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Efficient modelling of beam-like structures with general non-prismatic, curved geometry

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Abstract

The analysis of three-dimensional (3D) stress states can be complex and computationally expensive, especially when large deflections cause a nonlinear structural response. Slender structures are conventionally modelled as one-dimensional beams but even these rather simpler analyses can become complicated, *e.g.* for variable cross-sections and planforms (*i.e.* non-prismatic curved beams). In this paper, we present an alternative procedure based on the recently developed Unified Formulation in which the kinematic description of a beam builds upon two shape functions, one for the beam's axis, the other for its cross-section. This approach predicts 3D displacement and stress fields accurately and is computationally efficient in comparison with 3D finite elements. However, current modelling capabilities are limited to the use of prismatic elements. As a means for further applicability, we propose a method to create beam elements with variable planform and variable cross-section, *i.e.* of general shape. This method employs an additional set of shape functions which describes the geometry of the structure exactly. These functions are different from those used for describing the kinematics and provide local curvilinear basis vectors upon which 3D Jacobian transformation matrices are produced to define non-prismatic elements. The model proposed is benchmarked against 3D finite element analyses, as well as analytical and experimental results available in the literature. Significant computational efficiency gains over 3D finite elements are observed for similar levels of accuracy, for both linear and geometrically nonlinear analyses.

Keywords: Unified Formulation; Finite Element; Non-prismatic; Tapered Sandwich; Curved Structures.

1. Introduction

The accuracy and computational efficiency of static linear and nonlinear analyses of variable cross-section, variable planform, *i.e.* non-prismatic and curved beams is of great importance, particularly in mechanical, civil and aerospace engineering. In order to meet ever more stringent weight and/or efficiency requirements, many applications, such as wings, engine or wind turbine blades, demand the use of advanced composite materials and highly tailored geometries for functional and/or aerodynamic reasons. Such structures are also often sized to be slender, thin-walled and include variable-thickness semi-monocoque shells. Ultimately, these features compound to make stress/strength analyses both cumbersome and computationally expensive, which is a problem because accurate analyses are crucial for sizing and optimisation at all design stages.

At early design iterations, one-dimensional (1D) and two-dimensional (2D) finite element models are commonly employed for their efficiency and relative accuracy. However, their capabilities are limited to describing general, stiffness-driven, behaviour. For example, thin-shell formulations can predict in-plane bending and stretching response fairly well, but recovering local features, such as through-thickness transverse stresses or displacement field gradients close to singularities, may not be possible. These details are

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then addressed only at later stages of the design process by treating the structure as a three-dimensional (3D) continuum, *i.e.* utilising 3D equilibrium equations and associated point-wise boundary conditions to compute stresses and deflections. However, there is increasing evidence that localised 3D stress features extend their effects far field in sizing optimisation problems, especially when layered, anisotropic materials and the relevant manufacturing rules are involved [1]. This observation brings the adequacy of simplified, early-iteration models into question. Solving a 3D boundary value problem or employing 3D finite element (FE) models at early design iterations can be time consuming and computationally expensive. Therefore, there is a need to develop an efficient mathematical method that is able of capturing salient displacement and stress features in structures of generic, slender, thin-walled form for incorporation into early-stage design iterations.

Focusing on beam-like structures, it is known that even modelling relatively simple elements is a non-trivial task. For instance, consider a statically-loaded non-prismatic beam, behaving under the assumption of plane stress. Then, the most stressed cross-section generally does not coincide with the cross-section subjected to the maximal internal force [2]. Moreover, continuous variation of beam shape affects several aspects of its behaviour, such as the cross-sectional stress distribution [3] and constitutive relation [4, 5]. Therefore, several attempts have been made by researchers to develop tools for investigating deflection and stresses in non-prismatic beams. Hodges *et al.* [6] proposed a variational asymptotic method (VAM) for recovering stress, strain, and displacement fields for linearly tapered isotropic beams. Trinh and Gan [7] presented an energy based FE method and derived new shape functions for a linearly tapered Timoshenko beam. Balduzzi *et al.* [8] proposed a Timoshenko-like model for planar multilayer non-prismatic beams and solved the problem of the recovery of stress distribution within the cross-section of bi-symmetric tapered steel beams [9].

Developing models for accurate estimation of stress distributions in laminated structures of variable thickness/geometry is essential also for predicting failure initiation and propagation. Research on sandwich structures with variable thickness has been conducted since the eighties [10–14]. Hoa *et al.* [15] investigated interlaminar stresses in tapered laminates using 3D FE. Jeon and Hong [16] investigated the bending behaviour of tapered sandwich plates consisting of an orthotropic core of unidirectional linear thickness variation and two uniform anisotropic composite skins. The minimum total potential energy method was used to derive the governing equations and approximate solutions were obtained with the Ritz method. Recently, Ai and Weaver [17] developed a nonlinear layer-wise sandwich beam model to capture the static effects of a combination of geometric taper and variable core stiffness. In the model, the face sheets were assumed to behave as Euler beams and the core was modelled based on the first-order shear deformation theory. Though useful in their own merit, all of these models have the disadvantage of being fairly *ad hoc*.

Over the last few decades, significant efforts have been made also in the development of numerical tools for modelling curved beams. One of the earliest contributions was by Belytschko and Glaum [18], who extended their previous work on initially curved beams [19] and developed a nonlinear, higher-order, corotational FE formulation for analysing curved beams and arches. Surana and Sorel [20] developed a geometrically nonlinear framework for three-dimensional curved beam elements undergoing large rotations. A slightly different approach, using Reissner’s beam theory, was proposed by Ibrahimbegović and Frey [21, 22], where a hierarchical three-dimensional curved beam element was used to mitigate the shear and membrane locking caused by lower-order elements. Shahba *et al.* [23, 24] combined the stiffness- and flexibility-based methods and proposed a finite element model for analysing curved and non-prismatic beams. Within the element, the interpolating functions, derived in terms of basic displacement functions, are exact and are consistent with the mechanics of the problem. Petrov and Gradin [25] developed a geometrically nonlinear theory for curved and twisted beams. Their approach is based on the use of exact solutions for three-dimensional prismatic solids to formulate the kinematical hypothesis. Yu *et al.* [26] proposed a model for naturally curved anisotropic beams with thin-walled cross-sections. In their model, eigenfunctions were used as an expansion series to approximate the displacement field, allowing effects such as torsion and warping to be described accurately. Recently, the differential quadrature method [27–29] has received interest for the analysis of curved beams. An extensive review on the development of geometric nonlinear theories for the analysis of curved beams is available in a recently published paper by Ghuku and Saha [30].

In spite of the many approaches available for analysing tapered and curved structures, high-fidelity FE

models are often employed in practice, as they offer increased versatility in modelling complex shapes. For instance, modelling a wind turbine blade, with geometric features such as curvature, sweep, taper, twist and a variable section, is straightforward with the FE method. Still, analysts have to grapple with the trade-offs between the generality and versatility of the method and its computational cost, often favouring 1D/2D approximations for fast design iterations to accurate, yet expensive, 3D solutions. As an alternative to classic 3D FE analysis, the Unified Formulation (UF) developed by Carrera and coworkers [31, 32] is gaining credence.

The Unified Formulation relies on a displacement-based version of the finite element method applied to 3D field equations and provides one-dimensional beam models and two-dimensional plate/shell models. The displacement field is expressed classically along the reference axis (beam case) or surface (plate and shell cases), and by employing various hierarchical expansion functions over the cross-section (beam case) or through the thickness (plate and shell cases). As such, the method is able to recover 3D displacement and stress/strain fields in a computationally efficient manner. In recent years, several research papers have been published on the Unified Formulation framework to study linear and nonlinear deflections [33–36], 3D stress fields [37–40], failure and damage mechanics [41, 42], free vibrations [43, 44], buckling, and post-buckling behaviour [45, 46], of both metallic and composite structures. However, the existing modelling capabilities are largely limited to prismatic structures and, therefore, are not immediately suited for analysing complex, real-life, geometrical configurations.

When dealing with generic structural forms, a correct description of the geometry is of fundamental importance. Existing implementations of the Unified Formulation follow the common isoparametric FE methodology [47], where the same set of shape functions is used to approximate geometry and kinematics. However, unlike in conventional FE methods, in the Unified Formulation, the reference (beam axis or shell surface) and transverse directions are treated differently and decoupled adopting separate shape functions. This methodology leads to split volume integrals, which, in turn, limit current modelling capabilities to the use of prismatic elements. Therefore, in order to overcome this limitation, we propose to use an additional set of shape functions for describing the structural geometry that allow us to evaluate elemental stiffness properties by means of full volume integrals. With the proposed approach, the modelling capabilities of the classical Unified Formulation are extended so that elements of arbitrary geometry can be readily derived.

It is to be noted that modelling tapered beam structures with the Unified Formulation is currently possible if the beam width is treated as the longitudinal axis and the tapered plane as its cross-section [48, 49]. However, this approach is clearly a workaround and is associated with certain limitations, such as high aspect ratio of cross-section elements and compact placement of nodes along the width. Similarly, curved beam structures have been analysed employing the Unified Formulation using a Frenet-Serret description [50]. This model too has some limitations: (i) the cross-sectional shape and size cannot vary along the beam direction; (ii) it requires an exact description of the curved line defining the beam axis, and therefore, is not applicable to curved beams whose axes cannot be described by parametric equations.

To show the enhanced capabilities of the Unified Formulation in modelling real-life, geometrical configurations, we employ a finite element based on Serendipity Lagrange Expansions (SLE) [51]. SLE finite elements allow refinement by combined cross-sectional discretisation and hierarchical expansion, such that both local and global responses are obtained accurately in a computationally efficient manner. As mentioned previously, an extra set of shape functions are employed for the description of the element geometry. The proposed model is benchmarked against numerical results by means of static analyses of tapered I-section and sandwich beam structures. Furthermore, the recently developed geometrically nonlinear UF-SLE model [35] is employed for analysing corrugated structures under tensile loading, and the results obtained are validated with test results available in the literature. This particular example is chosen to assess the proposed methodology in predicting the response of curved beam-like structures.

The remainder of the paper is structured as follows. Sections 2.1 and 2.2 provide preliminary and introductory information on the Unified Formulation framework based on Serendipity Lagrange expansions, including Green-Lagrange nonlinear geometrical relations and elastic constitutive equations. Section 3 describes the 3D mapping technique required for modelling structures of general geometry. In Section 4, the efficacy of the proposed UF-SLE model is proven by means of linear and nonlinear static analyses of tapered and corrugated structures. Results are validated against numerical and experimental data. Finally,

Figure 1: Unified Formulation framework - 3D structure discretisation.

conclusions are drawn in Section 5.

2. Mathematical Formulation

2.1. Preliminaries

Consider a 3D beam-like structure with arbitrary geometry, as shown in Figure 1. A system of local curvilinear coordinates (r, s, t) is selected to describe the structure, such that the beam's axis is defined to be along the s -direction and its cross-section to lie in the rt -plane. The orientation of material points within the structure is determined by the direction cosines of the triad (r, s, t) with respect to the global Cartesian coordinate system (x, y, z) . Having established a coordinate system at the beginning of the formulation, the following maths has been cast in the global coordinate system. Since the derivation regarding the transformation is a standard procedure, it has not been included in the present article but can be found in [52].

For points in the structure's interior, the three-dimensional displacement field, in the global Cartesian coordinate system, is given by

$$\mathbf{U}(x, y, z) = \{u \quad v \quad w\}^\top, \quad (1)$$

and the displacement gradient vector Φ can be written as

$$\Phi = \{u_{,x} \quad u_{,y} \quad u_{,z} \quad v_{,x} \quad v_{,y} \quad v_{,z} \quad w_{,x} \quad w_{,y} \quad w_{,z}\}^\top, \quad (2)$$

where, here and henceforth, a subscript preceded by a comma denotes differentiation.

To account for large deformations, the Green-Lagrange strain vector $\mathbf{E}$ can be considered, which, in its matrix form, can be written as

$$\mathbf{E} = \left[\mathbf{H} + \frac{1}{2} \mathbf{A} \right] \Phi. \quad (3)$$

where

$$\mathbf{E} = \{E_{xx} \quad E_{yy} \quad E_{zz} \quad E_{yz} \quad E_{xz} \quad E_{xy}\}^\top, \\ \mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\mathbf{A} = \begin{bmatrix} u_{,x} & 0 & 0 & v_{,x} & 0 & 0 & w_{,x} & 0 & 0 \\ 0 & u_{,y} & 0 & 0 & v_{,y} & 0 & 0 & w_{,y} & 0 \\ 0 & 0 & u_{,z} & 0 & 0 & v_{,z} & 0 & 0 & w_{,z} \\ 0 & u_{,z} & u_{,y} & 0 & v_{,z} & v_{,y} & 0 & w_{,z} & w_{,y} \\ u_{,z} & 0 & u_{,x} & v_{,z} & 0 & v_{,x} & w_{,z} & 0 & w_{,x} \\ u_{,y} & u_{,x} & 0 & v_{,y} & v_{,x} & 0 & w_{,y} & w_{,x} & 0 \end{bmatrix}.$$

Virtual variations of equation (3) [53] are given by

$$\delta \mathbf{E} = \delta \left\{ \left[\mathbf{H} + \frac{1}{2} \mathbf{A} \right] \Phi \right\} = [\mathbf{H} + \mathbf{A}] \delta \Phi. \quad (4)$$

In the present work, the material is assumed to undergo deformation within the linear elastic range, and therefore, Hooke's law provides the constitutive relation

$$\mathbf{S} = \bar{\mathbf{C}} \mathbf{E}, \quad (5)$$

where $\mathbf{S} = \{S_{xx} \ S_{yy} \ S_{zz} \ S_{yz} \ S_{xz} \ S_{xy}\}^\top$ is the second Piola-Kirchhoff stress tensor and $\bar{\mathbf{C}}$ is the transformed material stiffness matrix expressed in the global Cartesian coordinate system,

$$\bar{\mathbf{C}} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{14} & \bar{C}_{15} & \bar{C}_{16} \\ & \bar{C}_{22} & \bar{C}_{23} & \bar{C}_{24} & \bar{C}_{25} & \bar{C}_{26} \\ & & \bar{C}_{33} & \bar{C}_{34} & \bar{C}_{35} & \bar{C}_{36} \\ & & & \bar{C}_{44} & \bar{C}_{45} & \bar{C}_{46} \\ & \text{Symmetric} & & & \bar{C}_{55} & \bar{C}_{56} \\ & & & & & \bar{C}_{66} \end{bmatrix}. \quad (6)$$

The coefficients $\bar{C}_{ij}$ relate to the elastic coefficients in material coordinates, C_{ij} , via the transformation matrix $\mathbf{Q}$, whose elements are obtained from the direction cosines of the material coordinate system projected onto the global (x, y, z) coordinate directions. Specifically,

$$\bar{\mathbf{C}} = \mathbf{Q} \mathbf{C} \mathbf{Q}^\top. \quad (7)$$

The coefficients C_{ij} and the matrix $\mathbf{Q}$ are not included here for sake of brevity, but can be found in Section 5.4 of [54].

The Cauchy stress tensor $\boldsymbol{\sigma}$ is derived from the Second-Piola stress tensor $\mathbf{S}$ using the deformation gradient $\mathbf{F}$. Specifically,

$$\boldsymbol{\sigma} = \frac{1}{\det \mathbf{F}} \mathbf{F} \mathbf{S} \mathbf{F}^\top, \quad (8)$$

where

$$\mathbf{F} = \begin{bmatrix} 1 + u_{,x} & u_{,y} & u_{,z} \\ v_{,x} & 1 + v_{,y} & v_{,z} \\ w_{,x} & w_{,y} & 1 + w_{,z} \end{bmatrix}. \quad (9)$$

The Cauchy stress tensor is referenced in the global coordinate system, which does not have a clear physical meaning in case of large deformations. Therefore, corotational Cauchy stresses are computed on the deformed configuration using the rotation tensor, $\mathbf{R}$:

$$\hat{\boldsymbol{\sigma}} = \mathbf{R}^\top \boldsymbol{\sigma} \mathbf{R}. \quad (10)$$

The rotation tensor $\mathbf{R}$ is obtained by polar decomposition of the deformation gradient as per [55].

Figure 2: Unified Formulation framework - 3D structure discretisation.

2.2. Serendipity-Lagrange-based Finite Element Model

Our model employs the Unified Formulation framework [31, 32], where a 3D beam-like structure is discretised with a finite number of transverse planes distributed along the principal axis. For simplicity, this axis can be thought of as the beam axis and the transverse planes as the beam cross-sections as shown in Figure 2. In order to maintain a consistent coordinate system throughout, the position of material points in the structure and all of the functions describing elasticity are expressed in global Cartesian coordinates through a mapping of the type $(x, y, z) = \phi(r, s, t)$. This transformation between global and local coordinates can be established either analytically or numerically.

The Unified Formulation relies on a displacement-based version of the finite element method. In the current setting, the longitudinal axis of the structure is discretised with N_e -noded, 1D finite elements, so that the displacement field can be approximated element-wise by means of local shape functions, $N_i(y)$, and generalised nodal displacements, $\mathbf{U}_i(x, z)$, such that

$$\mathbf{U}(x, y, z) = \sum_{i=1}^{N_e} N_i(y) \mathbf{U}_i(x, z). \quad (11)$$

The transverse, or cross-sectional deformations, are approximated using hierarchical Serendipity Lagrange expansion (SLE) functions, $F_\tau(x, z)$ [51]. Adopting this expansion model, cross-sections are further discretised using four-noded Lagrange sub-domains (SLE nodes). The displacement field within each sub-domain can be enriched by increasing the order of the local Serendipity Lagrange expansion. The cross-sectional displacement field at the i^{th} beam node is expressed as

$$\mathbf{U}_i(x, z) = \sum_{\tau=1}^m F_\tau(x, z) \mathbf{U}_{i\tau}, \quad (12)$$

where m is the number of terms depending on the order of SL expansion, and $\mathbf{U}_{i\tau}$ are generalized three-dimensional displacement vectors. The reader is referred to [37, 51] for a more detailed implementation and treatment of SLE models.

By introducing the cross-sectional approximation of Eq. (12) into the FE discretisation along the beam axis of Eq. (11), the displacement field reads

$$\mathbf{U}(x, y, z) = \sum_{i=1}^{N_e} \sum_{\tau=1}^m N_i(y) F_\tau(x, z) \mathbf{U}_{i\tau}. \quad (13)$$

For the sake of clarity, it is important to emphasise that, using Eq. (13), the cross-sectional mesh (indicated by discretisation variable τ) captures the warping of the cross-section with one set of 2D shape functions ($F_\tau(x, z)$), while the axial behaviour is modelled by a separate 1D mesh (indicated by discretisation variable i) with an independent set of 1D shape functions ($N_i(y)$). This approach differentiates the method from classic 3D FE models, where 3D shape functions are used over volumetric brick or tetrahedral elements that offer no separation of cross-sectional and axial deformations.

By substituting Eq. (13) in Eq. (2), the displacement gradient vector can be written as

$$\Phi = \mathbf{G}_{i\tau} \mathbf{U}_{i\tau} \quad (14)$$

where

$$\mathbf{G}_{i\tau} = \begin{bmatrix} N_i F_{\tau,x} & 0 & 0 \\ N_{i,y} F_\tau & 0 & 0 \\ N_i F_{\tau,z} & 0 & 0 \\ 0 & N_i F_{\tau,x} & 0 \\ 0 & N_{i,y} F_\tau & 0 \\ 0 & N_i F_{\tau,z} & 0 \\ 0 & 0 & N_i F_{\tau,x} \\ 0 & 0 & N_{i,y} F_\tau \\ 0 & 0 & N_i F_{\tau,z} \end{bmatrix}, \quad \mathbf{U}_{i\tau} = \begin{Bmatrix} u_{i\tau} \\ v_{i\tau} \\ w_{i\tau} \end{Bmatrix}.$$

In the previous expression and throughout the remainder of the paper, the Einstein summation convention is indicated with repeated indices.

Elastic equilibrium is enforced via the Principle of Virtual Displacements, which in a quasi-static setting, states that

$$\delta W_{\text{int}} = \delta W_{\text{ext}}, \quad (15)$$

where W_{int} and W_{ext} denote the internal and external work, respectively. By definition, the internal work is the work done by the internal stresses over the corresponding internal strains and is equivalent to the elastic strain energy. Noting that $W_{\text{int}} = \sum_e W_{\text{int}}^e$, where W_{int}^e represents the strain energy per element and letting V^e be the volume of the generic element in an undeformed state,

$$\delta W_{\text{int}}^e = \int_{V^e} \delta \mathbf{E}^\top \mathbf{S} dV. \quad (16)$$

In the notation of the Unified Formulation [51], the internal work can be re-written as

$$\delta W_{\text{int}}^e = \delta \mathbf{U}_{js}^\top \mathbf{K}_s^{\tau s i j} \mathbf{U}_{i\tau}, \quad (17)$$

where the term $\mathbf{K}_s^{\tau s i j}$ is referred to as the *Fundamental Nucleus* of the secant stiffness matrix. Its explicit form for an orthotropic lamina can be found in [56]. The main disadvantage of using the secant stiffness matrix is its low convergence rate when employed with the full Newton-Raphson method for solving the nonlinear governing equations. Moreover, the secant matrix is not uniquely defined and is generally non-symmetric. Therefore, we use the tangent stiffness matrix $\mathbf{K}_T$ obtained from the linearisation of the virtual variation of the strain energy,

$$d(\delta W_{\text{int}}^e) = \delta \mathbf{U}_{js}^\top \mathbf{K}_T^{\tau s i j} d\mathbf{U}_{i\tau}. \quad (18)$$

Fundamental nuclei are assembled into a global stiffness matrix following the standard finite element procedure. For the sake of brevity, the derivation and the explicit form of the fundamental nucleus of the tangent stiffness matrix is not reported here, but can be found in the authors' recent work [35].

3. 3D Mapping via Jacobian Transformation

As discussed in Section 2.2, the Unified Formulation employs two shape functions, $F_\tau(x, z)$ and $N_i(y)$, to describe the kinematics of the structure as given by Eq. (13). The cross-section expansion functions, F_τ ,

Figure 3: Geometric and kinematic description used for modelling non-prismatic beam-like structures.

are defined on a master surface element in $\Xi^{2D} = \{(\alpha, \beta) \in \mathbb{R}^2 : -1 \leq \alpha \leq 1, -1 \leq \beta \leq 1\}$, while the shape functions along the beam axis, N_i , are defined on a master line element in $\Xi^{1D} = \{(\eta) \in \mathbb{R} : -1 \leq \eta \leq 1\}$. These functions are then mapped onto (x, y, z) , *i.e.* the position in global coordinates, using the Jacobian of the transformation as defined by

$$\begin{Bmatrix} N_i F_{\tau, x} \\ N_{i, y} F_{\tau} \\ N_i F_{\tau, z} \end{Bmatrix} = \underbrace{\begin{bmatrix} x_{, \alpha} & y_{, \alpha} & z_{, \alpha} \\ x_{, \eta} & y_{, \eta} & z_{, \eta} \\ x_{, \beta} & y_{, \beta} & z_{, \beta} \end{bmatrix}}_{\mathbf{J}^{3D}}^{-1} \begin{Bmatrix} N_i F_{\tau, \alpha} \\ N_{i, \eta} F_{\tau} \\ N_i F_{\tau, \beta} \end{Bmatrix}, \quad (19)$$

the 3×3 matrix $\mathbf{J}^{3D}$ being the Jacobian accounting for the geometry description in (x, y, z) through its rows $(x_{, \alpha}, y_{, \alpha}, z_{, \alpha})$, $(x_{, \eta}, y_{, \eta}, z_{, \eta})$, $(x_{, \beta}, y_{, \beta}, z_{, \beta})$, which represent local curvilinear basis vectors, *i.e.* tangent, axial and normal directions, at any point in the structure.

The components of the Jacobian matrix are calculated using derivatives of shape functions with respect to the local coordinates α, η, β and global coordinates of element nodes. Following the classical Unified Formulation, if shape functions $F_{\tau}(\alpha, \beta)$ and $N_i(\eta)$ are used to describe the geometry, then the components of $\mathbf{J}^{3D}$ are evaluated as

$$\begin{aligned} x_{, \alpha} &= F_{\tau, \alpha} x_{\tau}, & y_{, \alpha} &= N_{i, \alpha} y_i, & z_{, \alpha} &= F_{\tau, \alpha} z_{\tau}, \\ x_{, \eta} &= F_{\tau, \eta} x_{\tau}, & y_{, \eta} &= N_{i, \eta} y_i, & z_{, \eta} &= F_{\tau, \eta} z_{\tau}, \\ x_{, \beta} &= F_{\tau, \beta} x_{\tau}, & y_{, \beta} &= N_{i, \beta} y_i, & z_{, \beta} &= F_{\tau, \beta} z_{\tau}. \end{aligned} \quad (20)$$

Clearly, the components $y_{, \alpha}, y_{, \beta}, x_{, \eta}$ and $z_{, \eta}$ of the Jacobian matrix become zero. As a result, only prismatic structures can be modelled. This procedure of evaluating the Jacobian matrix is followed in all the research published using the Unified Formulation [33–46], where 2D and 1D integrals—over the cross-section and along the beam—are performed separately by employing 2D and 1D Jacobians given by

$$\mathbf{J}^{2D} = \begin{bmatrix} x_{, \alpha} & z_{, \alpha} \\ x_{, \beta} & z_{, \beta} \end{bmatrix} \quad \text{and} \quad \mathbf{J}^{1D} = [y_{, \eta}]. \quad (21)$$

To overcome the above limitation and to model general beam-like structures (*e.g.* non-prismatic, curved beams), we propose to use an additional set of functions to represent the element geometry. These could be CAD basis functions, such as B-splines or Non-Uniform Rational B-Splines (NURBS), that guarantee an exact geometric representation or could be higher-order Lagrange functions. On the other hand, the Unified Formulation functions F_τ and N_i are used, in the usual manner, for approximating the displacement field. To demonstrate the validity of the proposed technique, in the present work, we discretise structural geometry using 27-noded brick elements, with 3D Lagrange shape functions, $\mathcal{N}^{3D}(x, y, z)$, to evaluate local curvilinear basis vectors (or Jacobian matrix components) at each point, as shown in Figure 3. Following this procedure, the components of $\mathbf{J}^{3D}$ are evaluated as

$$\begin{aligned} x_{,\alpha} &= \mathcal{N}_{,\alpha}^{3D} x_\tau, & y_{,\alpha} &= \mathcal{N}_{,\alpha}^{3D} y_i, & z_{,\alpha} &= \mathcal{N}_{,\alpha}^{3D} z_\tau, \\ x_{,\eta} &= \mathcal{N}_{,\eta}^{3D} x_\tau, & y_{,\eta} &= \mathcal{N}_{,\eta}^{3D} y_i, & z_{,\eta} &= \mathcal{N}_{,\eta}^{3D} z_\tau, \\ x_{,\beta} &= \mathcal{N}_{,\beta}^{3D} x_\tau, & y_{,\beta} &= \mathcal{N}_{,\beta}^{3D} y_i, & z_{,\beta} &= \mathcal{N}_{,\beta}^{3D} z_\tau. \end{aligned} \quad (22)$$

Furthermore, it is important to note that, the proposed methodology requires integration along the beam axis and over the cross-section to be performed simultaneously. For instance, the first component of the fundamental nucleus of the tangent stiffness matrix (for a geometrically linear model) is given as follows [35]:

$$\begin{aligned} \mathbf{K}_T^{tsij}(1, 1) &= \int_V \bar{C}_{11} F_{\tau,x} F_{s,x} N_i N_j dV + \int_V \bar{C}_{16} F_{\tau,x} F_s N_i N_{j,y} dV + \int_V \bar{C}_{15} F_{\tau,x} F_{s,z} N_i N_j dV \\ &+ \int_V \bar{C}_{16} F_\tau F_{s,x} N_{i,y} N_j dV + \int_V \bar{C}_{66} F_\tau F_s N_{i,y} N_{j,y} dV + \int_V \bar{C}_{56} F_\tau F_{s,z} N_{i,y} N_j dV \\ &+ \int_V \bar{C}_{15} F_{\tau,z} F_{s,x} N_i N_j dV + \int_V \bar{C}_{56} F_{\tau,z} F_s N_i N_{j,y} dV + \int_V \bar{C}_{55} F_{\tau,z} F_{s,z} N_i N_j dV. \end{aligned} \quad (23)$$

The integrals are evaluated numerically by employing Gaussian Quadrature. The first term of Eq. (23) can be evaluated as

$$\begin{aligned} \int_V \bar{C}_{11} F_{\tau,x} F_{s,x} N_i N_j dV &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \bar{C}_{11} (N_i F_{\tau,x}) (N_j F_{s,x}) |\mathbf{J}^{3D}| d\alpha d\eta d\beta \\ &= \sum_{l=1}^{l_{gp}} \sum_{k=1}^{k_{gp}} \sum_{m=1}^{m_{gp}} \bar{C}_{11}(\alpha_l, \eta_k, \beta_m) (N_i(\eta_k) F_{\tau,x}(\alpha_l, \beta_m)) (N_j(\eta_k) F_{s,x}(\alpha_l, \beta_m)) |\mathbf{J}^{3D}| w_l w_k w_m, \end{aligned} \quad (24)$$

where: w_l , w_k and w_m are the weights related to the Gauss points α_l , η_k and β_m ; l_{gp} , k_{gp} and m_{gp} are the number of Gauss points used; $\bar{C}_{11}(\alpha_l, \eta_k, \beta_m)$ is the material constant evaluated at a specific Gauss point; $|\mathbf{J}^{3D}|$ is the determinant of the Jacobian matrix.

In contrast, the classical Unified Formulation approach splits the beam and the cross-sectional integration as follows:

$$\begin{aligned} \int_V \bar{C}_{11} F_{\tau,x} F_{s,x} N_i N_j dV &= \bar{C}_{11} \int_{-1}^1 \int_{-1}^1 F_{\tau,x} F_{s,x} |\mathbf{J}^{2D}| d\alpha d\beta \int_{-1}^1 N_i N_j |\mathbf{J}^{1D}| d\eta \\ &= \bar{C}_{11} \sum_{l=1}^{l_{gp}} \sum_{m=1}^{m_{gp}} F_{\tau,x}(\alpha_l, \beta_m) F_{s,x}(\alpha_l, \beta_m) |\mathbf{J}^{2D}| w_l w_m \sum_{k=1}^{k_{gp}} N_i(\eta_k) N_j(\eta_k) |\mathbf{J}^{1D}| w_k. \end{aligned} \quad (25)$$

Clearly, Eq. (24) is computationally more expensive to evaluate than Eq. (25), as the number of loops required increases from $(l_{gp} \times m_{gp} + k_{gp})$ to $(l_{gp} \times k_{gp} \times m_{gp})$. Nevertheless, by performing the beam and the cross-sectional integration simultaneously and also by considering the 3D Jacobian matrix, enhances the Unified Formulation modelling capabilities.

Table 1: Dimensions of the tapered I-beam (in mm).

Length	$L = 10000$	Flange width	$b_f = 250$	Flange height	$h_f = 16$
Web width	$b_w = 6$	Web height	$h_w(0) = 900$	Web height	$h_w(L) = 100$

Figure 4: Tapered I-section beam: dimensions and load definition.

4. Numerical Results

In this section, the proposed UF-SLE model is assessed by means of static analyses of various non-prismatic and curved structures. The first example concerns a tapered I-beam. Then, a tapered sandwich beam-like structure is considered and the effect of increasing taper angle on the 3D stress distribution is studied. Finally, three corrugated structures of different depth are considered. The relative nonlinear force-displacement and stress responses are compared with numerical and experimental results.

4.1. Tapered I-beam

The tapered I-beam analysed here is shown in Figure 4. Its dimensions are shown in Table 1. The Young's modulus, E , and Poisson's ratio, ν , of the constituent material are 210 GPa and 0.3, respectively. The forces and moments acting on the beam ends are

$$\begin{aligned} V(0) &= 100 \text{ kN}, & M(0) &= 700 \text{ kNm}. \\ V(L) &= 100 \text{ kN}, & M(L) &= 300 \text{ kNm}. \end{aligned}$$

The example is adapted from [9], which employs a planar non-prismatic beam model [8] with enhanced stress recovery, based on a rigorous generalisation of Jourawsky's theory. Solving this problem with the UF-SLE model requires two input meshes as shown in Figure 5. The first mesh consists of 27-noded brick elements used to discretise the structure's geometry. Using this data and employing 3D Lagrange shape functions, local basis vectors representing the orientation of each structural component (web and flanges) can be computed at the grid nodes. The number of elements used to discretise the structure should be carefully chosen to ensure that the geometry is represented to the desired level of accuracy. Seven elements are sufficient in this case as shown in Figure 5a. These geometric elements contribute marginally in terms of computational cost as they do not contribute to the total structural degrees of freedom (DOFs). The second mesh is structural and contains the nodes of the Unified Formulation finite element discretisation, *i.e.* the nodes defining cross-sectional computational domains [51] and 1D beam elements. This I-section beam

Figure 5: Geometrical and structural meshes considered in the present study for tapered I-section beam.

is discretised using 20 B4 (four-noded cubic Lagrange) elements along its length and 25 SL4 (fourth-order Serendipity Lagrange) elements in the cross-section (Figure 5b). This discretisation and cross-sectional expansion order results in a total of 42,273 DOFs, which guarantees convergence. A 3D FE analysis, performed with commercial finite element software *ANSYS*, is used as a reference for verification of the proposed UF-SLE model. Structured meshes of $400 \times 3 \times 20$ and $400 \times 50 \times 4$ elements, as given in [9], are adopted for the flanges and web, respectively, resulting in a global count of 64,000 SOLID186 (3D 20-noded) elements with 1,038,687 DOFs. In both the UF-SLE and 3D FE models, the applied shear force at the beam ends are modelled as vertical forces per unit area uniformly distributed over the web section. Similarly, the bending moments are enforced using uniformly distributed surface tractions/compressions applied over the flanges. To avoid rigid body motion, we impede u_x and u_z at $(0, 0, \pm(h_f + h_w/2))$ and u_y at $(0, 0, 0)$.

Figures 6 to 8 show the through-thickness distribution of the axial normal, σ_{yy} , transverse shear, τ_{yz} , and von Mises stress, σ_{eq} , measured in global coordinates, at several locations along the beam length. From simple bending theory, the bending moment is predicted to be negative on the left hand part of the structure, vanish at $y = 7000$ mm and to then become positive progressing towards the right hand side. This effect can be observed by looking at the through-thickness distribution of σ_{yy} in panels (a) to (d) of Figure 6, where Figure 6c represents the section located at the theoretical contraflexure point. Moreover, the through-thickness distributions of τ_{yz} and σ_{eq} , which are influenced by the beam taper, is well captured by the UF-SLE model. It is to be noted from Figure 7a that the distribution of τ_{yz} appears to be constant along the web height at $y = 3000$ mm. However, by looking at the numerical values, it is observed that the variation is rather parabolic but is inwardly directed, *i.e.* the maximum shear stress is located at the web-flange junction and not at the middle of the web. This inwardly directed parabolic distribution of shear stress occurs across all the sections along the beam length where the internal force due to the bending moment is more dominating than the shear force. In general, the results obtained are in an excellent agreement with 3D FE solutions. Overall comparison with the analytical solution proposed in [9] is good, with slight discrepancies in shear and von Mises stresses observed particularly near the web-flange junction. The analytical method assumes a plane stress condition. Moreover, it is derived from Jourawsky's theory and is, indeed, not expected to predict the response in the vicinity of a geometric discontinuity accurately.

To further exploit the capabilities of the proposed model, the geometric nonlinear structural response of the tapered I-beam is presented. The beam is fixed at one end ($y = 0$) and is subjected to oppositely directed forces, V_f , in the two flanges at the end $y = L$, as shown in Figure 9. These forces act at a distance equal to the depth between the centroids of each flange and form a couple ($T = 11.6$ kN m). As one end of the beam is fixed, the warping deformation is constrained and the structure undergoes non-uniform torsion. This results in rotation of the section along with bending of flanges in their own plane. Each of the flanges can be visualized as a beam subjected to bending moments produced by the forces V_f . This leads to bending or warping normal stresses accompanied by the in-plane or warping shear stresses in the flanges. Figure 10 shows the nonlinear torque versus displacement (T vs u_x) curve at $(0, L, -h_f - \frac{h_w(y)}{2})$. Furthermore, the axial variation of displacements, u_x and u_y , are plotted in Figures 11a and 11b while the variation of bending and in-plane shear stresses are plotted in Figures 11c and 11d, respectively. All the elastic responses obtained herein are computed by employing both, the direct and the incremental UF-SLE model. The displacement and stress response obtained using the UF-SLE model correlate well with the 3D finite element model.

(a) σ_{yy} at $(0, 3000, z)$

(b) σ_{yy} at $(0, 5000, z)$

(c) σ_{yy} at $(0, 7000, z)$

(d) σ_{yy} at $(0, 9000, z)$

Figure 6: Through-thickness variation of axial normal stress evaluated at various locations along the length of the tapered I-section beam. Analytical results are taken from [9].

(a) τ_{yz} at $(0, 3000, z)$

(b) τ_{yz} at $(0, 5000, z)$

(c) τ_{yz} at $(0, 7000, z)$

(d) τ_{yz} at $(0, 9000, z)$

Figure 7: Through-thickness variation of transverse shear stress evaluated at various locations along the length of the tapered I-section beam. Analytical results are taken from [9].

(a) σ_{eq} at $(0, 3000, z)$

(b) σ_{eq} at $(0, 5000, z)$

(c) σ_{eq} at $(0, 7000, z)$

(d) σ_{eq} at $(0, 9000, z)$

Figure 8: Through-thickness variation of von Mises stress evaluated at various locations along the length of the tapered I-section beam. Analytical results are taken from [9].

Figure 9: Tapered I-section beam: under twisting load.

Figure 10: Tapered I-section beam: Applied twisting moment versus lateral displacement curve at $(0, L, -h_f - \frac{h_w(y)}{2})$.

(a) u_x at $(0, y, h_f + h_w(y)/2)$

(b) u_y at $(b_f/2, y, h_f + h_w(y)/2)$

(c) σ_{yy} at $(b_f/2, y, h_f + h_w(y)/2)$

(d) τ_{xy} at $(0, y, h_f + h_w(y)/2)$

Figure 11: Variation of displacement components, u_x and u_y , and in-plane normal and shear stress components, σ_{yy} and τ_{xy} , along the length of the tapered I-section beam.

Figure 12: Tapered sandwich beam: dimensions and load definition.

Table 2: Mechanical properties of the IM7/8552 graphite-epoxy composite [57].

E_1	E_2	E_3	G_{13}	G_{23}	G_{12}	ν_{13}	ν_{23}	ν_{12}
(GPa)								
150	9.08	9.08	5.6	2.8	5.6	0.32	0.5	0.32

4.2. Tapered Sandwich Beam-like 3D Structure

In this section, a tapered sandwich beam-like structure of length $L = 1000$ mm and width $b = 300$ mm is considered, as shown in Figure 12. The beam is fixed at $y = 0$ and subjected to a transverse force, P_z , applied as a uniformly distributed load across the section at $y = L$. The facings of the sandwich composite are made of IM7/8552 graphite-epoxy material of thickness $h_f = 10$ mm. Material properties in principal material coordinates are shown in Table 2. The core is assumed to be made of PVC foam with isotropic properties: Young's modulus, $E = 100$ MPa and shear modulus, $G = 38.5$ MPa. The core's thickness varies linearly along the beam length, starting from $h_c(0) = 300$ mm and ending at different values depending on the taper angle, ϕ .

Using the UF-SLE model, the beam is discretised with 20 B4 elements along its length. The nodes are distributed adopting a Chebyshev biased mesh. This distribution in the longitudinal direction captures boundary layer effects induced by the clamped support accurately [51]. The initial cross-section (at $y = 0$) is discretised with 3 SL5 (fifth-order Serendipity Lagrange) elements with one element for each outer facing and one for the core. This discretisation and order of expansion results in 10,431 DOFs. In addition,

Figure 13: Geometrical and structural meshes considered in the present study for tapered sandwich beam.

Figure 14: Axial variation of normalised extensional or longitudinal stress, $\bar{\sigma}_{ss}$, at the top surface of the tapered sandwich beam for various taper angles at $x = 0$.

three 27-noded brick elements are used to describe the geometry and to evaluate the local curvilinear basis vectors required during the analysis. Both, geometric and structural, meshes are depicted in Figure 13. Four different taper angles, $\phi = 0^\circ, 3^\circ, 5^\circ$, and 8° , are considered for analysis. For model verification, reference converged solutions are obtained by performing 3D FE analyses in **ANSYS**. The latter models are meshed with 25,600 SOLID186 elements (100 elements along the length, 15 elements across the width and 16 elements through the thickness, with 10 for the core and 3 for each face sheet), resulting in 339,099 DOFs.

In this case, stresses are presented in the local x - s - n coordinate system and are non-dimensionalised as follows:

$$\bar{\sigma}_{ss} = \frac{\sigma_{ss} b h_0^2}{P_z L}, \quad \bar{\tau}_{ns} = \frac{\tau_{ns} b h_0}{P_z}, \quad \bar{\sigma}_{nn} = \frac{\sigma_{nn} b h_0}{P_z}, \quad (26)$$

where h_0 is the total thickness at $y = 0$ and σ_{ss} , τ_{ns} and σ_{nn} are the longitudinal, transverse shear and peeling components of stress.

Figure 14 shows the axial variation of $\bar{\sigma}_{ss}$ on the top-most surface of the top facing for all taper angles under consideration. With an increase in the taper angle, the extensional stress redistributes along the length. For small ϕ angles, the stress distribution decreases monotonically from maxima occurring at the fixed end. These maxima decrease in magnitude for increasing ϕ . As ϕ increases, a hilltop forms on the right hand side of the beam. For $\phi = 8^\circ$, this region features a large gradient with a fully formed hilltop of value comparable to that at the fixed end. For completeness, Figure 15a shows the through-thickness variation of $\bar{\sigma}_{ss}$ at the mid-span of the structure.

Experiments have shown that transverse stresses can cause failure of tapered sandwich structures. Excessive inter-facial stresses could, indeed, cause delamination followed by debonding of the facings [58, 59]. Therefore, it is important to quantify the shear and peeling stresses at the interface between the core and the facings. These components of stress are not captured accurately by simple 1D or 2D models, which further demonstrates the need for a computationally efficient method for detailed 3D analysis.

The normalised transverse shear stress distribution, $\bar{\tau}_{ns}$, along $\bar{y}$ at the interface between the top facing and the core is depicted in Figure 16. For non-null taper angles, $\bar{\tau}_{ns}$ decreases monotonically towards the

Figure 15: Through-thickness distribution of normalised extensional and normalised transverse shear stress at the mid-span of the tapered sandwich beam for various taper angles obtained by the UF-SLE model.

free end, featuring sharp gradients towards both ends and reaching greater maximum absolute values for increasing angles. This contrasts with the observation that, at the mid-span of the beam, transverse shear stresses decrease as the taper angle increases, as shown in Figure 15b. For the prismatic case, the transverse shear stress is nearly constant, except in the vicinity of the clamped end, where the sharp gradient is maintained, and at the free end where a concentrated local minimum is observed.

Finally, Figure 17 shows the axial variation of normalised peeling stress, $\bar{\sigma}_{nn}$, between the core and the top facing. Compared to 3D FE solutions, adopting the same mesh as for the other two components of stress, small discrepancies are observed near the ends. Therefore, $\bar{\sigma}_{nn}$ is further investigated by using a refined mesh (40 B4 Chebyshev-biased beam elements and 9 SL5 elements in the cross-section: one per layer and three across the width). The peeling stress response obtained using the refined mesh correlates well with 3D FE solutions. Although, slight differences are still noticed near the ends, which are predominantly in the first and the last beam elements. The reason for this discrepancy could be the different order of expansions used in the cross-section and in the beam direction, which are quintic and cubic, respectively. The issue could be addressed by adopting higher-order beam elements. However, solving this disparity in results near the constraints, needs further investigation and will be explored in future work.

It is observed that, for all cases, peak stress values localise towards the ends and are negligible everywhere else. The large peeling stress near the ends could cause the facing to peel off from the core. The overall 3D stress response in the tapered sandwich structures is well predicted by the proposed UF-SLE model and is computationally efficient compared to the 3D FE model.

4.3. Corrugated Structures

In order to further exploit the capabilities of the UF-SLE model, the geometric nonlinear behaviour of corrugated structures in tensile loading is investigated. This example is adapted from the work done by Thurnherr *et al.* [60]. The objective is to show the proposed model's capability in analysing such complex curved structures and describing their nonlinear behaviour.

A corrugation pattern consisting of unit cells with circular sections is now described. The elementary unit cell is shown in Figure 18, together with the definition of its parameters: radius of curvature, R ; amplitude, c ; periodic cell length, p ; arc-length parameter, s ; and coordinates $(y(s), z(s))$ of points on the mid-surface.

(a) $\bar{\tau}_{ns}$ vs $\bar{y}$ for $\phi = 0^\circ$

(b) $\bar{\tau}_{ns}$ vs $\bar{y}$ for $\phi = 3^\circ$

(c) $\bar{\tau}_{ns}$ vs $\bar{y}$ for $\phi = 5^\circ$

(d) $\bar{\tau}_{ns}$ vs $\bar{y}$ for $\phi = 8^\circ$

Figure 16: Axial variation of normalised transverse shear stress between the core and the top facing of the tapered sandwich beam for various taper angles at $x = 0$.

(a) $\bar{\sigma}_{nn}$ vs $\bar{y}$ for $\phi = 0^\circ$

(b) $\bar{\sigma}_{nn}$ vs $\bar{y}$ for $\phi = 3^\circ$

(c) $\bar{\sigma}_{nn}$ vs $\bar{y}$ for $\phi = 5^\circ$

(d) $\bar{\sigma}_{nn}$ vs $\bar{y}$ for $\phi = 8^\circ$

Figure 17: Axial variation of normalised peeling stress between the core and the top facing of the tapered sandwich beam for various taper angles at $x = 0$.

Figure 18: Corrugated structure: unit cell geometry definition.

The latter are given by the parametric equations [61, 62]:

$$\begin{aligned} y(s) &= R [\sin \psi(s) + (2 + m) \sin \psi_0], \\ z(s) &= -mR [\cos \psi(s) - \cos \psi_0], \end{aligned} \quad (27)$$

where

$$m = \begin{cases} 1 & \text{if } y \leq p/2 \\ -1 & \text{if } y > p/2 \end{cases}, \quad (28)$$

$$\psi_0 = \begin{cases} \arcsin\left(\frac{p}{4R}\right) & \text{if } c \leq \frac{p}{4} \\ \arccos\left(\frac{p}{4R}\right) + \frac{\pi}{2} & \text{if } c > \frac{p}{4} \end{cases}, \quad (29)$$

representing the opening angle indicated in Figure 18, and

$$\psi(s) = \kappa s - (2 - m)\psi_0. \quad (30)$$

Finally, the radius of curvature, R , follows from p and c , and is given by

$$R = \frac{16c^2 + p^2}{32c}. \quad (31)$$

Three different structures, referred to as A, B and C, and each consisting of four unit cells, are studied. The dimensions for each case are shown in Figure 19. All configurations are clamped on both sides with an axial displacement, u_y , applied on one end. Thurnherr *et al.* [60] conducted an experimental study on these structures. The test samples shown in Figure 20 were fabricated using a 3D printer. All specimens were made from Polylactic acid (PLA). Due to the fabrication process the samples are printed layer by layer. This leads to anisotropic material behaviour where the direction parallel to the layers is the stiffest. However, under tensile loading the mechanical response is mainly governed by the stiffness in the axial direction. Therefore, the constituent material is assumed to be isotropic [60] with Young's modulus $E = 3.5$ GPa and Poisson's ratio $\nu = 0.346$.

Herein, the corrugated structures are analysed using the UF-SLE model. Configuration A is discretised with three SL3 (third-order expansion) elements in the cross-section and 50 B4 elements along the longitudinal direction. An increase in curvature demands a finer discretisation to capture nonlinear deflection and stress accurately. Hence, specimens B and C require respectively 80 and 150 B4 elements. These discretisations result in 12,684, 20,224 and 37,884 DOFs for configurations A, B and C, respectively. In order to verify the results obtained using the UF-SLE model, 3D FE analyses are performed using ANSYS. In all cases, the structures are discretised using 13,440 SOLID186 (20-noded brick) elements (with 2 elements through the thickness, 320 elements along the corrugation pattern and 20 elements across the width), resulting in 229,959 DOFs. It is to be noted that a third-order SLE model is employed in the cross-section, resulting

Figure 19: Corrugated structures modelled as curved beams.

Figure 20: Corrugated structure: test samples [60].

in a highly refined kinematic field, which may not be required for a thin structure under tensile load. Also, solid elements are employed in the FE analysis where shell elements could have been used instead. This expansion order and the type of finite element are used to ensure minimum error in the solutions. Moreover, the objective of this test case is to showcase the validity of the proposed model in analysing highly curved structures, and not to compare the computational efficiencies of the two approaches.

Figure 21 shows the nonlinear force-displacement curve for the three configurations. The responses obtained using the UF-SLE model correlate well with 3D FE solutions. The two numerical models are also validated against experimental results as given in [60]. There, the authors applied pre-stretching corrections of 2 mm, 2.6 mm and 8 mm, respectively, to their numerical models for specimens A, B and C. Here, we do the same. Thurnherr *et al.* justify the corrections with imperfections in the test set-up. In particular, they report that pre-stretching was induced by fitting the (thin) samples in the tensile machine. Post correction, the numerical results presented herein match the experiments well. Remaining discrepancies could be due to the assumed material properties, *i.e.* isotropy instead of layer-wise anisotropy, and other imperfections in the experimental set-up.

From the force-displacement curves, it is shown that the corrugated structures considered are sufficiently compliant to be readily deformed under tensile loading. Clearly, the initial response is linear and driven by bending deformations. However, as the structure deforms more, the corrugation flattens and the axial stiffness increases. The slope then changes drastically and overall response becomes dominated by axial deformations. Modelling this behaviour requires a geometrically nonlinear framework, and the proposed UF-SLE model is shown to capture the response accurately. In addition, the nonlinear stress predicting capability of the UF-SLE model is also tested. Figure 22 shows 3D contour plots of the axial stress, σ_{yy} , for the three corrugated structures, obtained from 3D FE and UF-SLE models, showing that the stress fields are almost the same. Small differences in the maximum and minimum values are observed, which are mainly due to the clamped support conditions.

5. Conclusions

Previous research based on the state-of-the-art Unified Formulation [33–46] has highlighted its ability in solving a wide range of structural mechanics problems in a computationally efficient manner. However, its modelling capabilities were limited to prismatic structures. This formulation is not able to analyse variable cross-section, variable planform geometries as used in many practical situations. To overcome this limitation and to enhance the capabilities of the Unified Formulation, this paper introduces separate shape functions to approximate structural geometry and displacement field. With this approach, the existing shortcomings of the state-of-the-art Unified Formulation are solved. Our representation is able to describe structures that are both non-prismatic and curved. In this work, Serendipity Lagrange expansions (UF-SLE) are used as Unified Formulation shape functions. Results on a tapered I-beam, a sandwich beam and corrugated structures have been presented. The hierarchical nature of the Serendipity Lagrange expansions allows the fidelity of the models to be tuned, such that low-fidelity and high-fidelity models can be used concurrently to assess global response and 3D stresses. In all cases analysed, the 3D stress distributions obtained indicate a good correlation with 3D FE solutions. In addition, the proposed model offers computational benefits over FE models. The nonlinear mechanics of corrugated structures under tensile loading was also investigated. The static response obtained is shown to match numerical and experimental results.

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(a) Corrugated Structure A.

(b) Corrugated Structure B.

(c) Corrugated Structure C.

Figure 21: Force vs axial displacement curve for the three corrugated structures A, B and C. Experimental results are taken from Thurnherr *et al.* [60].

Figure 22: Contour plot of axial normal stress σ_{yy} (in Pa) for the three corrugated structures A, B and C.

Data Statement

All data required to reproduce the figures in this paper can be found on the data repository of the University of Bristol via URL: <http://data.bris.ac.uk/data/>.

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