guaranteeing convergence. This problem is known as *pinning controllability* which provides a qualitative measure of the propensity of a

network to synchronize onto a reference trajectory, see e.g. [6], [7]. Pinning controllability was studied in [8], where local convergence conditions were derived, while in [9] conditions for pinning controllability were given in the global case. Moreover, in [10], [11] the authors proposed different methods for finding the optimal location of pinned nodes (i.e. the nodes on which control is directly exerted) by optimizing pinning controllability.

Most of the results on pinning controllability are based on the assumption that the dynamics of the network is deterministic and noise-free. Unfortunately, in many applications, networks are often subject to certain unavoidable disturbances and noise. For instance, in gene regulatory networks deterministic models are not able to capture cell-to-cell fluctuations in genetic switching [12], and also in engineered networks, communication between nodes might occur through quantized state variables [13] or fading communication links [14].

In this paper, which is motivated by the above observations, we study pinning controllability of networks subject to noise diffusion processes. In particular, we consider networks of diffusively coupled nonlinear agents, and investigate three different scenarios; namely, the cases where (i) noise affects the control action, or (ii) noise propagates through the interconnection links, or (iii) noise affects the intrinsic node dynamics. For such scenarios, we devise a set of sufficient conditions, which explicitly relate pinning controllability to the node dynamics, the network structure and the noise diffusion process. The results show that, as one would expect, pinning controllability becomes worse as the noise intensity increases, i.e. more control interventions and/or higher control gains are needed to guarantee the network reaches the target state. However, we have found that the pinning controllability can counterintuitively, be enhanced if noise propagates in a homogeneous manner. The effectiveness of the results is also illustrated via two application examples arising in the context of regulatory gene networks and chaotic systems.

A. Related work

The influence of noise on synchronization and consensus has recently gained much research attention, see e.g. [15], [16], [17], [18], [14], [19]. It has also been recently suggested [20], that noise can be beneficial for network coordination and that small random fluctuations, if properly injected in the network, can improve the collective performance of a group of agents.

The problem of controlling a network with stochastic terms has been studied in [21], [22]; however, convergence of the control strategies are proved assuming all nodes are controlled. In [23], noise has been used to implement a distributed speed

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advisory system, where vehicles are modeled as simple integrators. In [24] pinning control is investigated, via linearization of the network dynamics, in scenarios where the control action is applied to the network in a stochastic manner, while in [25] linear diffusively coupled networks with multiple disturbances and Lipschitz nonlinear terms were studied. More recently in [26] networks with nonlinear stochastic terms were analyzed assuming homogeneous noise, that is, the Brownian motion is assumed to be one-dimensional and identical for all the nodes.

When compared to previous literature, the work reported in this paper expands the state of the art in a number of different ways. In particular, (i) we study networks of diffusively coupled nodes affected by different types of possibly heterogeneous noise diffusion processes; (ii) our results, which are based on the use of stochastic Lyapunov functions, provide pinning controllability conditions in the case where only a fraction of the nodes is controlled; (iii) the nodes dynamics can be nonlinear, with their vector field satisfying the so-called QUAD (quadratic) condition, a more general condition than Lipschitz [27]; (iv) the coupling functions between nodes in the network can be nonlinear (preliminary results on networks with linear diffusive couplings were reported in [28]); and (v) under certain condition noise might be beneficial for enhancing the pinning controllability.

II. MATHEMATICAL PRELIMINARIES

A. Notation

We denote by $\mathbf{I}_N$ the identity matrix of dimension $N \times N$; by $\mathbf{1}_N$ a $N \times 1$ vector with unitary elements, while $\mathbf{0}$ is a matrix of zeros of appropriate dimensions. The symbol $\|\cdot\|$ denotes the Euclidean norm for vectors, and the spectral norm induced by the L^2 -norm for matrices. A diagonal matrix, say $\mathbf{D}$, with diagonal elements $d_1, \dots, d_N$ is denoted by $\mathbf{D} = \text{diag}\{d_1, \dots, d_N\}$; $\lambda_k(\mathbf{A})$ denotes the k -th eigenvalue of a squared matrix $\mathbf{A}$, and $\otimes$ denotes the Kronecker product. The triple $(\Sigma, \mathcal{F}, \mathbb{P})$ denotes a complete probability space where Σ is the sample space with elements denoted by ω , $\mathcal{F}$ is a σ -algebra and $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ is a probability measure. As usual, $\mathbb{P}(\omega)$ denotes the probability of an event ω . In what follows, $\mathcal{C}^{p \times q}$ represents the space of continuous functions $\mathbf{f} : \Omega_0 \subseteq \mathbb{R}^m \times \mathbb{R}^+ \cup \{0\} \mapsto \mathbb{R}^m$ that can be differentiated p times in $\mathbf{x}$ and q times in t . For instance $\mathcal{C}^{2 \times 1}$ denotes the space of functions $\mathbf{f}(\mathbf{x}, t)$ that are twice differentiable in $\mathbf{x}$ and differentiable in t . We also simply denote by $\mathcal{C}$ the space $\mathcal{C}^{1 \times 1}$.

Definition 2.1: A function $\varphi(\mathbf{x}, t) : \Omega_0 \subseteq \mathbb{R}^n \times \mathbb{R}^+ \cup \{0\} \mapsto \mathbb{R}^n$, is said to satisfy the Quadratic-Lipschitz condition if there exist constants $\kappa_1 \in \mathbb{R}^+$ and $\kappa_2 \in \mathbb{R}$ with $\kappa_2 \leq \kappa_1$, such that $\forall \mathbf{x}, \mathbf{y} \in \Omega_0$ and $\forall t \geq 0$

$$\|\varphi(\mathbf{x}, t) - \varphi(\mathbf{y}, t)\| \leq \kappa_1 \|\mathbf{x} - \mathbf{y}\| \quad (1)$$

$$(\mathbf{x} - \mathbf{y})^\top (\varphi(\mathbf{x}, t) - \varphi(\mathbf{y}, t)) \geq \kappa_2 (\mathbf{x} - \mathbf{y})^\top (\mathbf{x} - \mathbf{y}) \quad (2)$$

Remark 2.1: Note that a Quadratic-Lipschitz function is a generalization of the bi-Lipschitz condition. Indeed, using the triangle inequality, condition (2) can be upper-bounded by $\kappa_2 \|\mathbf{x} - \mathbf{y}\|^2 \leq \|\mathbf{x} - \mathbf{y}\| \|\varphi(\mathbf{x}, t) - \varphi(\mathbf{y}, t)\|$ for $\kappa_2 > 0$.

Hence, the quadratic-Lipschitz condition (1)-(2) reduces to the bi-Lipschitz condition $\kappa_2 \|\mathbf{x} - \mathbf{y}\| \leq \|\varphi(\mathbf{x}, t) - \varphi(\mathbf{y}, t)\| \leq \kappa_1 \|\mathbf{x} - \mathbf{y}\|$, which naturally arises in a number of applications, including filtering and state estimation [29], [30]. In addition, any linear function satisfies (1)-(2). For instance, let $\varphi(\mathbf{x}_i) = \mathbf{A}\mathbf{x}_i$, with $\mathbf{A}$ being a square matrix. Then, the Quadratic-Lipschitz condition is easily verified by setting $\kappa_1 = (1/2)\lambda_{\max}(\mathbf{A}^\top + \mathbf{A})$ and $\kappa_2 = (1/2)\lambda_{\min}(\mathbf{A}^\top + \mathbf{A})$.

B. Algebraic Graph Theory

An *undirected graph* $\mathcal{G}$ is defined by $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, 2, \dots, N\}$ is the finite set of N node indices; $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is the set containing the E edges between the nodes (i, j) for any $i, j \in \mathcal{N}$. The *adjacency matrix* $\mathbf{A}(\mathcal{G}) \in \mathbb{R}^{N \times N}$ (or simply $\mathbf{A}$ in what follows) of a graph $\mathcal{G}$ represents the topology of the network of interconnections and its elements a_{ij} are defined as $a_{ij} = 1$ for $i \neq j$ if there is an edge from node i to node j and zero otherwise. We assume that there are no self loops in the network, i.e. $a_{ii} = 0$, $\forall i = 1, \dots, N$ and that the edges are undirected, that is $a_{ij} = a_{ji}$ for all $i, j \in \mathcal{N}, i \neq j$. The *Laplacian matrix* $\mathbf{L}(\mathcal{G}) \in \mathbb{R}^{N \times N}$ (or simply $\mathbf{L}$ in what follows) of a graph $\mathcal{G}$ is given by $\mathbf{L} = \text{diag}\{\mathbf{A}\mathbf{1}_N\} - \mathbf{A}$, where the matrix $\text{diag}\{\mathbf{A}\mathbf{1}_N\}$ is often called the degree matrix. The elements of $\mathbf{L}$ are denoted by ℓ_{ij} , $i, j = 1, \dots, N$. A *multigraph*, is the set of M graphs $\mathcal{M} := \{\mathcal{G}_1, \dots, \mathcal{G}_M\}$ called layers of $\mathcal{M}$, where all the graphs in $\mathcal{M}$ share the same set of nodes, that is $\mathcal{G}_k = (\mathcal{N}, \mathcal{E}_k)$, for $k \in \{1, \dots, M\}$.

Lemma 2.1: [31] Let $\mathcal{G}(\mathcal{N}, \mathcal{E})$ be a connected undirected and unweighted graph. Then, the eigenvalues of its corresponding Laplacian matrix $\mathbf{L}$ can be ordered as $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$. Moreover, for any pair of vectors $\mathbf{x} = [x_1, \dots, x_N]$, $\mathbf{y} = [y_1, \dots, y_N] \in \mathbb{R}^N$, the following relation holds $\mathbf{x}^\top \mathbf{L} \mathbf{y} = \sum_{i,j \in \mathcal{E}} (x_j - x_i)(y_j - y_i)$.

C. Stability of Stochastic Differential Equations

Consider the m -dimensional stochastic differential equation of Itô type [32]

$$d\mathbf{x} = \mathbf{f}(\mathbf{x}, t)dt + \mathbf{g}(\mathbf{x}, t)d\mathbf{B}, \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (3)$$

where $\mathbf{x} \in \Omega \subseteq \mathbb{R}^m$ is the state variable, $\mathbf{x}_0 \in \Omega$, $\mathbf{f} \in \mathcal{C}^{2 \times 1}$, $\mathbf{f} : \Omega \times \mathbb{R}^+ \cup \{0\} \mapsto \mathbb{R}^m$, $\mathbf{g} \in \mathcal{C}$, $\mathbf{g} : \Omega \times \mathbb{R}^+ \cup \{0\} \mapsto \mathbb{R}^{m \times p}$ and $\mathbf{B}$ is a p -dimensional Brownian motion. Throughout this paper we will assume that the open set Ω is a forward invariant set for the dynamics (3). That is, for any given initial condition $\mathbf{x}_0 \in \Omega$, a unique solution of (3) exists, say $\mathbf{x}(t)$ such that, $\mathbf{x}(t) \in \Omega \forall t \geq 0$, see e.g. [33]. We will also assume that $\mathbf{f}(\mathbf{0}, t) = \mathbf{g}(\mathbf{0}, t) = \mathbf{0}$ and the solution $\mathbf{x} = \mathbf{0}$ is termed as the *trivial solution* of (3). Clearly, this implies by construction that $\mathbf{x} = \mathbf{0} \in \Omega$.

In this paper we are interested in exponential almost sure stability. Before giving the formal definition of stability, it is important to recall that a sequence of random variables, say $\{X_1, X_2, \dots\}$, converges almost surely (a.s.) to the random variable X if $\mathbb{P}(\{\lim_{n \rightarrow +\infty} X_n(w) = X(w)\}) = 1$; that is, convergence is attained with probability 1 ($\mathbb{P} = 1$). We are now ready to give the following definition.

Definition 2.2: [32] The trivial solution of (3) is said to be almost surely exponentially stable if for all $\mathbf{x}_0 \in \Omega$,

$$\lim_{t \rightarrow +\infty} \sup \frac{1}{t} \log(\|\mathbf{x}\|) < 0, \quad \text{a.s.} \quad (4)$$

The quantity whose limit is taken in (4) is called the *sample Lyapunov exponent* (see p. 63 [32]); therefore, its value can be used as an estimate of the rate of convergence towards the trivial solution of (3).

Definition 2.3: [32] Consider a non-negative function $V(\mathbf{x}, t) : \Omega \times \mathbb{R}^+ \cup \{0\} \mapsto \mathbb{R}^+$ belonging to $\mathcal{C}^{2 \times 1}$. The differential operator L associated to the stochastic Itô equation (3) is defined as

$$LV(\mathbf{x}, t) := V_t(\mathbf{x}, t) + V_x(\mathbf{x}, t)\mathbf{f}(\mathbf{x}, t) + V_{gx}(\mathbf{x}, t) \quad (5)$$

where $V_t(\mathbf{x}, t) := \partial V(\mathbf{x}, t)/\partial t$, $V_x(\mathbf{x}, t) := [V_{x_1}, \dots, V_{x_m}]$ with $V_{x_k} = \partial V(\mathbf{x}, t)/\partial x_k$ for $k \in \{1, \dots, m\}$, and $V_{gx} := (1/2)\text{trace}\{\mathbf{g}(\mathbf{x}, t)^\top V_{xx}\mathbf{g}(\mathbf{x}, t)\}$ where V_{xx} is an $m \times m$ matrix with entries $[V_{x_i x_j}] := \partial^2 V(\mathbf{x}, t)/\partial x_j \partial x_i$ for all $i, j \in \{1, \dots, m\}$.

Theorem 2.1: Consider a non-negative function $V(\mathbf{x}, t) : \Omega \times \mathbb{R}^+ \cup \{0\} \mapsto \mathbb{R}^+$ satisfying $V(\mathbf{x}, t) \in \mathcal{C}^{2 \times 1}$. Assume there exist arbitrary constants $\rho > 0$, $c_1 > 0$, $c_2 \in \mathbb{R}$, $c_3 \geq 0$, such that $\forall \mathbf{x} \in \Omega - \{0\}$, and $\forall t \geq 0$ the following conditions are fulfilled: (i) $c_1 \|\mathbf{x}\|^\rho \leq V(\mathbf{x}, t)$; (ii) $LV(\mathbf{x}, t) \leq c_2 V(\mathbf{x}, t)$; (iii) $\|V_x(\mathbf{x}, t)\mathbf{g}(\mathbf{x}, t)\|^2 \geq c_3 V^2(\mathbf{x}, t)$. Then, $\lim_{t \rightarrow +\infty} \sup (1/t) \log(\|\mathbf{x}(t)\|) \leq -(c_3 - 2c_2)/\rho$, a.s. In particular, if $c_3 > 2c_2$; then, the trivial solution of (3) is almost surely exponentially stable.

Proof: This is a straightforward extension of Theorem 3.3, page 121 in [32] (where $\Omega \equiv \mathbb{R}^m$). The proof follows identical steps as those used in [32] and hence it is omitted here for the sake of brevity. ■

III. PROBLEM FORMULATION

A. Network Model

Consider a network of $N > 1$ diffusively coupled identical nodes represented by an undirected and connected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ associated to the Laplacian matrix $\mathcal{L} = [\ell_{ij}]$, where each node is described by a set of nonlinear stochastic differential equations of Itô type

$$d\mathbf{x}_i = \left(\mathbf{f}(\mathbf{x}_i, t) - \sigma \sum_{j=1}^N \ell_{ij} \mathbf{h}(\mathbf{x}_j, t) + \mathbf{u}_i \right) dt + \phi_i(\mathbf{x}) d\mathbf{B}_i \quad (6)$$

with $\mathbf{x}_i \in \Omega_0 \subseteq \mathbb{R}^n$, $\mathbf{x}_i(0) = \mathbf{x}_{i0}$, $i \in \mathcal{N}$ being the initial conditions for the i th node. We assume that both $\mathbf{f}$ (i.e. the node dynamics) and $\mathbf{h}$ (i.e. the coupling function) belong to $\mathcal{C}^{2 \times 1}$ and that $\mathbf{f}, \mathbf{h} : \Omega_0 \subseteq \mathbb{R}^n \times \mathbb{R}^+ \cup \{0\} \mapsto \mathbb{R}^n$. It is implicit in the notation that the set Ω_0 is an open, forward-invariant subset of $\mathbb{R}^n$. In certain applications, the set Ω_0 is not an open set (for example, for a biochemical system, this is given by non-negativity constraints on the state variables). For a non-open set Ω_0 , the fact that $\mathbf{f}(\cdot, \cdot)$ and $\mathbf{h}(\cdot, \cdot)$ belong to $\mathcal{C}^{2 \times 1}$ means that the function $\mathbf{f}(\cdot, t)$ and $\mathbf{h}(\cdot, t)$ can be both extended as a twice differentiable function to some open set which includes Ω_0 , and that $\mathbf{f}(\mathbf{x}, \cdot)$ and $\mathbf{h}(\mathbf{x}, \cdot)$ are differentiable

on this open set. The non-negative constant σ represents the coupling strength, and $\mathbf{u}_i$ is an exogenous control input. Finally, $\phi_i(\mathbf{x})$ is a possibly nonlinear function modeling the diffusion of noise through the network, $\mathbf{x} := [\mathbf{x}_1^\top, \dots, \mathbf{x}_N^\top]^\top$, while $\mathbf{B}_i$ is a p -dimensional Brownian motion defined on the probability space $(\Sigma, \mathcal{F}, \mathbb{P})$.

Assumption 3.1: There exists a positive constant, say k_f , such that for all $\mathbf{x}, \mathbf{y} \in \Omega_0$ and for all $t \geq 0$

$$(\mathbf{x} - \mathbf{y})^\top (\mathbf{f}(\mathbf{x}, t) - \mathbf{f}(\mathbf{y}, t)) \leq k_f (\mathbf{x} - \mathbf{y})^\top (\mathbf{x} - \mathbf{y}) \quad (7)$$

Assumption 3.2: The coupling function $\mathbf{h}(\cdot, \cdot)$ satisfies the quadratic-Lipschitz condition for some $\kappa_1 = k_l$, $\kappa_2 = k_h$, with $k_l, k_h > 0$ (see Definition 2.1).

Remark 3.1: The quadratic condition (7) is well known as the QUAD or one-side Lipschitz condition, which is widely used within the synchronization literature and is closely related to the Lipschitz condition and the notion of contractive vector-fields [27]. We also note that Assumption 3.2 implies that the possibly nonlinear function $\mathbf{h}(\cdot, \cdot)$ is required to be Lipschitz and strongly monotone. Indeed, the monotonicity condition is crucial to guarantee convergence [34], [35]. We wish to emphasize that Assumption 3.2 encompasses a broader class of coupling functions that are not necessarily linear as often assumed in the existing literature [21], [22], [26].

The goal of this paper is that of designing the control law, $\mathbf{u}_i$, so that all the states of the network asymptotically converge onto a common desired state. To that aim we use the well known pinning control strategy, where the control action is only exerted on a fraction of nodes [4].

B. Pinning Control

We consider the control action $\mathbf{u}_i(t)$ to be given by a proportional feedback controller of the form

$$\mathbf{u}_i(t) := -p_i \alpha (\mathbf{h}(\mathbf{x}_i, t) - \mathbf{h}(\mathbf{x}_r, t)) \quad (8)$$

where: (i) $\alpha > 0$ represents the control strength; (ii) $\mathbf{x}_r \in \Omega_0$ is the reference (or target) state generated by a master (or pinner) node external to the network; (iii) p_i is a constant value equal to 1 if the i -th node is being controlled or 0 otherwise. We denote the set of pinned (or leader) nodes as a subset of node indices $\mathcal{N}_p \subset \mathcal{N}$ such that $p_i = 1$.

Definition 3.1: Under the control action (8), the stochastic network (6) is said to reach complete stochastic synchronization onto $\mathbf{x}_r(t)$ if

$$\lim_{t \rightarrow +\infty} \sup \frac{1}{t} \log(\|\mathbf{x}_i(t) - \mathbf{x}_r(t)\|) < 0, \quad \text{a.s.,} \quad i \in \mathcal{N} \quad (9)$$

Definition 3.2: We say that network (6) controlled by (8) is *pinning controllable in a stochastic sense* if there exist a value of the constant parameter α and a set of pinned nodes $\mathcal{N}_p$ for which complete stochastic synchronization is achieved onto the reference trajectory $\mathbf{x}_r(t)$.

Fig. 1. (a) Graph of a network with noisy communication links, (b) Multiplex representation of the noisy network

C. Noise on Networks

Next, we model different sources of noise in a network of nonlinear agents such as (6) by making different choices for the term $\phi_i(\mathbf{x})d\mathbf{B}_i$ therein. We first consider the case where noise enters the network through the control link between the pinner node and the nodes in the network which are directly controlled (or pinned). To model this scenario, we set

$$\phi_i(\mathbf{x})d\mathbf{B}_i = -\sigma^* p_i (\mathbf{h}(\mathbf{x}_i, t) - \mathbf{h}(\mathbf{x}_r, t)) db_i \quad (10)$$

where $\sigma^* > 0$ is a positive constant representing the intensity of noise and $b_i \in \mathbb{R}$ are independent Brownian motions. It is important to highlight that this type of perturbations might be used to model more realistic scenarios considering non-ideal sensors and actuators. For instance, consider the special case when $\mathbf{h}(\mathbf{x}_i, t) = \mathbf{x}_i$, and each pinned node has access to quantized state variables for computing the control action $\mathbf{u}_i$. Then, as pointed out in [13], [14], the quantized measurement of $\mathbf{x}_i - \mathbf{x}_r$ is given by $\mathbf{x}_i - \mathbf{x}_r + \sigma^* (\mathbf{x}_i - \mathbf{x}_r) w_i(t)$, with $w_i(t) = db_i/dt$, $i \in \mathcal{N}_p$ being independent white noises. Therefore, we can rewrite the whole network dynamics as in (6)-(10).

As a second type of noise diffusion, we consider the case where the communication links are noisy. In this case we set

$$\phi_i(\mathbf{x})d\mathbf{B}_i = \sigma^* \sum_{j=1}^N a_{ij}^* (\mathbf{h}(\mathbf{x}_j, t) - \mathbf{h}(\mathbf{x}_i, t)) db_{ij} \quad (11)$$

where db_{ij} are independent one-dimensional Brownian motions, and a_{ij}^* are the elements of an adjacency matrix $\mathcal{A}^*$ representing the structure of the network with noisy communication links. We denote such network structure by a graph $\mathcal{G}^* = (\mathcal{N}, \mathcal{E}^*)$. We wish to emphasize that, in general, $\mathcal{A} \neq \mathcal{A}^*$ and this situation may arise in the case where noise is only present (or the noise intensity is so small that can be neglected) on a subset of links of the network (6). See, for example, Figure 1(a), where $\mathcal{G}^* \subset \mathcal{G}$. Indeed, the presence of two different types of link (noisy and noise-free links) between the same nodes is known as multiplexity [36]; therefore, the structure of the network in (6) together with the noisy communication links (10) can be modeled as a multigraph of two layers $\mathcal{M} = \{\mathcal{G}, \mathcal{G}^*\}$ as shown in Figure 1(b). As noted in [14], [37] and references therein, the type of noise diffusion processes in (10) and (11) naturally arises when modeling communications among agents that are subject to quantization and/or the communication channel experiences fading.

Finally, we model the case of uncertain or noisy node dynamics by choosing

$$\phi_i(\mathbf{x})d\mathbf{B}_i = \sigma^* \mathbf{g}(\mathbf{x}_i, t) db \quad (12)$$

where b is a one-dimensional Brownian motion and $\mathbf{g} : \Omega_0 \subseteq \mathbb{R}^n \times \mathbb{R}^+ \cup \{0\} \mapsto \mathbb{R}^n$ is a nonlinear function that represents uncertainty on the vector-field $\mathbf{f}(\mathbf{x}_i, t)$, for instance where some parameters are subject to random environmental effects [32] (see application example in Section V-A).

Assumption 3.3: The function $\mathbf{g}(\mathbf{x}_i, t)$ in (12) satisfies the quadratic-Lipschitz condition (1) for some $\kappa_1 = k_{lg}$, $\kappa_2 = k_g$ with $k_{lg} \in \mathbb{R}^+$ and $k_g \in \mathbb{R}$ (see Definition 2.1).

Note that k_{lg} can also take negative values.

IV. MAIN RESULTS

We start by providing a sufficient condition for stochastic pinning controllability of network (6) under pinning control and subject to noise on the control links.

A. Noise on the control action

Theorem 4.1: Consider the stochastic network (6) controlled by (8) and assume noise is acting on the feedback control action, as given by (10). Let Assumptions 3.1, and 3.2 hold, and assume that: (i) there is at least one pinned node, i.e. $p_i \neq 0$, for some $i \in \mathcal{N}$, and $\mathcal{G}$ is undirected and connected; (ii) $\mathbf{x}_r(t)$ is a solution of the uncoupled dynamics, i.e. $d\mathbf{x}_r = \mathbf{f}(\mathbf{x}_r, t)dt$, with initial condition $\mathbf{x}_r(0) \in \Omega_0$. Then, the closed loop network is stochastic pinning controllable if

$$\tilde{\lambda} > \frac{2k_f}{k_h} + \frac{(\sigma^* k_l)^2}{k_h}, \quad (13)$$

where $\tilde{\lambda} := \lambda_{\min}(\tilde{\mathcal{L}})$ is the smallest eigenvalue of the matrix $\tilde{\mathcal{L}} := \sigma \mathcal{L} + \alpha \mathbf{P}$ with $\mathbf{P} := \text{diag}\{p_1, \dots, p_N\}$.

Proof: For the sake of clarity we divide the proof in two steps:

Step 1: We first define the error at each node as the difference between the i th node state and the reference trajectory as $\mathbf{e}_i := \mathbf{x}_i - \mathbf{x}_r$. Letting $\mathbf{x} := [\mathbf{x}_1^\top, \dots, \mathbf{x}_N^\top]^\top$, and $\mathbf{r} := \mathbf{1}_N \otimes \mathbf{x}_r$ be the stack vectors of the node states and reference trajectory respectively, the overall error dynamics $\mathbf{e} := [\mathbf{e}_1^\top, \dots, \mathbf{e}_N^\top]^\top$ can be recast in compact form as

$$d\mathbf{e} = \tilde{\mathbf{F}}(\mathbf{e}, t)dt + \tilde{\mathbf{G}}(\mathbf{e}, t)d\mathbf{B}, \quad (14)$$

where,

$$\begin{aligned} \tilde{\mathbf{F}}(\mathbf{e}, t) &= \mathbf{F}(\mathbf{e} + \mathbf{r}, t) - \mathbf{F}(\mathbf{r}, t) + \alpha(\mathbf{P} \otimes \mathbf{I}_n)\mathbf{H}(\mathbf{r}, t) \\ &\quad - (\sigma(\mathcal{L} \otimes \mathbf{I}_n) + \alpha(\mathbf{P} \otimes \mathbf{I}_n))\mathbf{H}(\mathbf{e} + \mathbf{r}, t)(15) \end{aligned}$$

with $\mathbf{F}(\mathbf{e} + \mathbf{r}, t) := [\mathbf{f}(\mathbf{e}_1 + \mathbf{x}_r, t)^\top, \dots, \mathbf{f}(\mathbf{e}_N + \mathbf{x}_r, t)^\top]^\top$, $\mathbf{H}(\mathbf{e} + \mathbf{r}, t) := [\mathbf{h}(\mathbf{e}_1 + \mathbf{x}_r, t)^\top, \dots, \mathbf{h}(\mathbf{e}_N + \mathbf{x}_r, t)^\top]^\top$. The diffusion term $\tilde{\mathbf{G}}(\mathbf{e}, t) = -\sigma^* \mathbf{G}(\mathbf{e}, t)$, where $\mathbf{G}(\mathbf{e}, t)$ is an $nN \times N$ matrix given by

$$\begin{bmatrix} p_1 (\mathbf{h}(\mathbf{x}_1, t) - \mathbf{h}(\mathbf{x}_r, t)) & \cdots & 0_{nN \times 1} \\ \vdots & \ddots & \vdots \\ 0_{nN \times 1} & \cdots & p_N (\mathbf{h}(\mathbf{x}_N, t) - \mathbf{h}(\mathbf{x}_r, t)) \end{bmatrix} \quad (16)$$

and $\mathbf{B} = [b_1, \dots, b_N]^\top$. Note that: (i) $\mathbf{x}_r$ is a solution of the uncoupled dynamics with initial conditions in Ω_0 ; then, $\mathbf{x}_r(t) \in \Omega_0, \forall t \geq 0$; (ii) $\mathbf{e} = \mathbf{0}$ is the trivial solution of (14). Therefore, Theorem 2.1 can be used to prove almost sure exponential stability of $\mathbf{e} = \mathbf{0}$. To that aim, consider the Lyapunov candidate function

$$V(\mathbf{e}, t) = V(\mathbf{e}) = \frac{1}{2} \mathbf{e}^\top \mathbf{e} \quad (17)$$

It is easy to verify that V is a positive definite function and satisfies condition (i) of Theorem 2.1 (where we assume $\rho = 2$).

Step 2: Next we calculate $LV(\mathbf{e})$ as defined in (5). It is easy to see that the first term is null, i.e. $V_t(\mathbf{e}) = 0$ and $V_e = \mathbf{e}^\top$. Calculating $V_e(\mathbf{e})\tilde{\mathbf{F}}(\mathbf{e}, t)$ yields

$$\begin{aligned} V_e(\mathbf{e})\tilde{\mathbf{F}}(\mathbf{e}, t) &= \mathbf{e}^\top (\mathbf{F}(\mathbf{e} + \mathbf{r}, t) - \mathbf{F}(\mathbf{r}, t)) \\ &\quad + \alpha \mathbf{e}^\top (\mathbf{P} \otimes \mathbf{I}_n) \mathbf{H}(\mathbf{r}, t) - \mathbf{e}^\top (\tilde{\mathcal{L}} \otimes \mathbf{I}_n) \mathbf{H}(\mathbf{x}, t) \end{aligned}$$

From Assumption 3.1 we have that $\mathbf{e}^\top (\mathbf{F}(\mathbf{x}, t) - \mathbf{F}(\mathbf{r}, t)) \leq k_f \mathbf{e}^\top \mathbf{e}$, therefore

$$\begin{aligned} V_e(\mathbf{e})\tilde{\mathbf{F}}(\mathbf{e}, t) &\leq k_f \mathbf{e}^\top \mathbf{e} - \mathbf{e}^\top (\tilde{\mathcal{L}} \otimes \mathbf{I}_n) \mathbf{H}(\mathbf{x}, t) \\ &\quad + \alpha \mathbf{e}^\top (\mathbf{P} \otimes \mathbf{I}_n) \mathbf{H}(\mathbf{r}, t) \end{aligned}$$

Next, adding and subtracting the term $\mathbf{e}^\top (\tilde{\mathcal{L}} \otimes \mathbf{I}_n) \mathbf{H}(\mathbf{r}, t)$ to the right-hand side of the last inequality, we obtain

$$\begin{aligned} V_e(\mathbf{e})\tilde{\mathbf{F}}(\mathbf{e}, t) &\leq k_f \mathbf{e}^\top \mathbf{e} - \mathbf{e}^\top (\tilde{\mathcal{L}} \otimes \mathbf{I}_n) (\mathbf{H}(\mathbf{x}, t) - \mathbf{H}(\mathbf{r}, t)) \\ &\quad - \mathbf{e}^\top (\tilde{\mathcal{L}} \otimes \mathbf{I}_n) \mathbf{H}(\mathbf{r}, t) + \alpha \mathbf{e}^\top (\mathbf{P} \otimes \mathbf{I}_n) \mathbf{H}(\mathbf{r}, t) \end{aligned}$$

and from the fact that $-(\tilde{\mathcal{L}} \otimes \mathbf{I}_n) + \alpha(\mathbf{P} \otimes \mathbf{I}_n) = -(\mathcal{L} \otimes \mathbf{I}_n)$, and $(\mathcal{L} \otimes \mathbf{I}_n) \mathbf{H}(\mathbf{r}, t) = (\mathcal{L} \otimes \mathbf{I}_n)(\mathbf{1}_N \otimes \mathbf{h}(\mathbf{x}_r, t)) = \mathbf{0}$ one has

$$V_e(\mathbf{e})\tilde{\mathbf{F}}(\mathbf{e}, t) \leq k_f \mathbf{e}^\top \mathbf{e} - \mathbf{e}^\top (\tilde{\mathcal{L}} \otimes \mathbf{I}_n) (\mathbf{H}(\mathbf{x}, t) - \mathbf{H}(\mathbf{r}, t)) \quad (18)$$

Next, defining $\zeta = -\mathbf{e}^\top (\tilde{\mathcal{L}} \otimes \mathbf{I}_n) (\mathbf{H}(\mathbf{x}, t) - \mathbf{H}(\mathbf{r}, t))$ we get:

$$\begin{aligned} \zeta &= -\sigma \sum_{i,j \in \mathcal{E}} (\mathbf{x}_j - \mathbf{x}_i)^\top (\mathbf{h}(\mathbf{x}_j, t) - \mathbf{h}(\mathbf{x}_i, t)) \\ &\quad - \alpha \sum_{i=1}^N p_i (\mathbf{x}_i - \mathbf{x}_r)^\top (\mathbf{h}(\mathbf{x}_i, t) - \mathbf{h}(\mathbf{x}_r, t)) \end{aligned} \quad (19)$$

where we used Lemma 2.1 [with $\mathbf{x} = \mathbf{e}$ and $\mathbf{y} = \mathbf{H}(\mathbf{x}, t) - \mathbf{H}(\mathbf{r}, t)$] and the fact that the matrix $\mathbf{P}$ is a diagonal matrix. Moreover, by means of Assumption 3.2 one has

$$\begin{aligned} \zeta &\leq -k_h \sigma \sum_{i,j \in \mathcal{E}} (\mathbf{x}_j - \mathbf{x}_i)^\top (\mathbf{x}_j - \mathbf{x}_i) \\ &\quad - k_h \alpha \sum_{i=1}^N p_i (\mathbf{x}_i - \mathbf{x}_r)^\top (\mathbf{x}_i - \mathbf{x}_r) \\ &= -k_h \mathbf{e}^\top (\sigma \mathcal{L} \otimes \mathbf{I}_n) \mathbf{e} - k_h \mathbf{e}^\top (\alpha \mathbf{P} \otimes \mathbf{I}_n) \mathbf{e} \\ &= -k_h \mathbf{e}^\top (\tilde{\mathcal{L}} \otimes \mathbf{I}_n) \mathbf{e} \leq -k_h \tilde{\lambda} \mathbf{e}^\top \mathbf{e} \end{aligned} \quad (20)$$

where $\tilde{\lambda}$ is the smallest eigenvalue of matrix $\tilde{\mathcal{L}}$. Therefore we can rewrite (18) as

$$V_e(\mathbf{e})\tilde{\mathbf{F}}(\mathbf{e}, t) \leq (k_f - k_h \tilde{\lambda}) \mathbf{e}^\top \mathbf{e} \quad (21)$$

Now, we calculate the last term of $LV(\mathbf{e})$, $V_{ge} = (1/2)\text{trace}\{\tilde{\mathbf{G}}(\mathbf{e}, t)^\top \mathbf{V}_{ee} \tilde{\mathbf{G}}(\mathbf{e}, t)\}$. Since $\mathbf{V}_{ee} = \mathbf{I}_{Nn}$ one has

$$V_{ge} = \frac{(\sigma^*)^2}{2} \sum_{i=1}^N p_i (\mathbf{h}(\mathbf{x}_i, t) - \mathbf{h}(\mathbf{x}_r, t))^\top (\mathbf{h}(\mathbf{x}_i, t) - \mathbf{h}(\mathbf{x}_r, t))$$

from Assumption 3.1, we have that $\mathbf{h}(\cdot, \cdot)$ satisfies the Lipschitz condition (1), yielding

$$V_{ge} \leq \frac{(k_l \sigma^*)^2}{2} \sum_{i=1}^N p_i (\mathbf{x}_i - \mathbf{x}_r)^\top (\mathbf{x}_i - \mathbf{x}_r) \quad (22)$$

$$\leq \frac{(k_l \sigma^*)^2}{2} \mathbf{e}^\top \mathbf{P} \mathbf{e} \leq \frac{(k_l \sigma^*)^2}{2} \mathbf{e}^\top \mathbf{e} \quad (23)$$

Next, using the upper bounds (21) and (23) we find that

$$LV(\mathbf{e}) \leq c_2 V(\mathbf{e}) \quad (24)$$

where $c_2 = 2(k_f - k_h \tilde{\lambda}) + (k_l \sigma^*)^2$. Then conditions (ii) and (iii) of Theorem 2.1 are fulfilled provided (13) holds and $c_3 = 0$. Therefore, the closed-loop network reaches stochastic synchronization onto the reference trajectory and the proof is complete. ■

B. Noise on the communication links

Next, we provide a result for pinning controllability of network (6), when this is controlled by (8) and the noise diffusion processes are given by (11).

Theorem 4.2: Consider the stochastic network (6) controlled by (8) where noise is acting on the communication links, i.e. $\phi_i(\mathbf{x}) d\mathbf{B}_i$ is given by (11). Let Assumptions 3.1, and 3.2 hold and assume that: (i) $p_i \neq 0$, for some $i \in \mathcal{N}$, and $\mathcal{G}$ is undirected and connected; (ii) $\mathbf{x}_r(t)$ is a solution of $d\mathbf{x}_r = \mathbf{f}(\mathbf{x}_r, t)dt$, with $\mathbf{x}_r(0) \in \Omega_0$. Then, the closed loop multiplex network with $\mathcal{M} = \{\mathcal{G}, \mathcal{G}^*\}$ is pinning controllable if

$$\tilde{\lambda} > \frac{k_f}{k_h} + \frac{(\sigma^* k_l \lambda_N^*)^2}{2k_h} \quad (25)$$

Proof: Considering the error $\mathbf{e}_i = \mathbf{x}_i - \mathbf{x}_r$ and following similar steps to those already presented in the proof of Theorem 4.1 yields (14) where $\tilde{\mathbf{F}}(\mathbf{e}, t)$ is given in (15) and $\tilde{\mathbf{G}}(\mathbf{e}, t) = \sigma^* \mathbf{G}(\mathbf{x}, t)$ with $\mathbf{G}(\mathbf{x}, t)$ being a $nN \times N^2$ matrix given by

$$\mathbf{G}(\mathbf{x}, t) = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{0}_{1 \times N} & \cdots & \mathbf{0}_{1 \times N} \\ \mathbf{0}_{1 \times N} & \mathbf{Z}_2 & \cdots & \mathbf{0}_{1 \times N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{1 \times N} & \mathbf{0}_{1 \times N} & \cdots & \mathbf{Z}_N \end{bmatrix} \quad (26)$$

where $\mathbf{Z}_i \in \mathbb{R}^{n \times N}$ matrix given by $\mathbf{Z}_i := [a_{i1}^*(\mathbf{h}(\mathbf{x}_1, t) - \mathbf{h}(\mathbf{x}_i, t)), \dots, a_{iN}^*(\mathbf{h}(\mathbf{x}_N, t) - \mathbf{h}(\mathbf{x}_i, t))]$. a_{ij}^* are the elements of the adjacency matrix $\mathcal{A}^*$ denoting the location of noisy communication links. Moreover, $\mathbf{B} = [\mathbf{b}_1^\top, \dots, \mathbf{b}_N^\top]$ with $\mathbf{b}_i = [b_{i1}, b_{i2}, \dots, b_{iN}]^\top$. Next, consider the candidate Lyapunov function (17) and calculating $LV(\mathbf{e})$ along the trajectories of (14)-(15)-(26) yields $LV(\mathbf{e}) = V_e(\mathbf{e})\tilde{\mathbf{F}}(\mathbf{e}, t) + V_{ge}$, where an upper-bound for $V_e(\mathbf{e})\tilde{\mathbf{F}}(\mathbf{e}, t)$ is given in (21), while $V_{ge} = (1/2)\text{trace}\{\tilde{\mathbf{G}}^\top \mathbf{G}\}$. In order to find an upper-bound for V_{ge} we start by noticing [from (26)] that $\mathbf{G}(\mathbf{r}, t) = \mathbf{0}$, so

that we can rewrite $\tilde{G}(e, t) = G(x) - G(r, t)$. Then V_{ge} can be rewritten as $V_{ge} = \frac{(\sigma^*)^2}{2} \text{trace}\{M\}$ with

$$M = (\tilde{G}(x, t) - \tilde{G}(r, t))^\top (\tilde{G}(x, t) - \tilde{G}(r, t))$$

Since M is a block-diagonal matrix we have that $\text{trace}\{M\} = \frac{(\sigma^*)^2}{2} \sum_{i=1}^N \text{trace}\{\tilde{Z}_i^\top \tilde{Z}_i\}$ where $\tilde{Z}_i := [a_{i1}^*(\tilde{h}_1 - \tilde{h}_i), \dots, a_{iN}^*(\tilde{h}_N - \tilde{h}_i)]$ with $\tilde{h}_i := h(x_i, t) - h(x_r, t)$ for $i \in \mathcal{N}$. Then, letting $\tilde{H} = H(x, t) - H(r, t) = [\tilde{h}_1, \dots, \tilde{h}_N]^\top$ one has

$$\begin{aligned} V_{ge} &= \frac{(\sigma^*)^2}{2} \sum_{i=1}^N \text{trace}\{\tilde{Z}_i^\top \tilde{Z}_i\} \\ &= \frac{(\sigma^*)^2}{2} \sum_{i=1}^N \sum_{j=1}^N (a_{ij}^*)^2 (\tilde{h}_j - \tilde{h}_i)^\top (\tilde{h}_j - \tilde{h}_i) \\ &= \frac{(\sigma^*)^2}{2} \tilde{H}^\top ((\mathcal{L}^*) \otimes I_n) \tilde{H} \leq \frac{(\sigma^* \lambda_N^*)^2}{2} \tilde{H}^\top \tilde{H} \\ &\leq c \sum_{i=1}^N (h(x_i, t) - h(x_r, t))^\top (h(x_i, t) - h(x_r, t)) \end{aligned} \quad (27)$$

with $c = (\sigma^* \lambda_N^*)^2/2$. Now, from Assumption 3.2, we find that

$$(h(x_i, t) - h(x_r, t))^\top (h(x_i, t) - h(x_r, t)) \leq k_l^2 (x - r)^\top (x - r)$$

so that $V_{ge} \leq (\sigma^* \lambda_N^* k_l)^2 V(e)$, and $LV(e) \leq c_2 V(e)$ where $c_2 = 2(k_f - k_h \tilde{\lambda}) + (\lambda_N^* \sigma^* k_l)^2$. Therefore, if condition (25) holds, setting $c_3 = 0$ in Theorem 2.1 guarantees complete stochastic synchronization of the closed-loop network onto x_r which completes the proof. ■

Remark 4.1: Theorems 4.1 and 4.2 provide sufficient algebraic conditions guaranteeing pinning controllability of a given network of interest. Such conditions depend on the node dynamics (via k_f), the structure of the network and the location of the pinned nodes (via $\tilde{\lambda}$), the coupling functions (via k_h and k_l), and on the intensity of the noise (via σ^* and $\sigma^* \lambda_N^*$ for noisy control and communication links respectively). Intuitively, the conditions indicate that, for a given network structure, noise can be compensated by either adding more control interventions or by increasing the feedback control gain α . We shall see that this is not the case for the next scenario –the case where noise affects the node dynamics itself– Indeed, it is shown that under certain conditions, noise can be useful for enhancing pinning controllability.

We also remark that the conditions in Theorems 4.1 and 4.2 depend on $\tilde{\lambda}$. Analytical estimates of $\tilde{\lambda}$ as a function of the number of pinned nodes, control strength α and network structure $\mathcal{L}$ can be obtained as shown in e.g. [9], [38]. In addition, $\tilde{\lambda}$ can also be computed in a distributed manner by using the distributed power iteration method [39].

C. Noise on the node dynamics

The following result gives a sufficient condition for pinning controllability of network (6)-(8) when noise diffusion processes are modeled as in (12).

Theorem 4.3: Consider the stochastic network (6) controlled by (8) and with noise propagating according to (12). Let

Assumptions 3.1, 3.2, and 3.3 hold, and assume that: (i) $p_i \neq 0$, for some $i \in \mathcal{N}$, and $\mathcal{G}$ is undirected and connected; (ii) x_r is a solution of $dx_r = f(x_r, t)dt + \sigma^* g(x_r, t)db$. Then, the closed-loop network is pinning controllable if

$$\tilde{\lambda} > \frac{k_f}{k_h} - \frac{(\sigma^*)^2}{2k_h} (2k_g^2 - k_{lg}^2) \quad (28)$$

Proof: Let $e_i := x_i - x_r$ be the error at each node. As in the proof of Theorem 4.1, we have that the overall error dynamics can be written as (14) where $\tilde{F}(e, t)$ is given by (15) and $\tilde{G}(e, t) = \sigma^* [(g(x_1, t) - g(x_r, t))^\top, \dots, (g(x_N, t) - g(x_r, t))^\top]^\top$. Considering the candidate Lyapunov function (17) and calculating $LV(e)$ yields $LV(e) = V_e(e, t) \tilde{F}(e, t) + V_{ge}$ where $V_e(e) \tilde{F}(e, t)$ is given by (21) while

$$\begin{aligned} V_{ge} &= \frac{1}{2} \text{trace}\{\tilde{G}^\top(e, t) \tilde{G}(e, t)\} \\ &= \frac{(\sigma^*)^2}{2} \sum_{i=1}^N (g(x_i, t) - g(x_r, t))^\top (g(x_i, t) - g(x_r, t)) \\ &\leq \frac{(\sigma^* k_{lg})^2}{2} \sum_{i=1}^N (x_i - x_r)^\top (x_i - x_r) \\ &\leq \frac{(\sigma^* k_{lg})^2}{2} e^\top e = (\sigma^* k_{lg})^2 V(e). \end{aligned} \quad (29)$$

Therefore, we have that $LV(e) \leq c_2 V(e)$ with $c_2 = 2(k_f - k_h \tilde{\lambda}) + (\sigma^* k_l)^2$. Next, we have that $e^\top \tilde{G}(e, t) = \sigma^* \sum_{i=1}^N (x_i - x_r)^\top (g(x_i, t) - g(x_r, t))$, and from Assumption 3.3 yields $e^\top \tilde{G}(e, t) \geq k_g e^\top e$ so that $\|e^\top \tilde{G}(e, t)\|^2 \geq 4(\sigma^* k_g)^2 V(e)$. If condition (28) holds; then, condition (ii)-(iii) of Theorem 2.1 are fulfilled with $c_3 > 2c_2$ so that the closed-loop network reaches complete stochastic synchronization onto x_r and the proof is complete. ■

Remark 4.2: Condition (28) reveals that noise can be beneficial to enhance the ability of a network to be controlled. Indeed, if $k_g/k_{lg} > (\sqrt{2}/2)$; then, pinning controllability can be improved for lower values of $\tilde{\lambda}$ by adding noise. This will be illustrated by some representative examples in Section V. We wish to highlight that noise can be potentially exploited for control purposes by injecting random signals on the nodes using noise generators. Interestingly, this observation is consistent with the recent findings in [40], [20], and also with previous studies in chaos control [41].

Remark 4.3: Note that the reference signal x_r is assumed to be the solution of an isolated node perturbed by noise. This is crucial to guarantee exact convergence of all nodes towards the noisy target evolution. If the reference signal does not have any stochastic term, i.e. it is the solution of $\dot{x}_r = f(x_r)$, then convergence is still achieved but a residual error will be present at steady state. Indeed, as for standard proportional control, the mean and variance of the steady state error might be attenuated as the control gain increases (see the example in the Applications Section V-A). Finding explicit bounds of the expectation and variance of the steady state error when the reference trajectory is noise-free is an open problem which is left for future work.

D. Control design

The location of pinned nodes and the control gain α can be properly designed according to the simple algebraic conditions obtained in the previous results. Indeed, the control design is based on the relation $\tilde{\lambda} > c_k$, where

$$c_k = \begin{cases} 2c_0 + (\sigma^* k_l)^2 / k_h, & k = 1 \quad \text{Control} \\ c_0 + (\sigma^* k_l \lambda_N^*)^2 / 2k_h, & k = 2 \quad \text{Links} \\ c_0 - (\sigma^*)^2 / 2k_h (2k_g^2 - k_{lg}^2), & k = 3 \quad \text{Nodes} \end{cases} \quad (30)$$

with $c_0 = k_f / k_h$. Note that given a certain network topology $\mathcal{L}$ and noise intensity σ^* , the control design consist on finding the matrix $\mathbf{P}$ and a positive gain α such that (30) is satisfied according to on of the three different scenarios. Moreover, in the third case ($k = 3$), noise might be beneficial and can be also used as an extra degree of freedom for designing the pinning control strategy (see Remark 4.2). We will illustrate in next Section the use of condition (30) for designing the pinning controller

V. APPLICATION EXAMPLES

A. Gene regulatory networks

As a first application example we study the pinning controllability of the genetic Toggle Switch, originally introduced (and engineered in *Escherichia coli*) in [42]. Such a biochemical circuit consists of two genes mutually inhibiting each others' promoters, see Figure 2(a).

1) *Node dynamics*: Following [42], the dynamics of a single Toggle Switch can be modeled with the following set of differential equations

$$\frac{dU}{dt} = \frac{\alpha_1}{1 + V^\beta} - U, \quad \frac{dV}{dt} = \frac{\alpha_2}{1 + U^\gamma} - V \quad (31)$$

where $U \in \mathbb{R}^+$ and $V \in \mathbb{R}^+$ are the state variables representing the concentration of repressor 1 and repressor 2 respectively. The parameters: (i) α_1 and α_2 are two positive constants denoting the effective rates of synthesis of repressor 1 and 2 respectively; (ii) β, γ are two nonnegative constants representing the cooperativity of repression of promoter 2 and 1, respectively. We consider the same parameter values as reported in [42], $\alpha_1 = 50$, $\alpha_2 = 20$, $\beta = 2.5$, and $\gamma = 1$. With this choice of parameters, the set onto which (31) evolves (i.e. Ω_0) is the positive orthant of $\mathbb{R}^2$. The basins of attraction of the toggle switch stable equilibria $(U, V) = (0.0301, 19.4161)$ (or $(U, V) = (\text{OFF}, \text{ON})$), $(U, V) = (44.2548, 0.4420)$ (or $(U, V) = (\text{ON}, \text{OFF})$) are also depicted in Figure 2(b).

2) *Stochastic network dynamics*: Different experimental results have confirmed that gene expressions are stochastic processes [43], where noise plays a very important role in the switching of such bistable systems [12], [44], and also it can be exploited for control [45]. We consider a network of $N > 1$ diffusively coupled toggle switches; that is, network (6) with $n = 2$,

$$\mathbf{f}(\mathbf{x}_i, t) = \mathbf{f}(\mathbf{x}_i) = \begin{bmatrix} \alpha_1 / (1 + V_i^\beta) - U_i \\ \alpha_2 / (1 + U_i^\gamma) - V_i \end{bmatrix}, \quad i \in \mathcal{N}, \quad (32)$$

Fig. 2. (a) Schematic of the toggle switch design [42], (b) Basins of attraction of the toggle switch model for $\alpha_1 = 50$, $\alpha_2 = 20$, $\beta = 2.5$, and $\gamma = 1$.

where $\mathbf{x}_i := [U_i, V_i]^\top$ with $U_i, V_i \in \mathbb{R}^+$ representing the concentrations of the i -th node. The coupling functions are assumed to be linear i.e., $\mathbf{h}(\mathbf{x}_i, t) = \mathbf{h}(\mathbf{x}_i) = \mathbf{x}_i$. We wish to emphasize that such couplings can be realized by an additional reaction $\mathbf{x}_j \rightleftharpoons \mathbf{x}_i$ with reaction rate $a_{ij} = a_{ji}$ [46], where a_{ij} are the entries of the adjacency matrix $\mathcal{A}$ (with associated Laplacian matrix $\mathcal{L}$) representing the network of interconnection between the toggle switches.

In this example we examine the scenario where nodes are affected by noise. Specifically, we consider the case where the effective rates of synthesis are subject to some random environmental effect, that is, $\alpha_1 = \hat{\alpha}_1 + \sigma^* w(t)$ and $\alpha_2 = \hat{\alpha}_2 + \sigma^* w(t)$, where $\hat{\alpha}_1 = 50$ and $\hat{\alpha}_2 = 20$ are the nominal values, $w(t) = db/dt$ is a white noise and σ^* represent the intensity of noise. Then the nonlinear stochastic term in (6) can be written as in (12) with $\mathbf{g}(\mathbf{x}_i) = [1/(1 + V_i^\beta), 1/(1 + U_i^\gamma)]^\top$.

3) *Control design*: Our control target is to drive the concentrations of all N units to the desired state $(U_i, V_i) = (44.2548, 0.4420)$ (or $U_i = \text{ON}$, $V_i = \text{OFF}$) for all $i \in \mathcal{N}$. To this aim, we use the pinning controller (8) where the reference signal $\mathbf{x}_r := [U_r, V_r]^\top$ is given by the solution of an isolated Toggle switch (31) with initial conditions (30, 1) (green region of Figure 2(b)).

To find the control gain α and the number and locations of pinned nodes we use condition (30) for $k = 3$ (or Theorem 4.3).

QUAD condition: In order to use condition (30) we need first to show that the vector-field $\mathbf{f}(\mathbf{x}_i)$ in (32) satisfies the QUAD condition (7). To that aim let us first define $\varphi(\theta) = \mathbf{f}(\mathbf{y} + \theta(\mathbf{x} - \mathbf{y}))$, for $\theta \in [0, 1]$ and $\mathbf{x}, \mathbf{y} \in \Omega_0$. Noticing that $\varphi(0) = \mathbf{f}(\mathbf{y})$ and $\varphi(1) = \mathbf{f}(\mathbf{x})$, from the Fundamental Theorem of calculus one has that $\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}) = \varphi(1) - \varphi(0) = \int_0^1 (d\varphi(\theta)/d\theta) d\theta$. Then, $\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}) = \int_0^1 \mathbf{D}\mathbf{f}(\mathbf{y} + \theta(\mathbf{x} - \mathbf{y})) d\theta$ where $\mathbf{D}\mathbf{f}$ is the Jacobian matrix given by

$$\mathbf{D}\mathbf{f} = \begin{bmatrix} -1 & Df_{12} \\ Df_{21} & -1 \end{bmatrix} \quad (33)$$

$$Df_{12} = \frac{-125(y_2 + \theta(x_2 - y_2))^{3/2}}{\left((y_2 + \theta(x_2 - y_2))^{5/2} + 1\right)^2} \quad (34)$$

$$Df_{21} = \frac{-20}{\left((y_1 + \theta(x_1 - y_1))^{5/2} + 1\right)^2} \quad (35)$$

Next, we have that

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq \sup_{\theta \in [0, 1]} \|\mathbf{D}\mathbf{f}(\mathbf{y} + \theta(\mathbf{x} - \mathbf{y}))\| \|\mathbf{x} - \mathbf{y}\|$$

from the triangle inequality we have $\|\mathbf{D}\mathbf{f}(\mathbf{y} + \theta(\mathbf{x} - \mathbf{y}))\| \leq 2 + \|Df_{12}\| + \|Df_{21}\|$ yielding $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq L \|\mathbf{x} - \mathbf{y}\|$ with $L = 59.5$, and from the fact that a Lipschitz function is also QUAD [27], $k_f = L$. The analysis above proves that the model of interest is described by a QUAD vector field but provides a conservative estimate of the Lipschitz constant k_f . Indeed using Matlab optimization

Fig. 3. Open-loop dynamics ($\mathbf{u}_i = 0$) for an all-to-all network of five toggle switches with noise acting on the nodes.

toolbox, we obtained numerically a better estimate for k_f which is equal to 21. Similarly, we find that the coupling function satisfies Assumption 3.2 with $k_h = 1$ and that the nonlinear function $\mathbf{g}(\mathbf{x}_i)$ satisfies Assumption 3.3 with $k_{lq} = 0.6$ and $k_g = -0.868$.

We consider the network topology $\mathcal{L}$ to be an all-to-all network of five nodes ($N = 5$, $\mathcal{N} = \{1, \dots, 5\}$) (note that in this case $\lambda_2 = 5$) and $\sigma = 14$. Then, we have from (30) with $k = 3$ that $\tilde{\lambda} > 21 - 0.5734\sigma^*$. Then in the absence of noise, i.e. $\sigma^* = 0$ we obtain the classic deterministic condition $\tilde{\lambda} > 21$ [9]. Then the pinning controllability condition is satisfied by setting $\alpha = 73$, and pinning three out of five nodes. If two nodes are pinned instead, the pinning controllability condition is not fulfilled.

Moreover, considering the noisy term with $\sigma^* = 3$, we find that pinning two nodes yields $\tilde{\lambda} = 16.1082 > 21 - 0.5734\sigma^* = 15.8392$ so that the condition is fulfilled and pinning controllability is guaranteed from Theorem 4.3. Note that increasing the noise intensity σ^* implies the network can be controlled for lower values of $\tilde{\lambda}$ so that less control interventions, and even lower values of the control strength α can guarantee the network reaches stochastic pinning synchronization. However, for large values of σ^* the variance of the reference trajectory increases.

4) *Numerical simulations:* In order to illustrate our results we first simulate the network dynamics in the absence of control, i.e. $\mathbf{u}_i = 0$ for all $i \in \mathcal{N}$. The time response of the open-loop stochastic network is shown in Figure 3 where the initial conditions of each node were randomly chosen on the yellow region of Figure 2(b). Note that all nodes reach the equilibrium $U = OFF, V = ON$ and the control target is not fulfilled. Then we apply the control feedback (8) pinning two nodes with $\alpha = 71$ and converge to the desired target trajectory is achieved as shown in Figure 3.

For the sake of completeness we further test the pinning control strategy when the reference signal does not have a stochastic term, i.e. $\dot{\mathbf{x}}_r = \mathbf{f}(\mathbf{x}_r)$. In this case, perfect agreement cannot be achieved (see remark 4.3), yet the error $\mathbf{e}(t)$ remain bounded at steady state. Indeed, we calculate the mean and variance of $\|\mathbf{e}(t)\|$ at steady state for different values of the control gain α and over one thousand trials starting from random initial conditions. The result is shown in Figure 5. Note that the mean of the error decreases as α increases.

B. Synchronization of chaotic oscillators

As an additional example, we now turn our attention to study pinning controllability of diffusively coupled chaotic systems. In

Fig. 4. Time response of an all-to-all network of five toggle switches with noise acting on the nodes and pinning control being exerted on one single node.

Fig. 5. Mean and variance of the error dynamics $\|\mathbf{e}(t)\|$ at steady state for different values of the control gain α .

particular we consider networks of chaotic Lorenz systems where the intrinsic nonlinear dynamics are given by

$$\mathbf{f}(\mathbf{x}_i, t) = \mathbf{f}(\mathbf{x}_i) = \begin{bmatrix} \mu(q_i - p_i) \\ p_i(\rho - r_i) - q_i \\ p_i q_i - \omega r_i \end{bmatrix}, \quad i \in \mathcal{N} \quad (36)$$

with $\mathbf{x}_i = [p_i, q_i, r_i]^\top$ and the parameters are set as $\mu = 10$, $\rho = 28$ and $\omega = 2$ (for such parameters the system exhibits a chaotic behavior [47]). The network topology is assumed to be given by the one shown in Figure 1(a) where $N = 6$ and $\sigma = 30$. In addition the coupling functions are assumed to be nonlinear and given by $\mathbf{h}(\mathbf{x}_i) = \mathbf{x}_i + \tan^{-1}(\mathbf{x}_i)$. Next we illustrate the effectiveness of our approach in two different scenarios; namely, noise on the communication and control links

1) *Noise on the communication links:* We first consider the case where noise is propagating thorough the communication links. Particularly we assume $\sigma^* = 1$ and noise propagating over a subset of links of the original network topology (illustrated in Figure 1(b)) for which $\lambda_N^* = 4.2143$. Next, we use Theorem 4.2 to design the pinning controller. To that aim, we first recall the fact that $\mathbf{f}(\mathbf{x}_i)$ satisfies the QUAD assumption (7) with $k_f = 14$ [47]. Next, we can easily verify that the nonlinear function $\mathbf{h}(\mathbf{x}_i)$ has bounded Jacobian and hence it can be shown to satisfy Assumption 3.2 with $k_l = 2$ and $k_h = 1$, respectively. From (30) with $k = 2$ we can conclude that the network of chaotic Lorenz systems is pinning controllable if $\tilde{\lambda} > 49.5206$.

Fig. 6. Time response of a multiplex network (Figure 1) of six chaotic Lorenz oscillators pinning the nodes 1, 4 and 5. The black dashed-line represents the desired trajectory x_r .

Fig. 7. Time evolution of the error dynamics for a network of chaotic Lorenz with noise acting on the control links.

We find that the network is pinning controllable if nodes 1, 4, and 5 are pinned with $\alpha = 350$. Hence, the closed-loop network converges to the desired target trajectory (see Figure V-B1) as theoretically predicted by Theorem 4.2.

2) *Noise on the control links:* We consider again the network of chaotic Lorenz systems presented above but, this time, we suppose the noise is acting on the control action via equation (10) with $\sigma^* = 2.2$. Now, from (30) for $k = 1$ we find the network is pinning controllable if $\bar{\lambda} > 47.41$. This condition is fulfilled by using the same set of pinned nodes as in the previous example and by setting $\alpha = 350$. Indeed, as shown in Figure 7, with this choice of the pinned nodes and control strength, the closed-loop network converges to the desired target trajectory provided all conditions of Theorem 4.1 are fulfilled.

VI. CONCLUSIONS

In this paper pinning controllability has been investigated in networks affected by different types of noise diffusion processes. The results leverage the use of appropriate stochastic Lyapunov functions and the notion of almost sure exponential stability to obtain sufficient condition guaranteeing convergence of the closed-loop network onto the desired target state. We find that, surprisingly, noise can enhance pinning controllability when it propagates across the network in an homogeneous fashion. The effectiveness of our theoretical results was illustrated via two representative applications arising in the context of gene regulatory networks and ensembles chaotic oscillators. In the former, we confirmed that noise can be beneficial for enhancing pinning controllability, while in the latter we showed that more control interventions and higher gains are needed to cope with the noisy term. Ongoing work is aimed at extending our approach to directed network topologies. Also, the effect of varying the topology of the noisy layer should be studied more in detail, since preliminary numerical results showed that this could enhance pinning controllability [28].

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