Jeffrey, M. (2017). An Update on that Singularity. In *Extended Abstracts Spring 2016: Nonsmooth Dynamics* (pp. 107-122). (Trends in Mathematics; Vol. 8). Birkhäuser Basel. https://doi.org/10.1007/978-3-319-55642-0_19

Peer reviewed version

Link to published version (if available):
[10.1007/978-3-319-55642-0_19](https://doi.org/10.1007/978-3-319-55642-0_19)

[Link to publication record in Explore Bristol Research](#)
PDF-document

This is the author accepted manuscript (AAM). The final published version (version of record) is available online via Springer at https://link.springer.com/chapter/10.1007%2F978-3-319-55642-0_19 . Please refer to any applicable terms of use of the publisher.

University of Bristol - Explore Bristol Research

General rights

This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available:
<http://www.bristol.ac.uk/red/research-policy/pure/user-guides/ebr-terms/>

An update on *that* singularity

Mike R. Jeffrey

University of Bristol, Department of Engineering Mathematics, Woodland Road, Bristol BS8 1UB, UK
mike.jeffrey@bristol.ac.uk

A bridge over troubled flows

Nothing epitomizes the intrigue of piecewise-smooth dynamics like the two-fold singularity. It is incredibly simple to describe — a point where a flow is tangent to a discontinuity threshold from both sides — yet intricate in its dynamics. Its understanding has pushed the boundaries of understanding in piecewise-smooth systems more than any other discontinuity-induced phenomenon.

It took nearly 30 years, from the translation of Filippov’s seminal book introducing the two-fold to the english speaking world, to resolving its switching layer behaviour, before we could say that the two-fold singularity was understood. And it *is* now understood, in wonderful detail: its structural and asymptotic stability [11, 9], its bifurcations including its local form and the affect of higher orders [11, 3, 4], the winding numbers when a flows rotates around it [6], the determinacy or determinacy-breaking that occurs when a flow passes through it [8], even its extension to multiple switches [10].

We now know that the two-fold singularity’s structural stability requires nonlinear switching or *hidden* terms, and that it comes in three main flavours, with numerous subclasses between which bifurcations can occur. We know that it is neither an attractor nor a repeller, but an organizing centre, a bi-directional bridge between attracting and repelling sliding on a switching surface, which can lead to the creation of a determinacy-breaking attractor (described as *non-deterministic chaos* in [3, 8, 1, 2]).

The developments towards understanding the two-fold singularity can be traced through the papers [7, 16, 11, 3, 4, 6, 9, 10]. Attempts to look beyond nonsmooth theory into the effects of regularization, introducing a non-ideal switch that is smooth, noisy, delayed, or hysteretic, have begun in [18, 13, 14, 12]. Finally, while attempts to explore its applications in electronics or mechanics have so far been somewhat unsatisfactory, hints of a deeper role in phase randomization can be found in [15].

To summarize the story so far, we must begin, of course, with . . .

Definition 1 A *two-fold* is a point $\mathbf{x}_p$ in a system

$$(1) \quad \dot{\mathbf{x}} = \begin{cases} \mathbf{f}^+(\mathbf{x}) & \text{if } \sigma(\mathbf{x}) > 0 \\ \mathbf{f}^-(\mathbf{x}) & \text{if } \sigma(\mathbf{x}) < 0 \end{cases} \quad \text{where} \quad \left. \begin{matrix} \sigma(\mathbf{x}_p) \\ \mathbf{f}^\pm(\mathbf{x}_p) \cdot \nabla \sigma(\mathbf{x}_p) \end{matrix} \right\} = 0,$$

and with certain non-degeneracy conditions satisfied at $\mathbf{x}_p$, namely $(\mathbf{f}^\pm \cdot \nabla)^2 \sigma \neq 0$, $0 \notin \mathbf{f}^\lambda \cdot \nabla \mathbf{x}$, and with transversality of the surfaces $\sigma = 0$, $\mathbf{f}^+ \cdot \nabla \sigma = 0$, $\mathbf{f}^- \cdot \nabla \sigma = 0$. We will introduce the combination $\mathbf{f}^\lambda$ below.

The local dynamics depends entirely on two parameters evaluated at $\mathbf{x}_p$,

$$(2) \quad \nu^+ = \frac{(\mathbf{f}^+ \cdot \nabla)(\mathbf{f}^- \cdot \nabla) \sigma}{\sqrt{|(\mathbf{f}^+ \cdot \nabla)^2 \sigma (\mathbf{f}^- \cdot \nabla)^2 \sigma|}} \quad \& \quad \nu^- = \frac{-\mathbf{f}^- \cdot \nabla \mathbf{f}^+ \cdot \nabla \sigma}{\sqrt{|(\mathbf{f}^+ \cdot \nabla)^2 \sigma (\mathbf{f}^- \cdot \nabla)^2 \sigma|}},$$

characterizing the local curvature of the flow. The product $\nu^+ \nu^-$ has a simple geometrical interpretation: it quantifies the jump in the vector field between $\mathbf{f}^\pm$ at the singularity.

Measuring angles from to the ‘+’ or ‘−’ folds respectively, letting $s^\pm = \text{sign}(\mathbf{f}^\pm \cdot \nabla)^2 \sigma$,

$$(3) \quad \nu^+ \nu^- = -s^+ s^- \frac{\cot \phi - \cot \theta_+^+}{\cot \phi - \cot \theta_+^-} = -s^+ s^- \frac{\cot \phi + \cot \theta_-^-}{\cot \phi + \cot \theta_-^+},$$

where ϕ is the angle between the folds, and θ_j^i is the angle of $\mathbf{f}^i$ from the ‘ j ’ fold, measured in the plane spanned by $\mathbf{f}^+$ and $\mathbf{f}^-$, with i and j denoting the labels + or −.

The leading order expansion of the two-fold singularity (sometimes called the ‘normal form’ in a somewhat loose usage of the terminology) is given [7, 4] by

$$(4) \quad (\dot{x}_1, \dot{x}_2, \dot{x}_3) = \begin{cases} (-x_2, -s^+, \nu^+) & \text{if } x_1 > 0 \\ (x_3, \nu^-, s^-) & \text{if } x_1 < 0 \end{cases} + (O(|\mathbf{x}|^2), O(|\mathbf{x}|), O(|\mathbf{x}|)),$$

where $s^\pm = \text{sign}[(\mathbf{f}^\pm \cdot \nabla)^2 \sigma(\mathbf{x}_p)]$, and in higher dimensions $\dot{x}_{i \geq 4} = O(|\mathbf{x}|)$ for $i = 4, 5, \dots$

Bifurcation diagrams

Almost everything we understood until the year 2009 could already be found in Filippov’s book [7], but much of it was presented in the form of unexplained diagrams whose original source is unknown (with their description emerging across [17, 4, 6]).

The wealth of information we have on the leading order dynamics (the truncation of (4)) is summarized in the figure below, see [4, 6] for detail.

FIGURE 1. Two-folds come in three flavours, formed by the different combinations of visible or invisible folds as determined by the signs of $s^\pm$. Top: Regions of attracting sliding (*att.*, shaded), repelling sliding (*rep.*, shaded), and crossing (unshaded) all meet at the singularity. Bottom: Their sliding and crossing topologies in the $\nu^\pm$ parameter plane are shown below; for the invisible two-fold, k is the number of windings between visits to the sliding regions, tending to infinity where $\nu^+ \nu^- \geq 1$ in $\nu^\pm < 0$. See [4, 6] for detail.

The folds are:

- both *visible* if $s^+ > 0$ and $s^- < 0$ at $\mathbf{x}_p$,
- both *invisible* if $s^+ < 0$ and $s^- > 0$ at $\mathbf{x}_p$,
- one *visible* and one *invisible* if $s^+ s^- > 0$ at $\mathbf{x}_p$,

(we sometimes refer to these as the *flavours* of two-fold).

Crossing maps and winding numbers

The distinguishing feature of the invisible two-fold is that the flow can wind repeatedly around the singularity, making repeated visits to the crossing regions, possibly between entry/exit points to/from the attracting/repelling sliding regions.

Let $\mathbf{y} = (x_2, x_3)$ denote a point on the switching surface $x_1 = 0$, and $\mathbf{y}_i$ denote an iterate of the return map to the switching surface under the flow. A single return to the surface is given by

$$(5) \quad \mathbf{y}_{2m+1} = \underline{\underline{B}}^\pm \mathbf{y}_{2m}, \quad \underline{\underline{B}}^+ = \begin{pmatrix} -1 & 0 \\ -2\nu^+ & 1 \end{pmatrix} \quad \& \quad \underline{\underline{B}}^- = \begin{pmatrix} 1 & -2\nu^- \\ 0 & -1 \end{pmatrix},$$

where $\underline{\underline{B}}^+$ and $\underline{\underline{B}}^-$ are applied in $x_2 < 0$ and $x_3 < 0$ respectively. The second return map, on $x_2 < 0$ or $x_3 < 0$, is therefore

$$(6) \quad \mathbf{y}_{2m+2} = \underline{\underline{A}}^\pm \mathbf{y}_{2m}, \quad \underline{\underline{A}}^\pm = \underline{\underline{B}}^\mp \underline{\underline{B}}^\pm.$$

Because the maps are associated with folds, they are involutions, so $(\underline{\underline{B}}^+)^2 = (\underline{\underline{B}}^-)^2 = \underline{\underline{1}}$ and $\underline{\underline{A}}^+ = (\underline{\underline{A}}^-)^{-1}$. The solutions to the difference equation (6) are now obviously

$$(7) \quad \mathbf{y}_{2m} = (\underline{\underline{A}}^+)^m \mathbf{y}_0 \quad \text{or} \quad \mathbf{y}_{2m} = (\underline{\underline{A}}^-)^m \mathbf{y}_0,$$

and a little trigonometry using the substitution $\nu^+ \nu^- = \cos^2 \Theta$ provides

$$(8) \quad (\underline{\underline{A}}^\pm)^m = \frac{\sin[2m\Theta]}{\sin 2\Theta} \underline{\underline{A}}^\pm - \frac{\sin[2(m-1)\Theta]}{\sin 2\Theta} \underline{\underline{1}}.$$

This is also the source of the crossing numbers k in the previous figure. The main dynamical features revealed by the map are shown in the figure below.

FIGURE 2. The nonsmooth diaboloid: invariant manifold (left) around an invisible two-fold. Right top: shown in the switching plane, the manifold bifurcates and disappears at $\nu^+ \nu^- = 1$, see [11]. Right bottom: the effect of higher order terms, showing a particular case leading to a determinacy-breaking attractor — as the flow exits the repelling sliding region, the crossing flow wraps it back around (via k windings) into the attracting sliding region, whereupon the sliding flow re-injects it back into the repelling region; when all local trajectories pass through the singularity, determinacy is broken; see [3].

Sliding dynamics and hidden instability

To derive sliding dynamics we need to define a *combination* of $\mathbf{f}^\pm$ on the switching surface. It turns out that Filippov's combination hides a structural instability, in

$$(9) \quad (\dot{x}_1, \dot{x}_2, \dot{x}_3) = \frac{1}{2} (1 + \lambda) (-x_2, -s^+, \nu^+) + \frac{1}{2} (1 - \lambda) (x_3, \nu^-, s^-),$$

essentially because the value $\lambda = \frac{x_3 - x_2}{x_3 + x_2}$ for which sliding occurs is singular at $x_2 = x_3 = 0$.

It is shown in [9] that a structurally stable combination is

$$(10) \quad (\dot{x}_1, \dot{x}_2, \dot{x}_3) = \frac{1}{2}(1 + \lambda)(-x_2, -s^+, \nu^+) + \frac{1}{2}(1 - \lambda)(x_3, \nu^-, s^-) + (1 - \lambda^2)(\alpha, 0, 0)$$

for small $\alpha \neq 0$. A well-defined manifold $\mathcal{M}$ of sliding solutions then exists,

$$(11) \quad \mathcal{M} = \left\{ (\lambda, x_2, x_3) : \frac{1}{2}(1 - \lambda)x_3 - \frac{1}{2}(1 + \lambda)x_2 + \alpha(1 - \lambda^2) = 0 \right\},$$

inside the layer $(\lambda, x_2, x_3) \in (-1, +1) \times \mathbb{R}^2$, with $\mathcal{M}$ normally hyperbolic except on

$$(12) \quad \mathcal{L} = \left\{ (\lambda, x_2, x_3) \subset \mathcal{M} : \lambda = 2 \frac{2\alpha + x_3 - x_2}{x_3 + x_2} = -\frac{x_3 + x_2}{4\alpha} \right\},$$

which corresponds to the two-fold magnified inside the switching layer $\lambda \in (-1, +1)$, $(x_2, x_3) \in \mathbb{R}^2$. The dynamics inside the layer is given by

$$(13) \quad (\varepsilon \dot{\lambda}, \dot{x}_2, \dot{x}_3) = \frac{1}{2}(1 + \lambda)(-x_2, -s^+, \nu^+) + \frac{1}{2}(1 - \lambda)(x_3, \nu^-, s^-) + (1 - \lambda^2)(\alpha, 0, 0),$$

for $\varepsilon \rightarrow 0$, which can be transformed into the well-known singularity of folded slow-manifolds associated with canards in smooth slow-fast systems,

$$(\varepsilon \dot{x}, \dot{y}, \dot{z}) = (y + x^2, pz + qx, r) + (\mathcal{O}(\varepsilon x, \varepsilon z, xz), \mathcal{O}(z^2, xz), \mathcal{O}(z, x))$$

provided $\alpha \neq 0$, where p, q, r , are real constants, and provided the conditions $\frac{1}{2}(\nu^+ - \nu^-) \leq 1 = -s^+ = s^-$ or $\frac{1}{2}(\nu^+ - \nu^-) \geq -1 = -s^+ = s^-$ do not hold.

References

- [1] M. Buchanan. Differentiating the discontinuous. *Nature Physics - Thesis*, 7:589, 2011.
- [2] M. Buchanan. Generating chaos in a new way. *Phys. Rev. Focus*, 28:1, 2011.
- [3] A. Colombo and M. R. Jeffrey. Non-deterministic chaos, and the two-fold singularity in piecewise smooth flows. *SIAM J. App. Dyn. Sys.*, 10:423–451, 2011.
- [4] A. Colombo and M. R. Jeffrey. The two-fold singularity: leading order dynamics in n-dimensions. *Physica D*, 263:1–10, 2013.
- [5] F. Dumortier, R. Roussarie, J. Sotomayor, and H. Zoladek. *Bifurcations of Planar Vector Fields: Nilpotent Singularities and Abelian Integrals*. Springer-Verlag, 1991.
- [6] S. Fernández-García, D. Angulo-García, G. Olivar-Tost, M. di Bernardo, and M. R. Jeffrey. Structural stability of the two-fold singularity. *SIAM J. App. Dyn. Sys.*, 11(4):1215–1230, 2012.
- [7] A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer Academic Publ. Dordrecht, 1988 (Russian 1985).
- [8] M. R. Jeffrey. Non-determinism in the limit of nonsmooth dynamics. *PRL*, 106(25):254103, 2011.
- [9] M. R. Jeffrey. Hidden degeneracies in piecewise smooth dynamical systems. *Int. J. Bif. Chaos*, 26(5):1650087(1–18), 2016.
- [10] M. R. Jeffrey. Exit from sliding in piecewise-smooth flows: deterministic vs. determinacy-breaking. *Chaos*, 26(3):033108:(1-20), 2016.
- [11] M. R. Jeffrey and A. Colombo. The two-fold singularity of discontinuous vector fields. *SIAM Journal on Applied Dynamical Systems*, 8(2):624–640, 2009.
- [12] K. U. Kristiansen and S. J. Hogan. Regularizations of two-fold bifurcations in planar piecewise smooth systems using blow up. *submitted*, 2015.
- [13] J. Llibre, P. R. da Silva, and M. A. Teixeira. Sliding vector fields via slow-fast systems. *Bull. Belg. Math. Soc. Simon Stevin*, 15(5):851–869, 2008.
- [14] D. J. W. Simpson. On resolving singularities of piecewise-smooth discontinuous vector fields via small perturbations. *Discrete Contin. Dyn. Syst. to appear*, 2014.
- [15] D. J. W. Simpson and M. R. Jeffrey. Fast phase randomisation via two-folds. *submitted*, 2015.
- [16] M. A. Teixeira. Stability conditions for discontinuous vector fields. *J. Differ. Equ.*, 88:15–29, 1990.
- [17] M. A. Teixeira. Generic bifurcation of sliding vector fields. *J. Math. Anal. Appl.*, 176:436–457, 1993.
- [18] M. A. Teixeira, J. Llibre, and P. R. da Silva. Regularization of discontinuous vector fields on R^3 via singular perturbation. *Journal of Dynamics and Differential Equations*, 19(2):309–331, 2007.

Abstract: It took nearly 30 years from the translation of Filippov's seminal book to be able to say that the two-fold singularity is understood. We now know that its structural stability requires nonlinear switching or *hidden* terms, and that it comes in three main flavours, with numerous subclasses between which bifurcations can occur. We know that it is neither an attractor nor a repeller, but a bridge between attracting and repelling sliding, and in certain cases is a source of determinacy-breaking.