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Chaotic rotations of a rigid ellipsoidal body exhibiting spin-orbit misalignment in a periodic orbit

James A. Kwiecinski^{*}, Stanisław W. Biber[†], and Robert A. Van Gorder[†]

^{*}*Mathematics, Mechanics, and Materials Unit*

Okinawa Institute of Science and Technology, Okinawa, 904-0495

[†]*Mathematical Institute, University of Oxford,*

*Andrew Wiles Building, Radcliffe Observatory Quarter,
 Woodstock Road, Oxford, OX2 6GG, United Kingdom*

^{*}*Corresponding author. email: james.kwiecinski@oist.jp*

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We study chaotic rotations of a rigid ellipsoidal body due to the effects of gravitational torques, in the case where this body exhibits spin-orbit misalignment. After first deriving a simple model of a rigid ellipsoid of uniform mass distribution with principal axis of rotation directed slightly out of the orbital plane, we prove using the Melnikov method that this perturbation is sufficient to excite the ellipsoid to chaotically rotate for a circular orbit. We further verify this analytical result with numerical time series and Poincaré sections for circular orbits. We then use numerical simulations to demonstrate that elliptical orbits provide further pathways to chaos. Our primary finding is that increasing the degree of spin-orbit misalignment will increase the prevalence of initial conditions leading to chaotic dynamics, for elliptical bodies on both circular and elliptical orbits. Indeed, our results suggest that chaotic rotation of elongated bodies exhibiting spin-orbit misalignment is somewhat common for large enough deviation of the principal axis of rotation from the normal to the orbital plane, provided the rigid body lacks rotational symmetry around its spin-axis and, furthermore, exhibits sufficiently small angular momentum so that the effects of gravitational torques are non-trivial.

Keywords: Celestial mechanics, rigid body rotation, spin-orbit misalignment, chaotic dynamics, Melnikov method

1. Introduction

Recent observations of the chaotic tumbling of Styx [Quillen *et al.*, 2017, 2018; Kwiecinski *et al.*, 2018a], one of the moons of Pluto, by the New Horizons probe [Weaver *et al.*, 2016], as well as other celestial bodies [Wisdom *et al.*, 1984; Wisdom, 1987b,a; Clifford & Bishop, 1995; Kuang *et al.*, 2002; Pravec *et al.*, 2008], suggest that the orbital dynamics of non-spherical bodies can often exhibit chaotic rotations. There are examples of N -body problems involving rigid bodies where the spin axis of one rigid body is tilted with respect to the normal axis to the orbital plane, with one such example being the comet 1P/Halley [Samarasinha & A'Hearn, 1991]. [Mysen *et al.*, 2006] found that if the spin axis of a regular solid body is tilted with respect to the normal axis to the orbital plane, the occurrence of initial conditions yielding chaos is significantly increased. We should note that such spin-orbit misalignment is observed in a variety of systems which do not often yield chaotic dynamics when the bodies are spherical [Huber *et al.*, 2013;

Spalding & Batygin, 2014, 2016; Lin & Ogilvie, 2017]. While there are cases where spin-orbit misalignment is present in chaotic systems of spherical bodies, there are often other mechanisms at play which influence the dynamics [Wu & Lithwick, 2011; Storch *et al.*, 2014; Grishin *et al.*, 2017]. On the other hand, when one of the bodies is irregular or elongated, such as discussed in [Mysen *et al.*, 2006], the coupling of spin-orbit misalignment with a sufficiently elongated (non-spherical) body may result in chaotic dynamics without additional mechanisms.

Motivated by the appearance of chaos in such N -body systems, and in order to better understand the mechanism behind the emergence of chaotic dynamics from elongated rigid bodies which are spin-orbit misaligned, in the present paper we formulate a relatively simple model to study the excitations of elongated celestial bodies to chaotically rotate due to the effects of gravitational torques. We take the rotating body to be a three-dimensional ellipsoid. Such a model is a natural generalization of a recent model of an N -body problem in which one body is a rigid rod whose rotation is confined to the orbital plane [Kwiecinski *et al.*, 2018a] to a model in which a fully three-dimensional ellipsoidal body rotates around a spin-axis that is slightly tilted out of its plane of rotation. One benefit of our model is that it is fairly simple compared to related models [Mysen *et al.*, 2006], yet still encapsulates the necessary features generating chaotic dynamics, making the analysis of chaotic dynamics more straightforward.

Due to the relative simplicity of our model, we are able to apply the Melnikov method [Melnikov, 1963], an analytical tool developed to study Hamiltonian systems undergoing periodic perturbations of first order [Perko, 2001], to demonstrate the emergence of Hamiltonian chaos from the model for certain parameter sets. The Melnikov method is used to establish a topological equivalence with the Horseshoe map [Smale, 1967], and hence to chaotic dynamics. In light of the KAM theorem for integrable Hamiltonian systems, the Melnikov method provides a tool to establish the breakdown of the invariant tori and thus chaos. Extension of the method allows for the study of dissipative systems [Bertozzi, 1988; Chen, 2002; Salam, 1987; Salam & Sastry, 1985]) or for almost-periodic perturbations [Wiggins, 1987; Scheurle, 1986] or even fairly general perturbations [Lu & Wang, 2010] of Hamiltonian systems. The dynamics of many systems have been examined by means of the Melnikov method, including the Cat's Eye flow [Bertozzi, 1988], three-body problems [Xia, 1992], a model of microcantilevers in scanning probe microscopy [Ashhab *et al.*, 1999], self-excited nonlinear oscillators [Belhaq *et al.*, 2000], parametrically forced pendulums [Koch & Leven, 1985], the dynamics of micro mechanical resonators [Haghighi & Markazi, 2010], models for the capsizing of ships MEOW [Falzarano *et al.*, 1992; Bikdash *et al.*, 1994], a model of two-dimensional thin panel subjected to subsonic flow and external excitation [Li *et al.*, 2011], and a model of a slender beam rotating about its longitudinal axis [Shaw, 1988], to name just a few such applications. Therefore, this analytical approach is sensible to apply in order to better understand the emergence of Hamiltonian chaos from our model.

The remainder of this paper is organized as follows. In Section 2 we outline the equations of motion for the planar gravitational N -body system in which one of the bodies is a tilted rotating ellipsoid. In Section 3 we give a brief review of the Melnikov method, which we then apply in Section 4 to our model in the case where the tilted ellipsoidal body travels along a circular orbit. The results of Section 4 will analytically demonstrate the existence of Hamiltonian chaos in our system. In order to verify the analytical results and to obtain a physical intuition for the results, in Section 5 we provide numerical simulations of our model. We discuss our results and give concluding remarks in Section 6.

2. Equations of motion for the planar gravitational N -body system with a tilted rotating ellipsoid

We study the gravitational N -body problem where $N - 1$ bodies are treated as point masses, whilst the N th body is a rigidly rotating ellipsoid of constant mass distribution. We assume that the orbital motion of all masses is confined to the x - y plane. For the case where all the gravitational bodies are separated on a length scale that is much larger than the size of the rigid ellipsoid, the dynamic equations describing the translational motion of the i th center of mass, given by $\mathbf{r}_i = (x_i, y_i)$ in Cartesian coordinates, at time t is

Fig. 1. Schematic diagram describing the tilted rotation of the rigid ellipsoid with semi-major axes lengths given by a and b . We suppose the ellipsoid rotates around a fixed spin-axis ω that points along the object's principal axis and is described by a time-dependent angular position $\phi(t)$ with respect to the positive x -axis (with the zero alignment shown by the dashed ellipse). We further tilt this spin-axis by a time-independent angle of $\sqrt{\varepsilon}$ with respect to the positive x axis in the x - z plane.

(see Appendix A for a detailed derivation):

$$\frac{d^2 \mathbf{r}_i}{dt^2} = \sum_{j \neq i}^N \frac{G m_j (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3}, \quad (1)$$

where m_i is the mass of the i th body and G is the universal gravitational constant.

Additionally, the N th body will also rotate due to the effects of gravitational torques. Although the orbital motion is confined to the x - y plane, the rigid ellipsoid is allowed to rotate out of this plane along the z -axis. For analytical tractability, we assume that this rotation always occurs along the principal axis of the ellipsoid and is fixed for all time. Realistically, one would expect that the spin-axis would precess in a manner similar to a spinning top particularly when the orbital dynamics are planar. However, proving the existence of chaos in this simplified rotation model would directly imply the existence of chaos when the body is allowed to freely tumble. With increased rotational degrees of freedom, we would expect that there would be more pathways for the ellipsoid to chaotically rotate.

For the simplified rotation model, we tilt the spin axis of the ellipsoid so that it makes an angle of $\sqrt{\varepsilon}$, for $\varepsilon \ll 1$, with the x -axis in the x - z plane and parameterize the rotation around this spin-axis by the time-dependent angular position $\phi(t)$ (see Fig. 1). In this case, the equation describing the rotation of the ellipsoid perpendicular to this tilted spin axis is (see Appendix A for a detailed derivation):

$$\begin{aligned} \frac{d^2 \phi}{dt^2} = \frac{3G}{2} \left(\frac{a^2 - b^2}{a^2 + b^2} \right) \sum_{j=1}^{N-1} \frac{m_i}{(\Delta x_j^2 + \Delta y_j^2)^{\frac{5}{2}}} & \left[2\Delta x_j \Delta y_j \cos 2\phi + (\Delta y_j^2 - \Delta x_j^2) \sin 2\phi \right. \\ & \left. + \varepsilon \Delta x_j (\Delta x_j \sin 2\phi - \Delta y_j \cos 2\phi) \right] + O(\varepsilon^2), \end{aligned} \quad (2)$$

where $\Delta x_j = x_j - x_N$ and $\Delta y_j = y_j - y_N$ are the differences in the x and y coordinates of the j th body and N th rigid ellipsoid and a and b are the lengths of the semi-major axes of the ellipsoid.

We can further simplify (2) by rescaling the angular position $\hat{\phi} = 2\phi$ and time:

$$\hat{t} = \sqrt{\frac{3G(a^2 - b^2)}{a^2 + b^2}} t, \quad (3)$$

so that the rotation equation becomes, upon dropping the hat symbol:

$$\begin{aligned} \frac{d^2\phi}{dt^2} = \sum_{j=1}^{N-1} \frac{m_i}{\left(\Delta x_j^2 + \Delta y_j^2\right)^{\frac{5}{2}}} [2\Delta x_j \Delta y_j \cos \phi + (\Delta y_j^2 - \Delta x_j^2) \sin \phi \\ + \varepsilon \Delta x_j (\Delta x_j \sin \phi - \Delta y_j \cos \phi)] + O(\varepsilon^2). \end{aligned} \quad (4)$$

Some comments are in order regarding (1), (2), and (4). First, for gravitational bodies which are sufficiently separated by a distance that is much larger than the size of the rotating body, the orbital dynamics decouple from the rotational dynamics to lowest order. Therefore, one can solve for the N -body problem and substitute the orbital dynamics into (2) to obtain the ellipsoid's corresponding rotation. Furthermore, we note that (2) is a generalization of the rotating rod model, as presented in [Kwiecinski *et al.*, 2018a], and reduces to the latter in the limit of the spin-axis tilt $\varepsilon \rightarrow 0^+$ and when the width of the rigid body becomes infinitesimally thin (i.e. $b \rightarrow 0^+$). This reduction explains why contributions from the moment of inertia did not appear in the rotating rod model.

After rescaling the time-scale, however, the two models are equivalent for the no-tilt case of $\varepsilon = 0$, provided that the rigid ellipsoid is not circular around its spin-axis, which would correspond to $a = b$. In this case, there would be no gravitational torque applied to the rotating body which would result in the body having a constant angular velocity for all time. Given that the two models are equivalent after rescaling when the body spins within the orbital plane $\varepsilon = 0$, the proof regarding the excitations to chaotic rotation when the orbital motion of the spinning object is perturbed from a circular orbit holds for our model presented here.

We shall use the model we have obtained in this section to study the effect of gravitational torques on rotation which is out of the orbital plane (i.e. $\varepsilon \neq 0$). First, we shall introduce the Melnikov method, which will provide us with an analytical approach to determine the emergence of Hamiltonian chaos under perturbations to this system.

3. The Melnikov method

In this section we review the Melnikov method, a powerful tool for establishing the existence of transverse homoclinic points of a perturbed dynamical system. The general form of such system can be represented by:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \varepsilon \mathbf{g}(\mathbf{x}, t), \quad (5)$$

with $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$.

Consider a system of a form (5) where $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{g}$ is periodic with period T in t . We pose the following two assumptions:

A1: For $\varepsilon = 0$ the system (5) admits a homoclinic orbit $\Gamma_0 : \mathbf{x} = \Gamma_0(t)$, $-\infty < t < \infty$ at a hyperbolic saddle point $\mathbf{x}_0$.

A2: For $\varepsilon = 0$ the system (5) has a one-parameter family of periodic subharmonic orbits $\gamma_\alpha(t)$ of period T_α on the interior of Γ_0 with $\partial\gamma_\alpha(0)/\partial\alpha \neq 0$.

Definition 3.1. The **Melnikov function** of system (5) is defined as:

$$M(t_0) = \int_{-\infty}^{\infty} e^{-\int_{t_0}^t \nabla \cdot \mathbf{f}(\gamma_0(s)) ds} \mathbf{f}(\gamma_0(t)) \wedge \mathbf{g}(\gamma_0(t), t + t_0) dt, \quad (6)$$

where for $\mathbf{u} = (u_1, u_2)^T$, $\mathbf{v} = (v_1, v_2)^T$ we define the wedge product $\mathbf{u} \wedge \mathbf{v} = u_1 v_2 - u_2 v_1$.

We note for $\varepsilon = 0$, the system (5) is Hamiltonian, provided:

$$\mathbf{f} = \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x} \right), \quad (7)$$

so that $\nabla \cdot \mathbf{f} = 0$. In this case, the Melnikov function takes the form:

$$M(t_0) = \int_{-\infty}^{\infty} \mathbf{f}(\gamma_0(t)) \wedge \mathbf{g}(\gamma_0(t), t + t_0) dt. \quad (8)$$

For the case where the perturbation $\mathbf{g}(\mathbf{x}, t)$ is periodic with period T in t , we note that:

$$\begin{aligned} M(t_0 + T) &= \int_{-\infty}^{\infty} \mathbf{f}(\gamma_0(t)) \wedge \mathbf{g}(\gamma_0(t), t + t_0 + T) dt \\ &= \int_{-\infty}^{\infty} \mathbf{f}(\gamma_0(t)) \wedge \mathbf{g}(\gamma_0(t), t + t_0) dt \\ &= M(t_0). \end{aligned} \quad (9)$$

Theorem 1. *Under conditions A1 and A2, if the Melnikov function $M(t_0)$ has a simple zero, not depending on ε , then for all sufficiently small $\varepsilon \neq 0$ the stable and unstable manifolds W^s and W^u of the Poincaré map intersect transversely, that is, the Poincaré map has a transverse homoclinic point.*

Furthermore, if $M(t_0) > 0$ (or $M(t_0) < 0$) for all t_0 then $W^s \cap W^u = \emptyset$.

A proof of Theorem 1 can be found in [Guckenheimer & Holmes, 1990], with an extension to the non-Hamiltonian case found in [Chen, 2002].

4. Application of the Melnikov method for purely circular orbits

We consider the simplest orbital system; namely, a two body system where the ellipsoid orbits in a purely circular orbit around a massive gravitational body. In this case, the separation between the two masses is parametrized by:

$$\Delta x_1 = R \cos(k\pi t), \quad (10)$$

$$\Delta y_1 = R \sin(k\pi t), \quad (11)$$

where R is the radius of the ellipsoid's orbit around the massive gravitational body and $k/2$ is the orbital period.

Substituting this orbital ansatz into the rotation equation (4), we find, under an additional rescaling of the angular variable $\hat{\phi} = \phi - 2k\pi t$ and time $\hat{t} = \sqrt{m_2}/R^3 t$, the equation, upon removing the hat symbol:

$$\frac{d^2\phi}{dt^2} = -\sin\phi + \frac{\varepsilon}{2} (\sin\phi + \sin(\phi + 2k\pi t)). \quad (12)$$

For the spin-axis being perpendicular to the orbital plane (i.e. $\varepsilon = 0$), the rotation equation is purely autonomous suggesting that there is no chaos. However, (12) admits an exact homoclinic connection for angular positions $\phi \in (-\pi, \pi]$ (see [Kwiecinski *et al.*, 2018a] for full derivation) of the form:

$$\phi^h \Big|_{\varepsilon=0} = \pm 2 \tan^{-1}(\sinh(t)), \quad (13)$$

$$\frac{d\phi^h}{dt} \Big|_{\varepsilon=0} = \pm 2 \operatorname{sech}(t), \quad (14)$$

where the positive solution of (13) tends to π as time $t \rightarrow \infty$ whilst the negative solution tends to $-\pi$ in the same limit.

We now apply the first order Melnikov method to prove that, perturbing the spin-axis of the ellipsoid out of the orbital plane is sufficient to give rise to chaotic rotations when the rotating object is in a purely circular orbit. The Melnikov function $M(t_0)$, which quantifies the distance between the stable and unstable manifolds to $O(\varepsilon)$ for some parameter t_0 , is, for the present case of (12)-(14):

$$M(t_0) = \int_{-\infty}^{\infty} \pm \operatorname{sech}(t) [\sin(\pm 2 \tan^{-1}(\sinh(t))) + \sin(\pm 2 \tan^{-1}(\sinh(t)) + 2k\pi(t + t_0))] dt. \quad (15)$$

To simplify the integral, we use the fact that $\operatorname{sech}(t)$ and $\sin(t)$ are odd functions so that the first term in the integral vanishes, and expand the second term using trigonometric identities to obtain, after

similar simplification:

$$M(t_0) = \pm \sin(2k\pi t_0) \int_{-\infty}^{\infty} \operatorname{sech}(t) \cos(\pm 2 \tan^{-1}(\sinh(t)) + 2k\pi t) dt. \quad (16)$$

Computing the integral, we obtain the result:

$$M(t_0) = A_{\pm}(k) \sin(2k\pi t_0), \quad (17)$$

where

$$A_{\pm}(k) = -4k^2\pi^3 (\operatorname{csch}(k\pi^2) \mp \operatorname{sech}(k\pi^2)). \quad (18)$$

We remark on (17) and (18). The Melnikov function has countably infinitely many simple zeros implying that the unstable and stable manifolds of the ellipsoid's rotation similarly transversely intersect infinitely many times. However, there are two degenerate cases which would cause the Melnikov function to vanish altogether; that is, when $k = 0$ or $k \rightarrow \pm\infty$, according to (18).

Provided the orbital period of the rotating ellipsoid k is non-zero and finite, then the Melnikov function always has countably infinitely many simple zeros independent of any of the physical parameters determining the orbital motion of the system or the rotational motion of the gravitational body. The only additional requirement is that the lengths of the ellipsoid's semi-major axes are not equal $a \neq b$ (i.e. the rigid body is not perfectly circular around its spin-axis). Therefore, the phase space of the ellipsoid's rotation is always topologically conjugate to the Smale horseshoe thereby proving that chaos exists for the case where the ellipsoid is confined to a circular orbit and spins at a non-zero tilt out of the orbital plane.

5. Numerical construction of time series and Poincaré sections

We numerically verify the previous Melnikov analysis showing the existence of chaotic rotations as the spin-axis tilt is increased for the case of the ellipsoid in a purely circular orbit. However, as was shown in [Kwiecinski *et al.*, 2018a], perturbing the orbital motion of the system should provide further pathways to chaos. Therefore, we also simulate the rotating object in an elliptical orbit and verify the excitation of chaotic rotations when both the periodic orbit deviates from a circle and the orientation of the rotational motion is perturbed out of the orbital plane.

In particular, we simulate a two-body system featuring a caricature of Saturn and one of its moons, Hyperion, by solving (1) and (2) using parameter values corresponding to the Saturn - Hyperion system (see Table 1 for a list of these parameter values). We first consider a hypothetical purely circular orbit followed by Hyperion's experimentally observed elliptical orbit (see Fig. 2). Previously, Hyperion was shown to exhibit chaotic rotations in [Wisdom *et al.*, 1984] using a resonance overlap argument together with a model whose rotation always occurred within the orbital plane. We show that tilting the rotation of the moon out of the plane creates larger ergodic regions in phase-space therefore providing more pathways to chaotic rotations.

Table 1. Parameter values used in the two-body numerical simulations of Hyperion and Saturn.

Parameter	Numerical value
Gravitational constant	$G = 6.674 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$
Mass of Saturn	$m_1 = 5.6834 \times 10^{26} \text{kg}$
Mass of Hyperion	$m_2 = 5.6199 \times 10^{18} \text{kg}$
Eccentricity	$e = 0.1042$
Orbital semi-major axis	$A = 1.481 \times 10^9 \text{m}$
Hyperion semi-major axis	$a = 1.80 \times 10^5 \text{m}$
	$b = 1.03 \times 10^5 \text{m}$

Fig. 2. Simulated orbits of the Hyperion-Saturn system with Saturn shown as the black circle (not to scale) and Hyperion shown as the black ellipse (not to correct scale and aspect ratio). Red line shows the hypothetical circular orbit of eccentricity $e = 0$, whilst the blue line shows the experimentally observed elliptical orbit of Hyperion with eccentricity $e = 1.024$. Vertical and horizontal units are in meters. (Color online)

The numerical parameters used in our simulations are given in Table 1. The universal gravitational constant is referenced from the National Institute of Standards and Technology (<https://physics.nist.gov/cgi-bin/cuu/Value?bg>) whilst the mass of Saturn is taken from the NASA Saturn Fact Sheet (<https://nssdc.gsfc.nasa.gov/planetary/factsheet/saturnfact.html>). Orbital and rotational data of Hyperion, including mass, eccentricity e , semi-major axis A of the orbit, and dimensions is referenced from the NASA Saturnian Satellites Fact Sheet (<https://nssdc.gsfc.nasa.gov/planetary/factsheet/saturniansatfact.html>). The periapsis P (closest distance to Saturn) was calculated using the formula

$$P = A\sqrt{1 - e^2}. \quad (19)$$

The initial orbital velocity v_0 of Hyperion was calculated from the eccentricity and semi-major axis of the orbit by

$$v_0 = \sqrt{\frac{(1 + e) G (m_1 + m_2)}{P}}. \quad (20)$$

To numerically solve (1) and (2), Mathematica’s “NDSolve” function was used together with absolute and relative error tolerances of 10^{-13} .

Poincaré sections were generated by simulating the Hyperion-Saturn system to a final time 0.5×10^9 s and collecting the angular displacement and rotational velocity when the orbital velocity of Hyperion satisfied the criterion $v_x = 0$ and $v_y > 0$. The number of equidistant initial angular displacements was 15 over $[0, 2\pi)$ and similarly 15 equidistant initial rotational velocities were taken over the range $[-8 \times 10^{-6} \text{rads}^{-1}, 2 \times 10^{-6} \text{rads}^{-1}]$.

We first simulate Hyperion in a purely circular orbit around Saturn and generate time series of the angular velocity $\omega = d\phi/dt$. Hyperion’s initial orbital position is

$$(x, y) = (0\text{m}, 1.481 \times 10^9\text{m})$$

and orbital velocities are

$$(v_x, v_y) = (5.060 \times 10^3 \text{ms}^{-1}, 0 \text{ms}^{-1})$$

whilst its initial angular displacement is $\phi = \pi/4 \text{rad}$ and rotational velocity is $\omega = 10^{-6} \text{rads}^{-1}$, which corresponds to approximately 72 days per revolution. We use a small initial angular velocity to ensure

Fig. 3. Numerical time series of Hyperion's rotational velocity in a hypothetical circular orbit around Saturn. From top to bottom, the spin-axis tilt is $\varepsilon = \{0, 0.1, 0.2, 0.3\}$. Vertical units are in radians per second whilst horizontal units are in seconds.

that the gravitational torques exerted by Saturn have a non-trivial effect on the rotational momentum of Hyperion. Saturn is taken to be at the origin $(0\text{m}, 0\text{m})$ with an initial velocity of

$$(-5.004 \times 10^{-5} \text{ms}^{-1}, 0 \text{ms}^{-1}).$$

The latter condition is determined by appealing to conservation of momentum.

We plot the time-series for various values of the spin-axis tilt in Fig. 3. We comment on some points: First, for a circularly orbiting Hyperion whose spin-axis is perpendicular to the x - y orbital plane (i.e. $\varepsilon = 0$), its rotational velocity is periodic. However, as the spin-axis is tilted (i.e. ε increases), the periodic rotational velocity features oscillations on a time-scale of 10^7 s, before transitioning into Hamiltonian chaos as $\varepsilon \geq 0.2$.

We accompany this time-series with corresponding Poincaré sections in Fig. 4. We observe that for the case where the spin-axis is unperturbed and perpendicular to the orbital plane, the Hyperion-Saturn system with rotation appears to be integrable, given the appearance of KAM tori in the rotational phase space. We note that perturbing the spin-axis out of the plane (i.e. increasing ε) pushes the system into a near-integrable state, with the KAM tori breaking apart into ergodic regions of Hamiltonian chaos. This numerical result coincides with the Melnikov analysis of Section 4. Increasing the angle at which rotation occurs out of the orbital plane corresponds to the ergodic region becoming larger, encompassing faster initial angular velocities.

We now take Hyperion to be orbiting around Saturn at its experimentally observed eccentricity and plot Poincaré sections corresponding to the spin-axis tilt being $\varepsilon = \{0, 0.1, 0.2, 0.3\}$ in Fig. 5. Saturn's initial orbital position is taken to be the same as for the purely circular case, however its initial orbital velocity is

$$(-5.273 \times 10^{-5} \text{ms}^{-1}, 0 \text{ms}^{-1}),$$

whilst Hyperion's initial position is now

$$(x, y) = (0 \text{m}, 1.473 \times 10^9 \text{m})$$

Fig. 4. Poincaré sections of Hyperion’s rotation as it circularly orbits Saturn. From top to bottom, the spin-axis tilt is $\varepsilon = \{0, 0.1, 0.2, 0.3\}$. Vertical units are in radians per second whilst horizontal units are in radians. (Color online)

and orbital velocities are

$$(v_x, v_y) = (5.332 \times 10^3 \text{ms}^{-1}, 0 \text{ms}^{-1}).$$

For the case where the spin-axis is not tilted (i.e. $\varepsilon = 0$), there is evidence of KAM tori break-up and ergodic regions in the rotational phase-space despite rotation being confined to the plane. This break-up is due to Hyperion’s orbit, which is now no longer perfectly circular, but a perturbation thereof, as shown in [Wisdom *et al.*, 1984] and for more general perturbations in [Kwiecinski *et al.*, 2018a]. As the spin-axis is tilted, so that rotation no longer occurs within the plane, the region of chaotic rotations becomes larger, as was shown previously for the hypothetical circular orbit of Hyperion around Saturn. This result suggests that perturbations to both the rotation orientation out of the plane and the orbit together provide further opportunities for chaotic rotations to exist within the system.

6. Discussion

In the present work, we studied the rotations of an ellipsoid whose spin-axis was always aligned with one of its principal axes but was able to tilt out of the orbital plane. We proved using Melnikov’s method that rotations out of the orbital plane resulted in the existence of chaotic rotations even if the rotating ellipsoid was confined to a purely circular orbit. We verified this analysis by numerically simulating a hypothetical Hyperion-Saturn system for a circular orbit, followed by the experimentally observed elliptical orbit, and varied the spin-axis orientation through the spin-orbit misalignment parameter, ε . We showed that perturbations to the spin-axis orientation and the circular orbit, both together and separately, led to chaotic rotations. Furthermore, as evidenced by Poincaré sections shown in Figs. 4-5, the prevalence of initial conditions leading to chaos increased as the spin-orbit misalignment parameter, ε , was increased.

The degree of deviation of the body from a sphere was shown to primarily influence the timescale of the dynamics, as seen in Eq. (3), with more elongated bodies losing regularity and falling into chaos more rapidly than bodies which were less elongated. Hence, so long as the difference $a - b$ was non-zero, the difference in dynamics (at the leading orders we considered) due to different choices of a and b was

Fig. 5. Poincaré sections of Hyperion’s rotation as it orbits Saturn at its experimentally observed elliptical orbit. From top to bottom, the spin-axis tilt is $\varepsilon = \{0, 0.1, 0.2, 0.3\}$. Vertical units are in radians per second whilst horizontal units are in radians. (Color online)

only to the timescale of the dynamics. As such, the actual choice of $a \neq b$ did not change the Poincaré sections shown in Figs. 4-5, and therefore did not influence the prevalence of initial conditions leading to chaos. Therefore, in the model we studied, the spin-orbit misalignment parameter, ε , was the primary contribution to the emergence of chaotic dynamics.

The analysis and numerical simulations discussed in this work, together with previous work in [Kwiecinski *et al.*, 2018a], suggest that chaotic rotations emerge when the celestial body is not spherical or nearly spherical. We again emphasize that although the tilted spin-axis model presented in Section 2 is not fully realistic, in that the spin-axis would not be fixed, but would rather precess, the model still demonstrates that the existence of chaotic rotations hold. Making the model more realistic by introducing further degrees of rotational freedom in the rotating body would result in additional pathways to chaotic rotation.

We finally highlight two caveats regarding our key result. First, the analysis will only hold provided that the separation between the gravitational bodies occurs on a length-scale much larger than the size of the rotating body. For “close-in” orbits, such as those discussed in [Correia *et al.*, 2014] and [Delisle *et al.*, 2017], this assumption no longer remains true and therefore our results are not applicable. Second, for the celestial body to chaotically rotate, it must have an angular velocity which is sufficiently small. For large rotational velocities, the body would have an angular momentum which is so significantly large that it would be unaffected by torques exerted on it by other gravitational bodies. An example of this is the hyperbolic flyby of the interstellar asteroid 1I/2017 (‘Oumuamua) [NASA, 2018; Jewitt *et al.*, 2017], with a simple model discussed in [Kwiecinski *et al.*, 2018b] demonstrating that torques exerted by gravitational bodies play a negligible role in the dynamics of 1I/2017 (‘Oumuamua). Provided these two caveats are not applicable, then celestial bodies which are elongated or non-circular about their spin axis tend to chaotically rotate for certain initial conditions when the bodies are confined to periodic orbits. Furthermore, the prevalence of initial conditions giving chaos increases with the spin-orbit misalignment parameter, ε .

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Appendix A: Derivation of equations of motion for a rotating ellipsoid under gravity

We derive equations which describe the translational motion of bodies in orbit and the 3D rotation of an ellipsoidal geometry due to the effects of gravity. We suppose there are N bodies in our system with $N - 1$ bodies treated as point masses and the ellipsoid taken to be the N th body which is rigid and undeformable. Note that treating the other bodies as point masses is valid provided that the bodies are separated by a distance which is significantly larger than the dimensions of the finite rigid body. This aspect, along with technical details related to deriving the balance of linear and angular momenta of the bodies, is further elaborated in Meow [Kwiecinski *et al.*, 2018a].

A1. Kinematic relations

We begin by outlining the kinematic relations necessary to model the orbital motion as well as 3D rotation. We suppose our system is described in terms of two bases: a fixed Cartesian basis spanned by unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, and a rotating basis such that the alignment of its orthogonal axes, described by unit vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, and $\mathbf{e}'_3$, line up with the principal semi-axes of the ellipsoid, which have lengths given by $2a$, $2b$, and $2c$, respectively (see Fig. 6). Any vector can be described in either the fixed basis, denoted by superscript F , or the rotational basis, denoted by superscript R . The Cartesian vectors described in the fixed basis are given by $\mathbf{e}_1^F = (1, 0, 0)$, $\mathbf{e}_2^F = (0, 1, 0)$, and $\mathbf{e}_3^F = (0, 0, 1)$ and we find the associated rotational vectors in the fixed basis $(\mathbf{e}'_i)^F$ using the relation:

$$(\mathbf{e}'_i)^F = T \mathbf{e}_i^F, \quad (21)$$

where T is an orthogonal 3×3 matrix whose entries are given by:

$$T = \begin{bmatrix} (\mathbf{e}'_{11})^F & (\mathbf{e}'_{21})^F & (\mathbf{e}'_{31})^F \\ (\mathbf{e}'_{12})^F & (\mathbf{e}'_{22})^F & (\mathbf{e}'_{32})^F \\ (\mathbf{e}'_{13})^F & (\mathbf{e}'_{23})^F & (\mathbf{e}'_{33})^F \end{bmatrix}, \quad (22)$$

with $(\mathbf{e}'_i)^F = ((\mathbf{e}'_{i1})^F, (\mathbf{e}'_{i2})^F, (\mathbf{e}'_{i3})^F)$ for $i = \{1, 2, 3\}$.

Similarly, the rotated vectors in the rotating basis are given by $(\mathbf{e}'_1)^R = (1, 0, 0)$, $(\mathbf{e}'_2)^R = (0, 1, 0)$, and $(\mathbf{e}'_3)^R = (0, 0, 1)$ which, by an inverse transformation, would give the Cartesian vectors in the rotated basis. Mathematically, this is written as the relation:

$$\mathbf{e}_i^R = T^T (\mathbf{e}'_i)^R, \quad (23)$$

for $i = \{1, 2, 3\}$, where superscript T denotes the transpose of the matrix and we note that the transformation matrix satisfies the orthogonality property so that $T^{-1} = T^T$.

We observe that equations (21) and (23) are the same, so that the transformation of unprimed and primed vectors is invariant regardless of the bases used to describe the vectors. Furthermore, by noting that $(\mathbf{e}'_i)^R = \mathbf{e}_i^F$ for $i = \{1, 2, 3\}$ and substituting this into (23), we have an equation that allows us to describe an arbitrary vector $\mathbf{k}$ in either the fixed or rotating bases, provided we can describe it in one of these two basis to begin with:

$$\mathbf{k}^R = T^T \mathbf{k}^F. \quad (24)$$

However, T will depend on time in general and will evolve based on the angular velocities of the ellipsoid as it rotates. We denote these angular velocities in the fixed and rotational axes as $\boldsymbol{\omega}^F = (\omega_1^F, \omega_2^F, \omega_3^F)$ and $\boldsymbol{\omega}^R = (\omega_1^R, \omega_2^R, \omega_3^R)$ respectively, which determines the velocity at which the rigid body rotates around the

Fig. 6. Diagram showing the relationship between the fixed basis, spanned by the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, and the rotating basis, spanned by the unit vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, and $\mathbf{e}'_3$, which always align with the principal axes of lengths $2a$, $2b$, $2c$ respectively of the ellipsoid. Note that the vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, and $\mathbf{e}'_3$ are only rotations with respect to the fixed basis and do not translate with the ellipsoid as it travels along its orbit (denoted by the blue body, not to scale), with the position of the center of mass and arbitrary material point given by $\mathbf{r}_N$ and $\mathbf{R}_t$ respectively. Vectors which are described in the fixed basis are denoted with a superscript F whilst vectors described in the rotating basis have the superscript R .

spanning unit vectors in the fixed and rotational bases. The vectors of the rotational basis evolve according to the relation [Landau & Lifshitz, 1976]:

$$\frac{d(\mathbf{e}'_i)^F}{dt} = \boldsymbol{\omega}^F \times (\mathbf{e}'_i)^F. \quad (25)$$

Therefore, if we wish to determine the time derivative of a quantity as observed in the rotating frame, we have the operator:

$$\left(\frac{d}{dt}\right)_{\text{inertial}} = \left[\left(\frac{d}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}^F \times\right], \quad (26)$$

where the first term on the right hand side is the rate of change as seen in the rotating basis and the second term corresponds to a Coriolis effect going from a non-inertial frame to an inertial one.

A2. Mechanical relations

We now derive relations describing the balancing of linear momentum for all masses in the system and the balancing of angular momentum for the N th rigid body of finite size. We denote the position vector of the i th body's center of mass by $\mathbf{r}_i$ for $i = \{1, \dots, N\}$ and $\mathbf{R}_t$ as the position vector of material points in the N th rigid body (see Fig. 6).

Consider the balance of linear momentum for the i th point mass where $i = \{1, \dots, N-1\}$. The acceleration of the point mass will be due to two contributing factors: the gravitational fields exerted by other point masses in the system as well as the gravitational field generated by the finite geometry of the N th body. For separation distances which are much larger than the dimensions of the rigid body, the equation of orbital motion becomes (see [Kwiecinski *et al.*, 2018a] for derivation):

$$\left(\frac{d^2 \mathbf{r}_i}{dt^2}\right)_{\text{inertial}} \approx \sum_{j \neq i}^N \frac{Gm_j}{|\mathbf{r}_j - \mathbf{r}_i|^3} (\mathbf{r}_j - \mathbf{r}_i) + G \int_B \left[\frac{\rho(\mathbf{R}_t)}{|\mathbf{r}_N - \mathbf{r}_i|^3} (\mathbf{R}_t - \mathbf{r}_N) + \frac{3(\mathbf{r}_N - \mathbf{r}_i) \cdot (\mathbf{r}_N - \mathbf{R}_t)}{|\mathbf{r}_N - \mathbf{r}_i|^2} (\mathbf{r}_N - \mathbf{r}_i) \right] dV, \quad (27)$$

where G is the gravitational constant, m_j is the mass of the j th point mass, $\rho(\mathbf{R}_t)$ is the density of the N th body, and the integral is taken over the entire volume of this rigid body $\mathcal{B}$.

We compute the balance of linear momentum in the fixed basis whereby the position vector of all material points in the rigid ellipsoid is given by $\mathbf{R}_t^F = \mathbf{r}_N^F + A(a\delta \cos \psi \sin \theta, b\delta \sin \psi \sin \theta, c\delta \cos \theta)^T$ for scaled radial coordinate $\delta \in [0, 1]$, spherical angles $\psi \in [-\pi, \pi]$ and $\theta \in [0, \pi]$, and 3×3 rotation matrix A , given that, in the fixed basis, the rigid ellipsoid will have an arbitrary orientation. In this case, the integral vanishes giving the point mass result:

$$\left(\frac{d^2 \mathbf{r}_i^F}{dt^2}\right)_{\text{inertial}} = \sum_{j \neq i}^N \frac{Gm_j}{|\mathbf{r}_j^F - \mathbf{r}_i^F|^3} (\mathbf{r}_j^F - \mathbf{r}_i^F), \quad (28)$$

which is valid regardless of the basis used to describe the vectors, but we have chosen the fixed frame for convenience. Note that we do not need to expand the inertial time-derivative using (26) because we have done the computation in the fixed coordinate system which is an inertial frame of reference.

Similarly, for the N th rigid body, the equation of orbital motion is:

$$\begin{aligned} \left(\frac{d^2 \mathbf{r}_N}{dt^2}\right)_{\text{inertial}} &\approx \sum_{j=1}^{N-1} \frac{Gm_j}{|\mathbf{r}_j - \mathbf{r}_N|^3} (\mathbf{r}_j - \mathbf{r}_N) \\ &+ \frac{G}{m_N} \int_{\mathcal{B}} \frac{\rho(\mathbf{R}_t)}{|\mathbf{r}_j - \mathbf{r}_N|^3} \left[(\mathbf{r}_N - \mathbf{R}_t) + \frac{3(\mathbf{r}_j - \mathbf{r}_N) \cdot (\mathbf{r}_N - \mathbf{R}_t)}{|\mathbf{r}_j - \mathbf{r}_N|^2} (\mathbf{r}_j - \mathbf{r}_N) \right] dV, \end{aligned} \quad (29)$$

where we have assumed that the translational acceleration of all material points is approximately the same. Note that this is true provided the body is undeformable and the translational acceleration is much larger than the rotational acceleration.

Using the same reasoning as above for the previous parameterization of the ellipsoid, the integral term vanishes giving the point mass result:

$$\left(\frac{d^2 \mathbf{r}_N^F}{dt^2}\right)_{\text{inertial}} = \sum_{j=1}^{N-1} \frac{Gm_j}{|\mathbf{r}_j^F - \mathbf{r}_N^F|^3} (\mathbf{r}_j^F - \mathbf{r}_N^F). \quad (30)$$

We now consider the balance of angular momentum of the rigid ellipsoid, which, in the rotating frame, is given by:

$$\begin{aligned} \left(\frac{d}{dt}\right)_{\text{inertial}} \int_{\mathcal{B}} \rho(\mathbf{R}_t^R) (\mathbf{R}_t^R - \mathbf{r}_N^R) \times \mathbf{v}_{\text{tan}}^R dV &\approx 3 \sum_{j=1}^{N-1} \frac{Gm_j}{|\mathbf{r}_j^R - \mathbf{r}_N^R|^5} \int_{\mathcal{B}} (\mathbf{R}_t^R - \mathbf{r}_N^R) \\ &\times \rho(\mathbf{R}_t^R) ((\mathbf{r}_j^R - \mathbf{r}_N^R) \cdot (\mathbf{R}_t^R - \mathbf{r}_N^R)) (\mathbf{r}_j^R - \mathbf{r}_N^R) dV, \end{aligned} \quad (31)$$

where $\mathbf{v}_{\text{tan}}^R$ is the tangential translational velocity of the rigid body as it rotates around its center of mass.

Using the relationship between the angular velocity vector and tangential translational velocity $\mathbf{v}_{\text{tan}}^R = \boldsymbol{\omega}^R \times (\mathbf{R}_t^R - \mathbf{r}_N^R)$, as well as the vector identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$, the left hand side of (31) becomes:

$$\begin{aligned} &\left(\frac{d}{dt}\right)_{\text{inertial}} \int_{\mathcal{B}} \rho(\mathbf{R}_t^R) (\mathbf{R}_t^R - \mathbf{r}_N^R) \times \mathbf{v}_{\text{tan}}^R dV \\ &= \left(\frac{d}{dt}\right)_{\text{inertial}} \int_{\mathcal{B}} \rho(\mathbf{R}_t^R) \left[|\mathbf{R}_t^R - \mathbf{r}_N^R|^2 \boldsymbol{\omega}^R - ((\mathbf{R}_t^R - \mathbf{r}_N^R) \cdot \boldsymbol{\omega}^R) (\mathbf{R}_t^R - \mathbf{r}_N^R) \right] dV \\ &= \left(\frac{d}{dt}\right)_{\text{inertial}} \int_{\mathcal{B}} \rho(\mathbf{R}_t^R) \left[|\mathbf{R}_t^R - \mathbf{r}_N^R|^2 \mathbb{I} - (\mathbf{R}_t^R - \mathbf{r}_N^R) \otimes (\mathbf{R}_t^R - \mathbf{r}_N^R) \right] dV \boldsymbol{\omega}^R, \end{aligned} \quad (32)$$

where $\mathbb{I}$ is the 3×3 identity matrix, $\otimes$ is the dyadic product, and we have assumed that the body is rigid, therefore implying that the angular velocity and angular acceleration is independent of the material position.

We note that the integral on the right hand side of (32) is the moment of inertia tensor which, upon evaluating in the ellipsoidal parameterization of the rotating basis:

$$\mathbf{R}_t^R = \mathbf{r}_N^R + (a\delta \cos \psi \sin \theta, b\delta \sin \psi \sin \theta, c\delta \cos \theta), \quad (33)$$

(i.e. the rotating ellipsoid is fixed and the coordinate axes align with the principal axes of the ellipsoid), gives a diagonal matrix I with entries given by:

$$I_{11} = \frac{1}{5} m_N (b^2 + c^2), \quad (34)$$

$$I_{22} = \frac{1}{5} m_N (a^2 + c^2), \quad (35)$$

$$I_{33} = \frac{1}{5} m_N (a^2 + b^2), \quad (36)$$

with m_N being the mass of the N th body. Note that we have assumed that the density of the celestial body is constant throughout, so that $m_N = 4\pi pabc/3$.

Therefore, the left hand side of the balance of angular momentum (31) gives:

$$\left(\frac{d(I\boldsymbol{\omega}^R)}{dt} \right)_{\text{inertial}} = I \left(\frac{d(\boldsymbol{\omega}^R)}{dt} \right)_{\text{rot}} + \boldsymbol{\omega}^R \times I\boldsymbol{\omega}^R, \quad (37)$$

where we have expanded the inertial time derivative using (26) and where we further note that the principal moments of inertia are independent of time in the rotating frame. Note that this relation is the left hand side of the Euler equations for rigid body motion [Goldstein *et al.*, 2012].

We now evaluate the integral on the right hand side of (31) using the same ellipsoidal parameterization in the rotating basis and combine this result with (37) to obtain:

$$I_{11} \left(\frac{d\omega_1^R}{dt} \right)_{\text{rot}} + (I_{33} - I_{22}) \omega_2^R \omega_3^R = \tau_1, \quad (38)$$

$$I_{22} \left(\frac{d\omega_2^R}{dt} \right)_{\text{rot}} + (I_{11} - I_{33}) \omega_3^R \omega_1^R = \tau_2, \quad (39)$$

$$I_{33} \left(\frac{d\omega_3^R}{dt} \right)_{\text{rot}} + (I_{22} - I_{11}) \omega_1^R \omega_2^R = \tau_3, \quad (40)$$

where the applied torques in the rotating frame (τ_1, τ_2, τ_3) are given by:

$$\tau_1 = \frac{3}{5} \sum_{j=1}^{N-1} \frac{Gm_j m_N}{|\mathbf{r}_j^F - \mathbf{r}_N^F|^5} (b^2 - c^2) \Delta y_j^R \Delta z_j^R, \quad (41)$$

$$\tau_2 = \frac{3}{5} \sum_{j=1}^{N-1} \frac{Gm_j m_N}{|\mathbf{r}_j^F - \mathbf{r}_N^F|^5} (c^2 - a^2) \Delta x_j^R \Delta z_j^R, \quad (42)$$

$$\tau_3 = \frac{3}{5} \sum_{j=1}^{N-1} \frac{Gm_j m_N}{|\mathbf{r}_j^F - \mathbf{r}_N^F|^5} (a^2 - b^2) \Delta x_j^R \Delta y_j^R, \quad (43)$$

where $\mathbf{r}_i^R = (x_i^R, y_i^R, z_i^R)$ is the position of the i th center of mass in the rotating basis, $\Delta x_j^R = x_j^R - x_N^R$, and similarly for Δy_j^R and Δz_j^R , and where we have again assumed that the density of the rigid body is constant so that $m_N = 4\pi pabc/3$. Note that under orthogonal transformation, the norms of vectors are preserved, therefore we calculate the 5-norm in the fixed basis for convenience.

A3. Reduction to the rotation of a tilted ellipsoid

Suppose we take the fixed basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to be the Cartesian directions (x, y, z) . We fix the rotations of the rigid ellipsoid so that it only rotates around the $\mathbf{e}_3'$ axis which is parameterized by a time-dependent angle $\phi(t)$. Furthermore, we take this spin axis to be tilted a time-independent angle of $\sqrt{\varepsilon}$ out of the x - y plane for $\varepsilon \ll 1$ (see Fig. 1). In this case, the rotating basis in the fixed frame is given by:

$$\mathbf{e}_1' = (\cos \phi \cos \sqrt{\varepsilon}, \sin \phi, \cos \phi \sin \sqrt{\varepsilon}), \quad (44)$$

$$\mathbf{e}_2' = (-\sin \phi \cos \sqrt{\varepsilon}, \cos \phi, -\sin \phi \sin \sqrt{\varepsilon}), \quad (45)$$

$$\mathbf{e}_3' = (-\sin \sqrt{\varepsilon}, 0, \cos \sqrt{\varepsilon}). \quad (46)$$

To determine the angular velocities in the fixed frame, we substitute this basis into the time-evolution relations (25) to find that $\omega_1^F = -\dot{\phi} \sin \sqrt{\varepsilon}$, $\omega_2^F = 0$, $\omega_3^F = \dot{\phi} \cos \sqrt{\varepsilon}$. Transforming these rotational velocities into the rotating frame by means of (24), we find a single non-zero angular velocity $\omega_3^R = \dot{\phi}$. As such, there is only a single non-trivial angular momentum equation, which is derived from (40):

$$I_{33} \frac{d^2 \phi}{dt^2} = \frac{3}{5} \sum_{j=1}^{N-1} \frac{G m_j m_N}{|\mathbf{r}_j^F - \mathbf{r}_N^F|^5} (a^2 - b^2) \Delta x_j^R \Delta y_j^R. \quad (47)$$

To determine the separation between the rigid ellipsoid and the other gravitational masses in the rotating frame, we suppose that, in the fixed frame, the masses are confined to the x - y plane for simplicity. In this case, we transform using (24):

$$\begin{bmatrix} \Delta x_j^R \\ \Delta y_j^R \\ \Delta z_j^R \end{bmatrix} = \begin{bmatrix} \cos \phi \cos \sqrt{\varepsilon} & \sin \phi & \cos \phi \sin \sqrt{\varepsilon} \\ -\sin \phi \cos \sqrt{\varepsilon} & \cos \phi & -\sin \phi \sin \sqrt{\varepsilon} \\ -\sin \sqrt{\varepsilon} & 0 & \cos \sqrt{\varepsilon} \end{bmatrix} \begin{bmatrix} \Delta x_j^F \\ \Delta y_j^F \\ 0 \end{bmatrix}, \quad (48)$$

and combine this result with (47) and the definition of the moments of inertia to find (2) upon expanding around $\varepsilon = 0$ and taking terms to $O(\varepsilon)$.

Note that we drop the superscripts F and R in the main-text and only work in the fixed Cartesian basis to avoid confusion.

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