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A Diophantine approach to the three and four exponentials conjectures

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Abstract Some special cases of Wilkie’s conjecture are shown to be equivalent to real versions of the three and four exponentials conjectures. Wilkie’s conjecture is an open problem originating in model theory that concerns the density of algebraic points in sets defined using the exponential function; the latter conjectures concern the algebraic nature of values of the exponential function.

Keywords transcendence, exponential function

Mathematics Subject Classification (2010) 03C64 · 11J81

1 Introduction

Transcendental number theory is an area of mathematics infamous for the ease with which conjectures can be formulated, and the difficulty in proving any of them. An example is Schanuel’s conjecture. Informally, the conjecture states that any algebraic relation between values of the exponential function can be explained by the addition formula, that is by the identity $e^{x+y} = e^x e^y$. The full statement of the conjecture concerns a collection of n complex numbers and the exponential function evaluated at these n numbers. Two natural corollaries arise if one assumes that the n complex numbers are algebraic, or that they are logarithms of algebraic numbers, and one might hope that these corollaries are easier to prove than the full conjecture. The first of these was indeed proved in the nineteenth century by Lindemann and Weierstrass.

Theorem 1 (Lindemann–Weierstrass) *Let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ be algebraic numbers that are $\mathbb{Q}$ -linearly independent. Then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are $\mathbb{Q}$ -algebraically independent.*

The latter case is the following, often referred to as the main conjecture for logarithms of algebraic numbers.

Conjecture 1 Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ have the property that e^{λ_i} is algebraic for each i . If $\lambda_1, \dots, \lambda_n$ are $\mathbb{Q}$ -linearly independent then they are $\mathbb{Q}$ -algebraically independent.

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This conjecture is still a long way from being proved. Algebraic independence over $\mathbb{Q}$ is the same as algebraic independence over $\overline{\mathbb{Q}}$, and with that in mind one of the best results so far towards this conjecture is Baker's theorem.

Theorem 2 (Baker) *Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ have the property that e^{λ_i} is algebraic for each i . If $\lambda_1, \dots, \lambda_n$ are $\mathbb{Q}$ -linearly independent then they are $\overline{\mathbb{Q}}$ -linearly independent.*

Beyond this not a great deal is known. It has still yet to be shown that there even exist two algebraically independent logarithms of algebraic numbers.

Since Baker's theorem rules out linear dependence between logarithms, an obvious next step is to rule out other polynomial relations between them. Various low-degree cases are tackled amongst a family of twelve statements known as the exponentials conjectures and theorems.

The earliest of these was established independently in the 1960s by Lang and Ramachandra and is usually stated as follows.

Theorem 3 ([Lang(1966)]; [Ramachandra(1967/1968)]) *Let $x_1, \dots, x_k$ be complex numbers that are linearly independent over $\mathbb{Q}$ and let $y_1, \dots, y_\ell$ also be complex numbers that are linearly independent over $\mathbb{Q}$. If $k\ell > k + \ell$ then at least one of the $k\ell$ numbers $\exp(x_i y_j)$ for $1 \leq i \leq k$ and $1 \leq j \leq \ell$ is transcendental.*

The case of interest is when $k = 2$ and $\ell = 3$ which gives rise to the theorem's name: the six exponentials theorem. Changing the strict inequality in the hypotheses of this theorem to $k\ell \geq k + \ell$ leads to what is probably the best known open problem in this family: the four exponentials conjecture. This conjecture's shadow first appeared in the 1940s when Alaoglu and Erdős mentioned a consequence of the conjecture and their difficulty in proving it ([Alaoglu and Erdős(1944)]). Indeed, the result they needed is a consequence of the real version of the four exponentials conjecture (conjecture 8 below), and if this conjecture is proved then it would prove that the ratio of every two successive colossally abundant numbers is a prime number, extending Ramanujan's corresponding result for superior highly composite numbers. An equivalent formulation of the four exponentials conjecture is as follows.

Conjecture 2 (Four exponentials conjecture) Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$ have the property that e^{λ_i} is algebraic for each i . Suppose that λ_1 and λ_3 are $\mathbb{Q}$ -linearly independent, and that λ_1 and λ_4 are $\mathbb{Q}$ -linearly independent. Then $\lambda_1 \lambda_2 \neq \lambda_3 \lambda_4$.

This conjecture follows from conjecture 1, albeit in a not entirely trivial manner (see [Waldschmidt(2000), Chapter 1]). It rules out a certain family of quadratic relations between logarithms of algebraic numbers, unless those relations exist for trivial reasons (so, for example, $\log 2 \times \log 9 = \log 4 \times \log 3$ should not be surprising). Another family of quadratic relations is dealt with by a conjecture posited by Waldschmidt that removes the homogeneity from the four exponentials conjecture.

Conjecture 3 (Three exponentials conjecture) Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ have the property that e^{λ_i} is algebraic for each i and let $\gamma \neq 0$ be an algebraic number. If $\lambda_1 \lambda_2 = \gamma \lambda_3$ then $\lambda_3 = 0$.

Unsurprisingly, this is a corollary of conjecture 1, as well as following from some of the other conjectures in the family, such as the sharp four exponentials conjecture. Despite being on quite a low rung of this ladder of implications, the three exponentials conjecture is strong enough to establish the transcendence of an abundance of numbers whose algebraic status is still open. Taking $\lambda_1 = i\pi$, $\lambda_2 = -i\pi$, and $\gamma = 1$ gives that e^{π^2} must be transcendental; at present it remains to be shown that it is even irrational. In this paper we will always be working in the real numbers, but even here new results follow. Taking $\lambda_1 = \lambda_2 = \log 2$ and $\gamma = 1$ implies that $2^{\log 2}$ must be transcendental. Similarly $\alpha^{\log \alpha}$ would be transcendental for any positive algebraic $\alpha \neq 1$; again, these numbers aren't yet known to be irrational.

Certain special cases of the four exponentials conjecture are known to hold unconditionally. It was shown in the early 1970s, independently and concurrently by Brownawell and Waldschmidt, that it holds when (with the above notation) the numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ generate a field with transcendence degree at least one over the rational numbers ([Brownawell(1974)], [Waldschmidt(1973)]). More recently Diaz made novel use of complex conjugation to prove several more special cases of the four exponentials conjecture ([Diaz(2004)]).

The purpose of this paper is to show that the real versions of the three and four exponentials conjectures are equivalent to special cases of an open problem in Diophantine geometry, namely Wilkie's conjecture. Although Wilkie's conjecture is best phrased using the language of o-minimal structures, this won't be necessary here, so we phrase it in more set-theoretic terms. The interested reader can do no better than to consult the paper [Pila and Wilkie(2006)] of Pila and Wilkie, wherein the conjecture first appears in print. (The conjecture was posited by both authors outside of print for several years beforehand, albeit for longer by Wilkie than by Pila, whence the name).

Loosely put, Wilkie's conjecture states that if one defines a subset of $\mathbb{R}^n$ using exponential polynomials and the set-theoretic operations of unions, complements, and projections, then the resulting set cannot contain many points all of whose coordinates are algebraic, unless the set in question actually contains a semi-algebraic set. This will be made precise in the next section, but in particular it applies to the set of (suitably positive) elements $(x, y, z, t) \in \mathbb{R}^4$ defined by the equation

$$(\log x)(\log y) = t \log z.$$

An algebraic point in such a set with $x, y, z \neq 1$ is clearly a counterexample to the three exponentials conjecture, and so that conjecture implies that this set should contain no non-trivial algebraic points. We will show that trivial algebraic points only lie in semi-algebraic sets contained in this set, and that if, as per Wilkie's conjecture, this set contains sufficiently few algebraic points, then the three exponentials conjecture follows.

We will also demonstrate a similar link between Wilkie's conjecture and the four exponentials conjecture. In closing we will discuss the problems in extending this work to the next tier of exponentials conjectures: the sharp three, four, and five exponentials conjectures. Fuller discussion of all these conjectures can be found in Waldschmidt's book [Waldschmidt(2000)] as well as his two papers [Waldschmidt(2005b)] and [Waldschmidt(2005a)].

Throughout the paper the set of strictly positive real numbers will be denoted $\mathbb{R}^+$. The nonzero elements of a field F will be denoted $F^\times$. The real algebraic numbers $\mathbb{R} \cap \mathbb{Q}$ will be denoted $\mathbb{R}_{\text{alg}}$. Both Landau's big O and Vinogradov's double arrow notation will be used for asymptotic upper bounds; that is, if $g(x) > 0$ for all $x \in \mathbb{R}^+$ then we write $f(x) \ll g(x)$ or $f(x) = O(g(x))$ if there are constants $c > 0$ and $x_0 > 0$ such that $|f(x)| \leq cg(x)$ for all $x > x_0$. We will also use the little o notation, whereby with g as before we write $f(x) = o(g(x))$ if $\lim_{x \rightarrow \infty} |f(x)/g(x)| = 0$. Model theorists may note that definable sets will be definable with parameters.

2 Wilkie's conjecture

Wilkie's conjecture posits an upper bound on the density of algebraic points in the transcendental part of a set definable in the o-minimal structure $\mathbb{R}_{\text{exp}}$. We will explain each of these terms in turn.

Given a subset of $X \subseteq \mathbb{R}^n$ and a number field $F \subset \mathbb{R}$ a common question in number theory is: how many points are there in $X(F) = X \cap F^n$? In most cases the answer is either none or infinitely many. Since this latter answer is hardly satisfactory one aims to get density results instead. This involves placing a height function on F . This is some function $\text{ht} : F \rightarrow \mathbb{R}$ that measures the size of elements of F and such that, crucially, given some finite value T there are only finitely many elements in F whose heights are bounded by T . The original question can then become: how many points are there in $X(F, T)$, where

$$X(F, T) = \{(x_1, \dots, x_n) \in X(F) : \text{ht}(x_i) \leq T, i = 1, \dots, n\}?$$

The number of points in $X(F, T)$ is denoted $N(X, F, T)$. Ideally one wants some asymptotic expression for $N(X, F, T)$ valid as T tends to infinity. In practice a non-trivial upper bound is significant progress.

Sometimes trivial upper bounds are the best one can do. If one is counting the density of points in $\mathbb{R}^n(F)$ then one is really just counting the number of points in F of bounded height. While this is not necessarily a simple problem, it is not really the kind of problem we are interested in. In fact any semi-algebraic set will contain rather a lot of algebraic points and will tend to dominate any asymptotic result or upper bound one cares to find. For our purposes, where transcendence is the name of the game, the solution is to define the algebraic part of a set to be the union of all connected, positive dimensional, semi-algebraic subsets of that set,

$$X^{\text{alg}} = \bigcup_{\substack{U \subseteq X \\ U \text{ semi-algebraic} \\ U \text{ connected} \\ \dim(U) > 0}} U.$$

This part is then unceremoniously thrown away, and the question becomes: what is the density of algebraic points in the transcendental part of the set X , i.e. in $X^{\text{trans}} = X \setminus X^{\text{alg}}$?

How interesting and difficult this question is relies predominantly on what the set X is. If it is semi-algebraic then we've just discarded the whole set, so the question has become very easy and not very interesting. On the other hand if the

set is so pathologically horrific that it defies investigation then the question may become extremely difficult, though perhaps interesting. Somewhere between these two possibilities lies a broad class of sets that are demonstrably not horrifying but which also encapsulate many questions of interest; this class is the collection of sets that are definable in o-minimal structures. Amongst these are those definable in the particular structure $\mathbb{R}_{\text{exp}}$. Exactly what an o-minimal structure is is not that important in this case because a theorem of Wilkie ([Wilkie(1996)], second main theorem) gives the following definition that entirely captures the notion of being definable in $\mathbb{R}_{\text{exp}}$.

Definition 1 *A set $X \subset \mathbb{R}^n$ is called definable in $\mathbb{R}_{\text{exp}}$ if there is a polynomial $P \in \mathbb{R}[t_1, \dots, t_{n+m}, y_1, \dots, y_{n+m}]$ in $2(n+m)$ variables such that if $V \subseteq \mathbb{R}^{n+m}$ is given by*

$$V = \{(x_1, \dots, x_{n+m}) : P(x_1, \dots, x_{n+m}, e^{x_1}, \dots, e^{x_{n+m}}) = 0\}$$

and the map $\pi : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ is the projection map onto the first n variables, then $X = \pi(V)$.

To state Wilkie's conjecture we also need to pick a height function. The actual choice is not too important – most natural choices of height function can be bounded in terms of one another. Here we use the multiplicative height H . For an algebraic number α this is defined by $H(\alpha) = \exp(h(\alpha))$ where h is the logarithmic height, namely

$$h(\alpha) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} D_v \log \max\{1, |\alpha|_v\},$$

where K is any number field containing α , M_K is the set of places of K , and D_v is the local degree at $v \in M_K$. At first sight this is can be a rather daunting definition, but we only need the facts that for algebraic α, β and $n \in \mathbb{Z}$ we have $H(\alpha^n) = H(\alpha)^{|n|}$ and that $H(\alpha\beta) \leq H(\alpha)H(\beta)$ (see, for example, [Waldschmidt(2000), Chapter 3]). We shall also use the easily checked fact that if a and b are co-prime nonzero integers then $H(a/b) = \max\{|a|, |b|\}$. If $\mathbf{x} = (x_1, \dots, x_n) \in F^n$ for some number field F then we write $H(\mathbf{x})$ for $\max_i H(x_i)$. Armed with this function we can state Wilkie's conjecture.

Conjecture 4 (Wilkie's conjecture) Let $X \subseteq \mathbb{R}^n$ be definable in $\mathbb{R}_{\text{exp}}$ and $F \subset \mathbb{R}$ be a number field of degree f . There are constants $c_1 = c_1(X, f)$ and $c_2 = c_2(X)$ such that for any $T \geq e$,

$$N(X^{\text{trans}}, F, T) \leq c_1 (\log T)^{c_2}.$$

So far the conjecture is only known in cases where the dimension of X is 1 and for certain surfaces in $\mathbb{R}^3$ ([Pila(2010)], [Jones and Thomas(2012)], [Butler(2012)]). In all of these cases the exponents c_2 that have been proved are rather large and presumably a long way from best possible. Quite what size the exponent should actually be is unclear, although a family of examples gives a possibility.

Theorem 4 *Let $n \geq 1$ and let $P_n \subset \mathbb{R}^{n+1}$ be the set*

$$P_n = \left\{ (y, x_1, \dots, x_n) : y = \prod_{i=1}^n p_i^{x_i} \right\}$$

where p_i is the i th prime number. Then this set is definable in $\mathbb{R}_{\text{exp}}$ and

$$(\log T)^n \ll N(P_n, \mathbb{Q}, T) \ll (\log T)^n.$$

Proof The definability part is not strenuous to show, if not particularly instructive. Let $V \subset \mathbb{R}^{2n+1}$ be the set

$$V = \{(y, x_1, \dots, x_n, u_1, \dots, u_n) : \sum_{i=1}^n (u_i - x_i \log p_i)^2 + (y - e^{u_1} \dots e^{u_n})^2 = 0\}.$$

Then a little thought shows that $P_n = \pi(V)$ where π projects onto the first $n+1$ coordinates.

Next we need to show that $P_n(\mathbb{Q})$ is actually the same as $P_n(\mathbb{Z})$, that is any rational point in the set is actually an integral point. This amounts to showing that if $p_1, \dots, p_n$ are distinct primes and $x_1, \dots, x_n \in \mathbb{Q} \setminus \mathbb{Z}$ then $\prod p_i^{x_i}$ is irrational. The proof is essentially the same as the best known proof that $\sqrt{2}$ is irrational and so we leave it as an exercise for the reader.

So the problem now is to estimate $N(P_n, \mathbb{Z}, T)$. As noted earlier the height function is sub-multiplicative, so that $H(ab) \leq H(a)H(b)$ for all a and b , and also $H(a^n) = H(a)^{|n|}$ for integers n . So in particular

$$H\left(\prod_{i=1}^n p_i^{x_i}\right) \leq \prod_{i=1}^n p_i^{|x_i|}.$$

Hence the height of $(y, x_1, \dots, x_n) \in P_n(\mathbb{Z})$ being at most T is assured if the x_i satisfy

$$\sum_{i=1}^n |x_i| \log p_i \leq \log T.$$

The number of n -tuples $(x_1, \dots, x_n)$ satisfying this inequality is clearly $\gg (\log T)^n$, which thus acts as a lower bound for $N(P_n, \mathbb{Z}, T)$.

For the upper bound it is enough to observe that

$$H\left(\prod_{i=1}^n p_i^{x_i}\right) \geq \prod_{i=1}^n p_i^{|x_i|/2}$$

and so $N(P_n, \mathbb{Q}, T)$ is asymptotically bounded above by the number of integer n -tuples $(x_1, \dots, x_n)$ satisfying

$$\sum_{i=1}^n |x_i| \log p_i \leq 2 \log T.$$

Again, it is clear that the number of such n -tuples is $\ll (\log T)^n$, whence we have the result. $\square$

With this result in mind one might postulate that the correct exponent in Wilkie's conjecture should be at most $\dim(X)$. We don't dare to state this as a conjecture, but mention it here to justify the later results which require particularly low exponents in special cases of Wilkie's conjecture.

3 The real three exponentials conjecture

The version of the three exponentials conjecture stated in the introduction was notably lacking in exponentials. The exponential formulation of the conjecture is the following.

Conjecture 5 (Three exponentials conjecture) Let $x_1, x_2, y \in \mathbb{C}^\times$ and $\gamma \in \overline{\mathbb{Q}}^\times$, then at least one of the following three exponentials is transcendental:

$$e^{x_1 y}, \quad e^{x_2 y}, \quad e^{\gamma x_1 / x_2}.$$

The equivalence to the logarithmic formulation is straightforward via an appropriate choice in the values of x_1, x_2 and y in terms of the λ_i , and vice versa. We will be considering the real version of the conjecture which, for completeness, is as follows.

Conjecture 6 Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ have the property that e^{λ_i} is algebraic for each i and let $\gamma \neq 0$ be an algebraic number. If $\lambda_1 \lambda_2 = \gamma \lambda_3$ then $\lambda_3 = 0$.

As stated in the introduction, this conjecture is intricately linked to the following three-fold in $\mathbb{R}^4$:

$$X_3 = \{(x, y, z, t) \in \mathbb{R}^4 : x, y, z > 0, t \neq 0, (\log x)(\log y) = t \log z\}.$$

This set is definable in $\mathbb{R}_{\text{exp}}$, being the projection of the zero set of the polynomial

$$(x - e^u)^2 + (y - e^v)^2 + (z - e^w)^2 + (uv - tw)^2 + (t^2 - e^s)^2.$$

If we want to see what Wilkie's conjecture says about this set X_3 then we first need to understand its algebraic part. For this, it suffices to find all those algebraic curves that intersect X_3 in an arc. We will make substantial use of Ax's proof of Schanuel's conjecture in the setting of differential fields, usually known as the Ax-Schanuel theorem.

Theorem 5 ([Ax(1971)], theorem 3) Let $\mathbb{Q} \subset E \subset F$ be a tower of fields and Δ be a set of derivations on F with $\bigcap_{D \in \Delta} \ker D = E$. Let $y_1, \dots, y_n, z_1, \dots, z_n \in F^\times$ satisfy:

(a) for every $D \in \Delta$ and $i = 1, \dots, n$,

$$Dy_i = \frac{Dz_i}{z_i},$$

and either

- (b1) no non-trivial power product of the z_i is in E , or
- (b2) the y_i are $\mathbb{Q}$ -linearly independent modulo E .

Then

$$\text{trdeg}_E E(y_1, \dots, y_n, z_1, \dots, z_n) \geq n + \text{rank}_{D \in \Delta} (Dy_i)_{i=1, \dots, n}.$$

We will be working in the situation where $E = \mathbb{R}$ and $F = \mathbb{R}\langle\langle t \rangle\rangle$ is the field of Puiseux series over $\mathbb{R}$, that is the field of formal power series with fractional exponents:

$$\mathbb{R}\langle\langle t \rangle\rangle = \bigcup_{n \geq 1} \left\{ \sum_{k=-N}^{\infty} a_k t^{k/n} : N \in \mathbb{Z}, a_k \in \mathbb{R} \right\}.$$

The usual derivation will be the only one used and will be denoted d/dt . Using the formal power series for $\log(1+t)$, if $\gamma(t) \in \mathbb{R}\langle\langle t \rangle\rangle$ is non-zero then one can expand $\log(\gamma(t))$ as a Puiseux series too and the resulting element of $\mathbb{R}\langle\langle t \rangle\rangle$ (which will just be called $\log(\gamma(t))$) satisfies

$$\frac{d}{dt} \log(\gamma(t)) = \frac{1}{\gamma(t)} \frac{d}{dt} \gamma(t).$$

In particular then, part (a) of Ax's theorem applies if we take $z_i = \gamma_i(t)$ and $y_i = \log(\gamma_i(t))$. Moreover, in this case conditions (b1) and (b2) are equivalent, so if either of them applies then so does the conclusion. Since our set Δ of derivations consists of a single element, the rank of the matrix in the right hand side of the theorem's conclusion is 1, unless each γ_i is an element of $\mathbb{R}$, a situation that won't come up. So the consequence of consequence is that if

$$\text{trdeg}_{\mathbb{R}} \mathbb{R}(\gamma_1, \dots, \gamma_n, \log(\gamma_1), \dots, \log(\gamma_n)) < n + 1$$

then the $\log(\gamma_i)$ are $\mathbb{Q}$ -linearly dependent modulo $\mathbb{R}$, and the γ_i are $\mathbb{Q}$ -multiplicatively dependent modulo $\mathbb{R}$.

We can now establish X_3^{alg} .

Lemma 1 *For $p, q \in \mathbb{R}$ let*

$$\Gamma_{1,p,q} = \{(e^q, s^p, s, pq) : s > 0\}$$

and

$$\Gamma_{2,p,q} = \{(s^p, e^q, s, pq) : s > 0\},$$

and for any algebraic function $f : I \rightarrow \mathbb{R}$, with $I \subset \mathbb{R}$ an interval, let

$$\Gamma_{3,f} = \{(1, f(s), 1, s) : s \in f^{-1}(\mathbb{R}_{>0}) \setminus \{0\}\}$$

and

$$\Gamma_{4,f} = \{(f(s), 1, 1, s) : s \in f^{-1}(\mathbb{R}_{>0}) \setminus \{0\}\}.$$

Then

$$X_3^{\text{alg}} = \bigcup_{\substack{p \in \mathbb{Q}^\times \\ q \in \mathbb{R}^\times}} (\Gamma_{1,p,q} \cup \Gamma_{2,p,q}) \bigcup_{\text{algebraic } f} (\Gamma_{3,f} \cup \Gamma_{4,f}).$$

Proof Let Γ be an arc of an algebraic curve contained in X_3 . There are two broad cases we need to consider: either t varies on Γ or it is constant. We start with the case that t varies.

In this case we may write

$$\Gamma = \{(\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_0(t)) : t \in I\}$$

where each $\gamma_i(t) \in \mathbb{R}\langle\langle t \rangle\rangle$ and $\gamma_0(t) = t$ (we will usually suppress the argument t in what follows). Since each of $\gamma_1, \gamma_2, \gamma_3$ is an algebraic function in t we have

$$\begin{aligned} \text{trdeg}_{\mathbb{R}} \mathbb{R}(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \log \gamma_0, \log \gamma_1, \log \gamma_2, \log \gamma_3) \\ = \text{trdeg}_{\mathbb{R}} \mathbb{R}(\gamma_0, \log \gamma_0, \log \gamma_1, \log \gamma_2, \log \gamma_3). \end{aligned}$$

But $\Gamma \subset X_3$ so there is an algebraic relation between $\gamma_0, \log \gamma_1, \log \gamma_2$, and $\log \gamma_3$ given by the defining equation of X_3 . Hence

$$\text{trdeg}_{\mathbb{R}} \mathbb{R}(\gamma_0, \log \gamma_0, \log \gamma_1, \log \gamma_2, \log \gamma_3) < 5$$

and so we can apply Ax–Schanuel to deduce that the γ_i are $\mathbb{Q}$ -multiplicatively dependent modulo $\mathbb{R}$, or their logarithms are $\mathbb{Q}$ -linearly dependent modulo $\mathbb{R}$. In what follows we write g_i for $\log \gamma_i$ for $0 \leq i \leq 3$. We split into cases depending on the multiplicative degree of the γ_i over $\mathbb{Q}$, modulo $\mathbb{R}$.

Case 0 $\text{muldeg}_{\mathbb{Q}}(\gamma_0, \gamma_1, \gamma_2, \gamma_3)/\mathbb{R} = 0$.

This can only be the case if each γ_i is actually constant, but we know that at least γ_0 varies so this cannot be the case.

Case 1 $\text{muldeg}_{\mathbb{Q}}(\gamma_0, \gamma_1, \gamma_2, \gamma_3)/\mathbb{R} = 1$.

In this case we may write $\gamma_i = c_i \gamma_0^{a_i}$ for $1 \leq i \leq 3$ with $c_i > 0$ and $a_i \in \mathbb{Q}$. Since $\Gamma \subset X_3$ we can plug these values into the defining equation to arrive at the following identity

$$(a_1 g_0 + \log c_1)(a_2 g_0 + \log c_2) = \gamma_0(a_3 g_0 + \log c_3).$$

This is an algebraic relation between γ_0 and its logarithm, so by Ax–Schanuel γ_0 must be constant, which by assumption it is not. The only way around this is if the above formula is actually the identity $0 = 0$ in disguise. Multiplying out the above this gives us the system of equations

$$\begin{aligned} a_1 a_2 &= 0 \\ c_2^{a_1} c_1^{a_2} &= 1 \\ c_1 &= 1 \text{ or } c_2 = 1 \\ a_3 &= 0 \\ c_3 &= 1. \end{aligned}$$

The first solution to this system has $a_1 = a_3 = 0$, $c_1 = c_3 = 1$, and any $a_2 \in \mathbb{Q}$ and $c_2 > 0$, thus giving $\Gamma \subset \Gamma_{3,f(s)}$ where $f(s) = c_2 s^{a_2}$. The second solution has $a_2 = a_3 = 0$, $c_2 = c_3 = 1$, and any $a_1 \in \mathbb{Q}$ and $c_1 > 0$, thus giving $\Gamma \subset \Gamma_{4,f(s)}$ where $f(s) = c_1 s^{a_1}$.

Case 2 $\text{muldeg}_{\mathbb{Q}}(\gamma_0, \gamma_1, \gamma_2, \gamma_3)/\mathbb{R} = 2$.

There are only two substantially different cases here.

Case 2A If γ_0 and γ_3 are multiplicatively independent then we can write $\gamma_1 = c_1 \gamma_0^{a_1} \gamma_3^{a_2}$ and $\gamma_2 = c_2 \gamma_0^{b_1} \gamma_3^{b_2}$ for $c_1, c_2 > 0$ and $a_1, a_2, b_1, b_2 \in \mathbb{Q}$. Expanding the equation of X_3 with these values gives

$$(\log c_1 + a_1 g_0 + a_2 g_3)(\log c_2 + b_1 g_0 + b_2 g_3) = \gamma_0 g_3.$$

This means $\text{trdeg}_{\mathbb{R}} \mathbb{R}(\gamma_0, \gamma_3, g_0, g_3) < 3$, which with Ax–Schanuel contradicts the multiplicative independence of γ_0 and γ_3 . Again, this could be sidestepped if the above identity was secretly $0 = 0$ but no judicious choice of constants can eliminate the $\gamma_0 g_3$ term and so this case cannot occur.

Case 2B If, on the other hand, γ_0 and γ_1 are multiplicatively independent then we can write $\gamma_2 = c_1 \gamma_0^{a_1} \gamma_1^{a_2}$ and $\gamma_3 = c_2 \gamma_0^{b_1} \gamma_1^{b_2}$ for $c_1, c_2 > 0$ and $a_1, a_2, b_1, b_2 \in \mathbb{Q}$. We can again expand the equation of X_3 with these values to obtain

$$g_1(\log c_1 + a_1 g_0 + a_2 g_1) = \gamma_0(\log c_2 + b_1 g_0 + b_2 g_1).$$

As before, Ax–Schanuel would then contradict the multiplicative independence of γ_0 and γ_1 unless this identity collapses to $0 = 0$. For this to happen we need $c_1 = c_2 = 1$, $a_1 = a_2 = 0$, and $b_1 = b_2 = 0$. This means $\gamma_2 = \gamma_3 = 1$ and $\gamma_1 = f(t)$ for some algebraic function f . Hence $\Gamma \subset \Gamma_{4,f}$. Similarly, if instead we have that γ_0 and γ_2 are multiplicative independent then $\Gamma \subset \Gamma_{3,f}$.

Case 3 $\text{muldeg}_{\mathbb{Q}}(\gamma_0, \gamma_1, \gamma_2, \gamma_3)/\mathbb{R} = 3$.

Again there are only two cases up to symmetries.

Case 3A If $\gamma_0, \gamma_1, \gamma_2$ are multiplicatively independent then we can write $\gamma_3 = c \gamma_0^{a_0} \gamma_1^{a_1} \gamma_2^{a_2}$ for some $c > 0$ and $a_0, a_1, a_2 \in \mathbb{Q}$. Putting this into the equation of X_3 gives

$$g_1 g_2 = \gamma_0(\log c + a_0 g_0 + a_1 g_1 + a_2 g_2).$$

This algebraic relation between γ_0, g_0, g_1 , and g_2 forces

$$\text{trdeg}_{\mathbb{R}} \mathbb{R}(\gamma_0, \gamma_1, \gamma_2, g_0, g_1, g_2) < 4,$$

and so Ax–Schanuel applies implying the multiplicative dependence of γ_0, γ_1 , and γ_2 , a contradiction that can't be avoided by any choice of the constants c, a_0, a_1, a_2 .

Case 3B If $\gamma_0, \gamma_1, \gamma_3$ are multiplicatively independent then we can write $\gamma_2 = c \gamma_0^{a_0} \gamma_1^{a_1} \gamma_3^{a_2}$ for some $c > 0$ and $a_0, a_1, a_2 \in \mathbb{Q}$. The same working as the previous case then leads to a contradiction.

That deals with the cases when t varies on Γ . Next we consider what happens if t is actually constant on Γ and equal to, say, $r \neq 0$. This case is mostly taken care of in [Pila(2010), proposition 4.1] but we whizz through the proof using the above methods for completeness.

First we consider the case that z is also constant on Γ . Clearly x and y must both vary and on a small enough neighbourhood we may write $x = \gamma_1(x) \in \mathbb{R}\langle\langle x \rangle\rangle$ and $y = \gamma_2(x) \in \mathbb{R}\langle\langle x \rangle\rangle$. As before we write $g_i = \log(\gamma_i)$ and then we note that $\text{trdeg}_{\mathbb{R}} \mathbb{R}(\gamma_1, \gamma_2, g_1, g_2) \leq 2 < 3$ so by Ax–Schanuel γ_1 and γ_2 are $\mathbb{Q}$ -multiplicatively

dependent modulo $\mathbb{R}$. So we may write $\gamma_2 = c\gamma_1^a$ for $c > 0$ and $a \in \mathbb{Q}$ and, with the equation for X_3 , deduce

$$g_1(\log c + ag_1) = b$$

for some constant b . This is a quadratic equation for g_1 that we could simply solve to attain a constant value for g_1 unless the equation is actually $0 = 0$. But then we would require $a = 0$ and $c = 1$. This would mean $\gamma_2 = 1$ is constant, another contradiction. Clearly, then, z must vary on Γ .

We now pick a small enough neighbourhood so that the coordinates of Γ can be expressed as Puiseux series in z . We write $z = \gamma_0(z) \in \mathbb{R}\langle\langle z \rangle\rangle$ and then we have

$$\Gamma = \{(\gamma_1(z), \gamma_2(z), \gamma_0(z), r) : z \in I\}$$

for some interval I . As earlier we can observe that $\text{trdeg}_{\mathbb{R}}(\gamma_0, \gamma_1, \gamma_2, g_0, g_1, g_2) \leq 3 < 4$ so that $\gamma_0, \gamma_1, \gamma_2$ are $\mathbb{Q}$ -multiplicatively dependent modulo $\mathbb{R}$. There are now two cases to consider.

Case 1 $\text{muldeg}_{\mathbb{Q}}(\gamma_0, \gamma_1, \gamma_2)/\mathbb{R} = 1$.

In this case we can write $\gamma_1 = c_1\gamma_0^{a_1}$ and $\gamma_2 = c_2\gamma_0^{a_2}$ for $c_1, c_2 > 0$ and $a_1, a_2 \in \mathbb{Q}$. Then

$$(\log c_1 + a_1 g_0)(\log c_2 + a_2 g_0) = r g_0.$$

This is a quadratic equation for g_0 and so γ_0 is forced to be constant unless the equation reduces to $0 = 0$. Multiplying out the above this means

$$\begin{aligned} a_1 a_2 &= 0 \\ c_1^{a_2} c_2^{a_1} &= e^r \\ c_1 &= 1 \text{ or } c_2 = 1. \end{aligned}$$

There are two families of solutions to these equations. The first is the family $a_1 = 0$, $c_2 = 1$, and $c_1^{a_2} = e^r$ which leads to $\gamma_1 = c_1$, $\gamma_2 = z^{a_2}$, and $r = a_2 \log c_1$, so that $\Gamma \subset \Gamma_{1, a_2, \log c_1}$. The second family has $a_2 = 0$, $c_1 = 1$, and $c_2^{a_1} = e^r$ which leads to $\gamma_1 = z^{a_1}$, $\gamma_2 = c_2$, and $r = a_1 \log c_2$, so that $\Gamma \subset \Gamma_{2, a_1, \log c_2}$.

Case 2 $\text{muldeg}_{\mathbb{Q}}(\gamma_0, \gamma_1, \gamma_2)/\mathbb{R} = 2$.

There are two possibilities here but the symmetry of the equation means we need only consider, say, the case when γ_0 and γ_1 are multiplicatively independent so that $\gamma_2 = c\gamma_0^{a_0}\gamma_1^{a_1}$. Plugging this into the equation of X_3 proffers

$$g_1(\log c + a_0 g_0 + a_1 g_1) = r g_0.$$

Ax-Schanuel then implies the multiplicative dependence of γ_0 and γ_1 unless we force this equation to be $0 = 0$. This happens if $c = 1$ and $a_0 = a_1 = r = 0$. But $r \neq 0$ by hypothesis, so this cannot occur. $\square$

With the algebraic part of X_3 understood, we can see the link between Wilkie's conjecture for the set X_3 and conjecture 6.

Theorem 6 *Suppose that for any number field $F \subset \mathbb{R}$ we have*

$$N(X_3^{\text{trans}}, F, T) = o((\log T)^3).$$

Then conjecture 6 is true. Conversely, if conjecture 6 is true then for every number field $F \subset \mathbb{R}$, $X_3^{\text{trans}}(F) = \emptyset$.

This suggests a nice repulsion result. Wilkie's conjecture doesn't forbid X_3 from containing lots of algebraic points from the same number field, as long as their height grows quickly enough. But combined with transcendence properties of the exponential function, if $X_3^{\text{trans}}(F)$ isn't empty then there is a lower bound on the density of its points.

Proof First suppose that conjecture 6 is false, so there are three real numbers $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^\times$ such that $e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_3} \in \mathbb{R}_{\text{alg}}$, and a real algebraic number $\gamma \neq 0$ such that $\lambda_1 \lambda_2 = \gamma \lambda_3$. Let $x = e^{\lambda_1}$, $y = e^{\lambda_2}$, $z = e^{\lambda_3}$, and $t = \gamma$. Let F be the number field obtained by adjoining these numbers to $\mathbb{Q}$, so $F = \mathbb{Q}(x, y, z, t)$. Then we have that

$$(\log x)(\log y) = \lambda_1 \lambda_2 = \gamma \lambda_3 = t \log z.$$

And so $(x, y, z, t) \in X_3(F)$. We can raise each of these numbers to integer powers without expelling them from F , and so for any integers $a, b, c \in \mathbb{Z}$ with $c \neq 0$ we also have

$$(x^a, y^b, z^c, abt/c) \in X_3(F).$$

We now need to check two things: that by varying a, b , and c we get enough algebraic points to confound the $o((\log T)^3)$ upper bound, and that the points we get aren't contained in X_3^{alg} . The second of these issues is easy to deal with.

Recall that X_3^{alg} consists of the four families $\Gamma_{1,p,q}$, $\Gamma_{2,p,q}$ for $p \in \mathbb{Q}$, $q \in \mathbb{R}$, and $pq \neq 0$, and $\Gamma_{3,f}$ and $\Gamma_{4,f}$ for algebraic functions f . In the first two families, two of the coordinates are of the form (e^q, pq) for our nonzero rational parameter p and real parameter q . But by Hermite's theorem at least one of these two numbers is transcendental, and so none of x, y , or z can be of this form, and nor can any of their powers. In the latter two families of curves, either the x or y coordinate is fixed at 1, but this would correspond to either λ_1 or λ_2 being zero, which we ruled out by hypothesis. So our points and the derived points cannot lie in X_3^{alg} .

We now show that varying a, b, c gives $\gg (\log T)^3$ points in $X_3^{\text{trans}}(F, T)$. As just stated, neither x, y , nor z can be 1, so, since we are in $\mathbb{R}^+$, raising them to different powers always results in different numbers. Now we check how many different values of a, b , and c we can use without exceeding height T . First let $\eta_1 = h(x)$, $\eta_2 = h(y)$, $\eta_3 = h(z)$, and $\eta_4 = H(t)$, where h is the logarithmic height defined earlier and $H = e^h$ is the multiplicative height. Recall that $H(\alpha^n) = H(\alpha)^{|n|}$ for algebraic α and integral n , and so if we want $H(x^a) \leq T$ then it suffices to have $|a| \leq \log T / \log H(x)$, and similarly for y^b and z^c . And so we can take

$$|a| \leq \frac{\log T}{\eta_1}, \quad |b| \leq \frac{\log T}{\eta_2}, \quad |c| \leq \frac{\log T}{\eta_3}.$$

But we still need to check that the height in the t -coordinate doesn't grow too quickly. The multiplicative height H is sub-multiplicative, so we have

$$H\left(\frac{ab}{c}t\right) \leq H\left(\frac{ab}{c}\right)H(t) = \max\{|ab|, |c|\}\eta_4.$$

In particular, if a, b, c range over the domains specified above then the above height is $\ll (\log T)^2$ which is definitely less than T after some small, fixed value of T depending on the η_i s. So if we let $\eta = \max_i \eta_i$ then we can let $|a|, |b|, |c|$ vary between 1 and $\lfloor (\log T)/\eta \rfloor$ and keep all our points of height at most T . Hence we have

$$N(X_3^{\text{trans}}, F, T) \gg (\log T)^3.$$

We now prove the converse implication. So suppose that for some number field F we have a point $(x, y, z, t) \in X_3^{\text{trans}}(F)$. We show this gives a counterexample to conjecture 6. By definition this point gives $\lambda_1 = \log x$, $\lambda_2 = \log y$, $\lambda_3 = \log z$, and $\gamma = t$ that satisfy the hypotheses of conjecture 6 but such that $\lambda_1 \lambda_2 = \gamma \lambda_3$. And so we are done unless $\lambda_1 \lambda_2 = \gamma \lambda_3 = 0$. Since $\gamma = t \neq 0$ this only leaves the possibility that $\lambda_3 = 0$ so that $z = 1$, and $\lambda_1 = 0$ or $\lambda_2 = 0$ so that x or y is 1. But these two possibilities both lie in X_3^{alg} , specifically the families referred to as $\Gamma_{3,f}$ and $\Gamma_{4,f}$. This contradicts $(x, y, z, t) \in X_3^{\text{trans}}$, and so we are done. $\square$

The set X_3 tackles the three exponentials conjecture all at once, as it were. An alternative approach would be to try to prove the conjecture individually for each value of γ . This corresponds to proving Wilkie's conjecture for the algebraic fibres of X_3 if we view it as a definable family parametrised by t . So we define

$$X_{3,\gamma} = \{(x, y, z) \in (\mathbb{R}^+)^3 : (\log x)(\log y) = \gamma \log z\}.$$

The case $\gamma = 1$ is the surface considered in [Pila(2010)], where it is shown that

$$N(X_{3,1}^{\text{trans}}, F, T) \ll (\log T)^{44+\varepsilon}.$$

We show below how proving $N(X_{3,\gamma}^{\text{trans}}, F, T) \ll \log T$ for all algebraic $\gamma \neq 0$ would prove conjecture 6.

Theorem 7 *Let $\gamma \neq 0$ be real and algebraic. Suppose that for any number field $F \subset \mathbb{R}$*

$$N(X_{3,\gamma}^{\text{trans}}, F, T) = o(\log T \log \log T).$$

Then if λ_1, λ_2 , and λ_3 are nonzero logarithms of positive, real algebraic numbers then $\lambda_1 \lambda_2 \neq \gamma \lambda_3$.

Proof This is essentially the same as the previous proof. The set $X_{3,\gamma}^{\text{alg}}$ is comprised of the sections of X_3^{alg} under $t = \gamma$. In particular, for any $(x, y, z) \in X_{3,\gamma}^{\text{alg}}$, either $x = 1$ or $y = 1$ or x is transcendental or y is transcendental. So if we have nonzero logarithms of algebraic numbers $\lambda_1, \lambda_2, \lambda_3$ such that $\lambda_1 \lambda_2 = \gamma \lambda_3$ then letting $x = e^{\lambda_1}, y = e^{\lambda_2}, z = e^{\lambda_3}$ gives us a point in $X_{3,\gamma}^{\text{trans}}(F)$ for $F = \mathbb{Q}(x, y, z)$. As above we can then take integral powers of the exponents, this time parametrised by two integers, so we get

$$(x^a, y^b, z^{ab}) \in X_{3,\gamma}^{\text{trans}}(F)$$

for any integers a and b . This time we can't simply vary a and b between $-\lfloor \log T \rfloor$ and $\lfloor \log T \rfloor$ since then the height of the third coordinate, z^{ab} would grow about as quickly as $T^{\log T}$, faster than the T permitted. So we need $|a|, |b| \ll \log T$ and $|ab| \ll \log T$. If we take the lattice points $(a, b) \in \mathbb{Z}^2$ below the curve $xy = \log T$ then our conditions are met. The number of such points is bounded by

$$\int_1^{\log T} \frac{\log T}{x} dx = \log T \log \log T.$$

And so taking all such a and b gives us

$$N(X_{3,\gamma}^{\text{trans}}, F, T) \gg \log T \log \log T.$$

□

The preceding way of considering X_3 as a definable family of surfaces in $\mathbb{R}^3$ is arguably the most natural, and has the benefit that Wilkie's conjecture has been proved for the individual fibres (as a consequence of the results in [Jones and Thomas(2012)]), albeit with a much larger exponent than required. However, since $X_{3,\gamma}$ has dimension 2 one might hope that an upper bound of $o((\log T)^2)$ would be sufficient to prove the three exponentials conjecture. And this is precisely what happens if one considers X_3 as a definable family parametrised by the other variables.

Theorem 8 *For $a \in \mathbb{R}$ let*

$$X'_{3,a} = \{(x, y, t) \in \mathbb{R}^3 : x, y > 0, t \neq 0, (\log x)(\log y) = at\}.$$

Let $\alpha \in \mathbb{R}_{\text{alg}}$ be positive. Suppose that for any number field $F \subset \mathbb{R}$ with $\alpha \in F$ it is the case that

$$N(X'_{3,\log \alpha}, F, T) = o((\log T)^2).$$

Then, if β_1, β_2 , and γ are real and algebraic with $\beta_1, \beta_2 > 0$ and $\gamma \neq 0$, and if $\log \beta_1 \log \beta_2 = \gamma \log \alpha$, then $\alpha = 1$.

Proof Suppose that $\alpha > 0$ is algebraic and $\alpha \neq 1$. In the formulation of Wilkie's conjecture given in the hypotheses, the transcendental part of $X'_{3,\log \alpha}$ is conspicuous by its absence. That is simply because the algebraic part of $X'_{3,\log \alpha}$ is empty. Indeed, suppose $\Gamma \subset X'_{3,\log \alpha}$ is an arc of an algebraic curve. If one of the variables x, y , or t is constant on Γ then what's left is clearly a transcendental function. Hence all three variables really do vary. Writing x and y as Puiseux series in t leads to the same kind of contradiction as was seen in the proof of lemma 1. And so no such Γ can exist.

Now suppose we are given real algebraic numbers β_1, β_2 , and γ with $\beta_1, \beta_2 > 0$ and $\gamma \neq 0$, and such that $\log \beta_1 \log \beta_2 = \gamma \log \alpha$. Let $F = \mathbb{Q}(\beta_1, \beta_2, \gamma, \alpha)$. So $(\beta_1, \beta_2, \gamma) \in X'_{3,\log \alpha}(F)$. Moreover, for any integers a and b with $ab \neq 0$ we have that $(\beta_1^a, \beta_2^b, \gamma/ab) \in X'_{3,\log \alpha}(F)$. As in the proof of theorem 6 we may let a and b vary between $-\lfloor \log T \rfloor$ and $\lfloor \log T \rfloor$ (up to some constant multiple) without the heights of the coordinates of $(\beta_1^a, \beta_2^b, \gamma/ab)$ growing beyond T . This gives $N(X'_{3,\log \alpha}, F, T) \gg (\log T)^2$, a contradiction. And so $\alpha = 1$. □

4 The four exponentials conjecture

We now move onto the better known member of the exponentials conjectures family: the four exponentials conjecture, posited independently by Lang and Ramachandra in the 1960s ([Lang(1966)], [Ramachandra(1967/1968)]). Again there is a logarithmic formulation, given in the introduction, and the following better known exponential formulation.

Conjecture 7 (Four exponentials conjecture) Let $x_1, x_2 \in \mathbb{C}$ be $\mathbb{Q}$ -linearly independent, and let $y_1, y_2 \in \mathbb{C}$ be $\mathbb{Q}$ -linearly independent. Then at least one of the following four exponentials is transcendental:

$$e^{x_1 y_1}, \quad e^{x_1 y_2}, \quad e^{x_2 y_1}, \quad e^{x_2 y_2}.$$

The real version of the conjecture that we consider is as follows.

Conjecture 8 Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ have the property that e^{λ_i} is algebraic for each i . Suppose that λ_1 and λ_3 are $\mathbb{Q}$ -linearly independent. Moreover, suppose that λ_1 and λ_4 are $\mathbb{Q}$ -linearly independent. Then $\lambda_1 \lambda_2 \neq \lambda_3 \lambda_4$.

The equivalence of the two versions is not hard to see using a suitable choice of the λ_i in terms of the x_i and y_i , and vice versa.

One way of tackling the four exponentials conjecture that is in the vein of this paper is to look at the definable sets $U_\alpha = \{(x, y) \in \mathbb{R}^2 : y = x^\alpha\}$ for irrational numbers α . If $N(U_\alpha, F, T) = o((\log T)^2)$ for every number field F and every irrational α then the above real version of the four exponentials conjecture follows. Note that the exponent of 2 is higher than one might expect, namely an exponent of 1. These curves are unusual in that Wilkie's conjecture has been proved to hold for them, albeit with a higher exponent. Nonetheless, in this case *any* exponent gives a result, with the current best exponent of 20 implying the “42 exponentials theorem”, a weak version of the six exponentials theorem ([Pila(2010)]).

Here we consider an approach akin to that used in the previous section on the three exponentials conjecture. Again, we consider a three-fold in $\mathbb{R}^4$ and will show that a sufficient paucity of algebraic points implies the four exponentials conjecture. The set in question is

$$X_4 = \{(x, y, z, w) \in \mathbb{R}^4 : x, y, z, w > 0, (\log x)(\log y) = (\log z)(\log w)\}.$$

This set is definable, being the projection of the zero set of the exponential polynomial

$$(x - e^s)^2 + (y - e^t)^2 + (z - e^u)^2 + (w - e^v)^2 + (st - uv)^2.$$

Again we need to understand this set's algebraic part before we can draw any conclusions about it. And again, we will make substantial use of the Ax–Schanuel theorem.

Lemma 2 *For every algebraic function $f : \mathbb{R} \rightarrow \mathbb{R}^+$ and for every $a, b \in \mathbb{Q}^\times$ and $c \in \mathbb{R}^+$ let*

$$\begin{aligned} \Gamma_{1,a,b,c} &= \{(x, y, z, w) : x > 0, z = x^a, w = cx^b, y = c^a x^{ab}\} \\ \Gamma_{2,a,b,c} &= \{(x, y, z, w) : x > 0, z = cx^b, w = x^a, y = c^a x^{ab}\} \\ \Gamma_{3,a,f} &= \{(x, y, z, w) : z > 0, x = z^a, w = f(z)^a, y = f(z)\} \\ \Gamma_{4,a,f} &= \{(x, y, z, w) : w > 0, x = w^a, z = f(w)^a, y = f(w)\} \\ \Gamma_{5,a,f} &= \{(x, y, z, w) : w > 0, y = w^a, z = f(w)^a, x = f(w)\} \\ \Gamma_{6,a,f} &= \{(x, y, z, w) : z > 0, y = z^a, w = f(z)^a, x = f(z)\}. \end{aligned}$$

Then

$$X_4^{\text{alg}} = \bigcup_{\substack{a, b \in \mathbb{Q}^\times \\ c \in \mathbb{R}^+ \\ f \text{ algebraic}}} (\Gamma_{1,a,b,c} \cup \Gamma_{2,a,b,c} \cup \Gamma_{3,a,f} \cup \Gamma_{4,a,f} \cup \Gamma_{5,a,f} \cup \Gamma_{6,a,f}).$$

Proof Let $\Gamma \subset X_4^{\text{alg}}$ be an arc of an algebraic curve. Without loss of generality we may assume that x varies on Γ . So we may write

$$\Gamma = \{(t, \gamma_2(t), \gamma_3(t), \gamma_4(t)) : t \in I\}$$

where each $\gamma_i \in \mathbb{R}\langle\langle t \rangle\rangle$, $I \subset \mathbb{R}$ is some interval, and on I each γ_i is either injective or constant, but they are not all constant. We will let $\gamma_1(t) = t \in \mathbb{R}\langle\langle t \rangle\rangle$.

To make use of the Ax–Schanuel theorem we need to show that

$$\text{trdeg}_{\mathbb{R}} \mathbb{R}(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \log(\gamma_1), \log(\gamma_2), \log(\gamma_3), \log(\gamma_4)) < 5.$$

Each of the γ_i is an algebraic function of t , so any two of them are algebraically dependent. Hence the above transcendence degree is equal to

$$\max_{1 \leq i \leq 4} \text{trdeg}_{\mathbb{R}} \mathbb{R}(\gamma_i, \log(\gamma_1), \log(\gamma_2), \log(\gamma_3), \log(\gamma_4)).$$

But $\Gamma \subset X_4$ so there is an algebraic relation between the four $\log(\gamma_i)$, and so at most three of them can be algebraically independent. Hence

$$\text{trdeg}_{\mathbb{R}} \mathbb{R}(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \log(\gamma_1), \log(\gamma_2), \log(\gamma_3), \log(\gamma_4)) \leq 4.$$

So, by the Ax–Schanuel theorem, the γ_i are $\mathbb{Q}$ -multiplicatively dependent modulo $\mathbb{R}$. For brevity we now write $g_i = \log(\gamma_i)$. The rest of the proof splits into four cases depending on the multiplicative degree of the γ_i over $\mathbb{Q}$, modulo $\mathbb{R}$.

Case 0 $\text{muldeg}_{\mathbb{Q}}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)/\mathbb{R} = 0$.

This is the case only if each $\gamma_i \in \mathbb{R}$, which contradicts our assumption that Γ was a curve.

Case 1 $\text{muldeg}_{\mathbb{Q}}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)/\mathbb{R} = 1$.

In this case we can write $\gamma_i = c_i \gamma_1^{a_i}$ for $2 \leq i \leq 4$ where $c_i \in \mathbb{R}^+$ and $a_i \in \mathbb{Q}$. Then the defining equation for X_4 implies that

$$g_1(\log c_2 + a_2 g_1) = (\log c_3 + a_3 g_1)(\log c_4 + a_4 g_1).$$

Expanding this gives a quadratic equation for g_1 , namely

$$(a_2 - a_3 a_4) g_1^2 + (\log c_2 - a_4 \log c_3 - a_3 \log c_4) g_1 - \log c_3 \log c_4 = 0.$$

But then this implies that g_1 is constant, a contradiction only avoided if this equation just says $0 = 0$. In that case $a_2 = a_3 a_4$, $c_2 = c_3^{a_4} c_4^{a_3}$, and either $c_3 = 1$ or $c_4 = 1$. If $c_4 = 1$ then we have

$$\gamma_4 = \gamma_1^{a_4}, \quad \gamma_3 = c_3 \gamma_1^{a_3}, \quad \gamma_2 = c_3^{a_4} \gamma_1^{a_3 a_4}.$$

And so $\Gamma \subset \Gamma_{2, a_4, a_3, c_3}$. Similarly, if $c_3 = 1$ then $\Gamma \subset \Gamma_{1, a_3, a_4, c_4}$. In the special case that $a_2 = 0$ then either $a_3 = 0$ or $a_4 = 0$ and $\Gamma \subset \Gamma_{3, a_4, f}$ or $\Gamma \subset \Gamma_{4, a_3, f}$ respectively where f is the constant function $f(x) = c$.

Case 2 $\text{muldeg}_{\mathbb{Q}}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)/\mathbb{R} = 2$.

There are only really two different possibilities here: either $\mathbb{R}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is multiplicatively spanned by elements from the same side of the defining equation of X_4 , i.e. by γ_1 and γ_2 , or by γ_3 and γ_4 ; or else it is spanned by elements from different sides of the equation. First we suppose that γ_1 and γ_2 are $\mathbb{Q}$ -multiplicatively independent modulo $\mathbb{R}$.

Case 2A $\gamma_3 = c_3 \gamma_1^{a_1} \gamma_2^{a_2}$ and $\gamma_4 = c_4 \gamma_1^{b_1} \gamma_2^{b_2}$ with $c_3, c_4 \in \mathbb{R}^+$ and $a_1, a_2, b_1, b_2 \in \mathbb{Q}$.

In this situation we have

$$g_1 g_2 = (\log c_3 + a_1 g_1 + a_2 g_2)(\log c_4 + b_1 g_1 + b_2 g_2).$$

This means that $\log(\gamma_1)$ and $\log(\gamma_2)$ are algebraically dependent, and so by Ax–Schanuel they are $\mathbb{Q}$ -multiplicatively dependent modulo $\mathbb{R}$. This is a contradiction unless the above equation is actually $0 = 0$. This gives the system of equations

$$\begin{aligned} a_1 b_2 + a_2 b_1 &= 1 \\ a_1 b_1 &= a_2 b_2 = 0 \\ a_1 \log c_4 + b_1 \log c_3 &= 0 \\ a_2 \log c_4 + b_2 \log c_3 &= 0 \\ \log c_3 \log c_4 &= 0. \end{aligned}$$

The two possible solutions are

$$a_1 = b_2 = 0, \quad a_2 b_1 = 1, \quad c_3 = c_4 = 1,$$

and

$$a_2 = b_1 = 0, \quad a_1 b_2 = 1, \quad c_3 = c_4 = 1.$$

In the first case $\gamma_3 = \gamma_2^{a_2}$ and $\gamma_1 = \gamma_4^{a_2}$, and so $\Gamma \subset \Gamma_{6,1/a_2,f}$ for some algebraic function f . Similarly, in the second case $\Gamma \subset \Gamma_{3,1/a_1,f}$.

Case 2B $\gamma_2 = c_2 \gamma_1^{a_1} \gamma_3^{a_2}$, and $\gamma_4 = c_4 \gamma_1^{b_1} \gamma_3^{b_2}$ with $c_2, c_4 \in \mathbb{R}^+$ and $a_1, a_2, b_1, b_2 \in \mathbb{Q}$.

This time we have

$$g_1(\log c_2 + a_1 g_1 + a_2 g_3) = g_3(\log c_4 + b_1 g_1 + b_2 g_3).$$

Again, Ax–Schanuel then contradicts the multiplicative independence of γ_1 and γ_3 unless this expression reduces to $0 = 0$. In that case we have

$$\begin{aligned} a_1 &= b_2 = 0 \\ a_2 &= b_1 \\ \log c_2 &= \log c_4 = 0. \end{aligned}$$

This in turn means $\gamma_2 = \gamma_3^{a_2}$ and $\gamma_4 = \gamma_1^{a_2}$ and so $\Gamma \subset \Gamma_{4,1/a_2,f}$ for some algebraic f .

Assuming that other pairs of the γ_i are multiplicatively independent in this case leads to other instances of the Γ_i , in particular the “missing” Γ_5 , but since this working just involves permuting indices we don’t carry it out here.

Case 3 $\text{muldeg}_{\mathbb{Q}}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)/\mathbb{R} = 3$.

In this instance we would have, say, $\gamma_4 = c\gamma_1^{a_1}\gamma_2^{a_2}\gamma_3^{a_3}$. Then

$$g_1g_2 = g_3(\log c + a_1g_1 + a_2g_2 + a_3g_3).$$

Ax–Schanuel then contradicts the multiplicative independence of $\gamma_1, \gamma_2, \gamma_3$ unless this equation is trivial. But there is no choice of c, a_1, a_2, a_3 that forces this identity to be $0 = 0$ and so this case cannot occur. $\square$

Before we demonstrate the equivalence between Wilkie’s conjecture and the four exponentials conjecture it will be useful to reformulate membership of X_4^{trans} in terms of matrices.

Lemma 3 *Let $x, y, z, w \in \mathbb{R}_{\text{alg}}^+$ and let*

$$\Gamma = \begin{pmatrix} \log x & \log z \\ \log w & \log y \end{pmatrix}.$$

Then $(x, y, z, w) \in X_4^{\text{trans}}$ if and only if the following conditions hold:

- (1) *Any non trivial right null vector of Γ is not in $\mathbb{Q}^2$.*
- (2) *Any non trivial left null vector of Γ is not in $\mathbb{Q}^2$.*
- (3) *$\det(\Gamma) = 0$.*

Proof Condition (3) is simply the defining equation of X_4 . We just need to show that conditions (1) and (2) rule out any points in X_4^{alg} . One can observe from lemma 2 that if $(x, y, z, w) \in X_4^{\text{alg}}$ then either $\log x$ and $\log z$ are $\mathbb{Q}$ -linearly dependent or else $\log x$ and $\log w$ are $\mathbb{Q}$ -linearly dependent. So suppose $\log x$ and $\log z$ are $\mathbb{Q}$ -linearly independent and let $v_1, v_2 \in \mathbb{Q}$ not both be zero. We have

$$\Gamma \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} v_1 \log x + v_2 \log z \\ v_1 \log w + v_2 \log y \end{pmatrix}.$$

This result isn’t the zero vector since $\log x$ and $\log z$ are $\mathbb{Q}$ -linearly independent, and so $\mathbf{v}$ cannot be a null vector, giving condition (1). A similar calculation shows that if $\log x$ and $\log w$ are $\mathbb{Q}$ -linearly independent then Γ cannot have a rational non trivial left null vector, that is condition (2). $\square$

We can now relate the four exponentials conjecture to Wilkie’s conjecture.

Theorem 9 *Suppose that for every number field $F \subset \mathbb{R}$ we have*

$$N(X_4^{\text{trans}}, F, T) = o((\log T)^4).$$

Then conjecture 8 is true. Conversely, suppose that conjecture 8 is true, then for any number field $F \subset \mathbb{R}$, $X_4^{\text{trans}}(F) = \emptyset$.

Proof Suppose that the real four exponentials conjecture is true. So if $(x, y, z, w) \in X_4(F)$ for some number field $F \subset \mathbb{R}$ then either x and z are $\mathbb{Q}$ -multiplicatively dependent or x and w are $\mathbb{Q}$ -multiplicatively dependent. In the former case the equation for X_4 implies that if, say, $x = z^a$ then $w = y^a$, and so $(x, y, z, w) \in X_4^{\text{alg}}$, specifically in $\Gamma_{3,a,f}$ as defined in lemma 2, for some f . In the latter case we get $x = w^a$ and $z = y^a$, and so $(x, y, z, w) \in \Gamma_{4,a,f} \subset X_4^{\text{alg}}$. In particular, $X_4^{\text{trans}}(F) = \emptyset$.

Now suppose the real four exponentials conjecture has a counterexample, say $\log x \log y = \log z \log w$ with $x, y, z, w \in \mathbb{R}_{\text{alg}}^+$ and with $\log x, \log z$ linearly independent over $\mathbb{Q}$ and the same for $\log x$ and $\log w$. This means $(x, y, z, w) \in X_4^{\text{trans}}(F)$ for some number field F , which we may as well take to be $\mathbb{Q}(x, y, z, w)$. Now let a, b, c, d be integers, each bounded in absolute value by $c \log T$ for some constant $c > 0$ and with $ad - bc \neq 0$. We claim that for a suitable choice of c , each such four-tuple gives rise to a new point in $X_4^{\text{trans}}(F, T)$. There are clearly $\gg (\log T)^4$ such four-tuples of integers so if this is true we will have $N(X_4^{\text{trans}}, F, T) \gg (\log T)^4$, which is what we need to prove. To this end, for each four-tuple (a, b, c, d) we associate the point $(x^a w^b, z^c y^d, z^a y^b, x^c w^d) \in F^4$. As in the proof of lemma 6 if the constant c is suitably chosen in terms of the heights of x, y, z , and w then these new points will have height $\leq T$. To check these points are in X_4^{trans} we note that

$$\Lambda = \begin{pmatrix} \log(x^a w^b) & \log(z^a y^b) \\ \log(x^c w^d) & \log(z^c y^d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \log x & \log z \\ \log w & \log y \end{pmatrix}.$$

We can now appeal to lemma 3. The determinant of Λ is certainly zero since the rightmost matrix has determinant zero. Now suppose Λ has a rational left null vector, say (v_1, v_2) . But then by the above identity we have

$$\mathbf{0} = (v_1, v_2) \Lambda = (v_1 a + v_2 c, v_1 b + v_2 d) \begin{pmatrix} \log x & \log z \\ \log w & \log y \end{pmatrix}.$$

This contradicts $(x, y, z, w) \in X_4^{\text{trans}}$ unless $v_1 a + v_2 c = v_1 b + v_2 d = 0$. But then we have $ad = bc$, which was ruled out by construction. So Λ has no rational left null vectors. Similarly if Λ has a right null vector $\mathbf{w}$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \log x & \log z \\ \log w & \log y \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathbf{0}.$$

But since $(x, y, z, w) \in X_4^{\text{trans}}$ the rightmost product cannot be the null vector so that, necessarily, the determinant of the left most matrix is zero, which again it is not. So, by lemma 3, for each choice of a, b, c, d the point $(x^a w^b, z^c y^d, z^a y^b, x^c w^d)$ is in $X_4^{\text{trans}}(F, T)$. Now we just need to check these points are distinct. That follows readily from the fact that $(x, y, z, w) \in X_4^{\text{trans}}$. Indeed, if two such points coincided then, since $ad - bc \neq 0$, we would have that $x = w^r$ for some $r \in \mathbb{Q}$ and $z = y^r$ for the same r . But then $(x, y, z, w) \in X_4^{\text{alg}}$, specifically the set called Γ_4 in lemma 2. $\square$

5 The sharp exponentials conjectures

More refined versions of the exponentials conjectures first appeared in the 1980s when Michel Waldschmidt proved what would eventually be known as the sharp six exponentials theorem ([Waldschmidt(1988)]). One as-yet-unproven strengthening of this conjecture is the sharp four exponentials conjecture, the real version of which is as follows.

Conjecture 9 Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ have the property that e^{λ_i} is algebraic for each i and let $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{R}$ be algebraic. Suppose that λ_1 and λ_3 are $\mathbb{Q}$ -linearly independent. Moreover, suppose that λ_1 and λ_4 are $\mathbb{Q}$ -linearly independent. Then

$$(\lambda_1 + \beta_1)(\lambda_2 + \beta_2) \neq (\lambda_3 + \beta_3)(\lambda_4 + \beta_4).$$

The usual four exponentials conjecture clearly follows upon taking $\beta_i = 0$ for each i . There's an obvious strategy to extend our earlier work to try to deal with this conjecture: we introduce the set $X_{\#4} \subset \mathbb{R}^8$ consisting of all $(x, y, z, w, t_1, t_2, t_3, t_4)$ with $x, y, z, w > 0$ satisfying

$$(\log x + t_1)(\log y + t_2) = (\log z + t_3)(\log w + t_4).$$

The strategy to relate conjecture 9 to Wilkie's conjecture via the set $X_{\#4}$ would then, presumably, be the same as in the previous section: show that the algebraic part of $X_{\#4}$ contains precisely those points that the hypotheses of conjecture 9 rule out, then show that $X_{\#4}^{\text{trans}}$ cannot contain too many algebraic points without foiling the conjecture.

The only problem with using this strategy is that it does not work. The complication is the first part: we would like the algebraic part of $X_{\#4}$ to consist of points $(x, y, z, w, \mathbf{t})$ where either x and z are $\mathbb{Q}$ -multiplicatively dependent or x and w are $\mathbb{Q}$ -multiplicatively dependent. But this is not the case. For any fixed values of $x, y, z, w > 0$ there is an algebraic curve that lies on the four-fold cut out by this fixed quadruple. In particular, one can pick values of (x, y, z, w) where x, z , and w are $\mathbb{Q}$ -multiplicatively independent, but find $\mathbf{t}$ such that $(x, y, z, w, \mathbf{t}) \in X_{\#4}^{\text{alg}}$. In fact by solving the resulting equation in the t_i variables one sees that $X_{\#4}(\mathbb{R}_{\text{alg}}) \subset X_{\#4}^{\text{alg}}$, so we can definitely count the algebraic points in $X_{\#4}^{\text{trans}}$ and it will definitely tell us nothing about the sharp four exponentials conjecture. The same issue arises when trying to tackle the sharp three and five exponentials conjectures using this method.

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