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Analytical Solution for Axially Loaded Piles in Two-Layer Soil

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ABSTRACT

An analytical elastic continuum model is developed for the settlement of end-bearing piles in a two-layer soil over a rigid stratum. The model has its roots in the point-load solution of Westergaard which was later extended by Tajimi to deep foundations and lies on the assumption of a vanishing soil stress or displacement component. For piles in homogeneous soils such solutions were elaborated by Nogami and Novak. Contrary to these solutions, the proposed generalized formulation can handle layered soils using, for the first time, two sets of eigenfunctions (static “modes”) which are different for the soil and the pile. Stresses and displacements are determined in the form of Fourier series with coupled coefficients obtained by solving a system of algebraic equations of rank equal to the number of modes considered. This is in contrast with existing models, where the Fourier coefficients are obtained individually. Pile head stiffnesses obtained from this model are verified against results from rigorous finite-element analyses and other solutions. Results for pile settlement, pile stresses, side friction and Winkler moduli are presented.

Keywords: soil-pile interaction, end-bearing piles, elasticity, analytical model, layered soil

INTRODUCTION

Pile response in non-homogeneous soils has been the subject of significant research effort in both classical and earthquake geotechnical engineering. Available models to predict pile response can be classified in two main groups, based on the way of simulating soil resistance: (A) simple numerical and analytical formulations that treat the soil as a series of independent (Winkler) springs which are uniformly distributed along the pile shaft and can be determined through either theoretical or experimental means (e.g., Seed and Reese 1957, Coyle and Reese 1966, Murff 1975, Guo and Randolph 1977, Randolph and Wroth 1978, Kraft Ray and Kagawa 1981, Armaleh and Desai 1987, Kodikara and Johnston 1994, Motta 1994, Mylonakis and Gazetas 1998, Guo 2000) and (B) more rigorous solutions (e.g., boundary integral method, thin layer method, finite-element method, elastic and elasto-plastic solutions) which treat the soil as a continuum (e.g., Poulos and Davis 1968, Mattes and Poulos 1969, Butterfield and Banerjee 1971, Nogami and Novak 1976, Poulos 1979, Akiyoshi 1982, Rajapakse, 1990, Lee and Small 1991, Liu et al. 2004, Ai and Han 2009).

While spring models have attracted significant research interest, their accuracy and performance depend on the selection of their stiffness (Winkler modulus). In this regard, continuum solutions offer the advantage of being theoretical sound and self-standing; yet are more difficult to implement. Further, while the latter solutions may perform well for homogeneous half-space and soils with stiffness varying linearly with depth (e.g., Butterfield and Banerjee (1971), Banerjee (1978), Poulos and Davis (1980)), it has been found that they may not give acceptable results for piles embedded in layered soils having strong stiffness contrast between successive layers (Poulos 1979, Yamashita et al. 1987).

The proposed elastic model falls into category (B). It extends the static counterpart of the pioneering Tajimi type model of Nogami and Novak (1976) (Mylonakis 2001) to piles in a two-layer soil. Fundamental to such models is the assumption that the vertical normal and vertical shear stresses in the soil are controlled by the vertical soil displacement component. Thus, it is essentially an approximate continuum solution, which by adopting this assumption, reduces the two equations of classical axisymmetric elasticity to one (“Tajimi approach”). As will be shown in detail in the mathematical derivation sections, this extension is not straightforward in non-homogeneous soils and demonstrates one of the novelties of the model. A special treatment is required to derive the solution in terms of Fourier series (i.e., ensure compatibility of stresses and displacements along the soil-pile interface in generalized coordinates).

In this light, the following innovations are implemented: (i) “static” pile modes (eigenfunctions) and associated eigenvalues are different to the corresponding soil modes and eigenvalues; and (ii) the second derivative of the pile modes, shown in the pile equilibrium equation, is expressed in terms of the soil modes using the orthogonality identity. Such innovative physical (i.e., (i)) and mathematical (i.e., (ii)) modelling allows the calculation of pile response (e.g., settlements, axial stresses), as well as the displacement and stress fields in the surrounding soil for layered soil media of the “Tajimi” type.

As a consequence of (i) and (ii), the Fourier coefficients of the solution cannot be obtained independently. A set of simultaneous linear equations, of rank equal to the number of pre-selected modes, must be solved instead. These “modes” are continuous depth-dependent functions, independent of the horizontal spatial variable and are similar to those used in classical structural dynamics. Herein, they are derived under the more realistic assumption of zero radial stress (i.e., $\sigma_r = 0$) and tangential strain (i.e., $\varepsilon_\theta = 0$) (Mylonakis 2001, Anoyatis and Mylonakis 2012). This

is in contrast with the case of a homogeneous (Mylonakis 2001) where such coefficients are obtained in closed-form.

Such formulations are advantageous over existing simple analytical models of the Winkler type as they are essentially three-dimensional (i.e., account for the thickness of the soil medium which is critical in dynamic analysis to evaluate resonance effects) and are self-standing (i.e., free of empirical constants, such as the spring modulus in the commonly used Winkler models). Further, they retain a mathematical simplicity (i.e., well-known method of separation of variables and orthogonality identity of modes) over rigorous analytical models which focus mainly on a homogeneous half-space and use more complex techniques (e.g., Green's functions and/or integral equations).

The mathematical core of the proposed model can, with pertinent modifications, be further extended to tackle a variety of problems in geotechnical engineering such as retaining walls (under harmonic-seismic excitation the base), cylindrical embedded foundations (under static and dynamic loads) as well as floating piles in non-homogeneous soils. This can be viewed as an additional benefit of the solution over more recent continuum-based analytical solutions to predict pile settlement (e.g., Vallabhan and Mustafa 1996, Lee and Xiao 1999, Seo and Prezzi 2007, Basu et al. 2008, Seo et al. 2009). This versatility further highlights the importance of the solution and demonstrates the future potential, rendering this solution a starting point.

PROBLEM AT HAND

The problem considered in this study is presented in Fig. 1: an axially-loaded single vertical solid cylindrical pile is embedded in a two-layer deposit, which rests over a rigid base and is subjected

to an axial head load. The upper soil layer of thickness h_1 and the bottom layer of thickness h_2 form a deposit of total thickness H (equal to the pile length L). The pile is modelled as an elastic rod using the classic strength-of-materials assumption and is described by its Young's Modulus E_p , length L and diameter d . Each soil layer is described by its Young's modulus E_{si} (or equivalently its shear modulus G_{si}), and Poisson's ratio ν_{si} . The subscript i ($= 1, 2$) denotes the layer number starting with the upper one. The soil medium is modelled as an approximate 3D continuum ("Tajimi" medium) as presented in previous works by the authors (e.g., Mylonakis 2001, Anoyatis and Mylonakis 2012, Anoyatis et al. 2013) and perfect bonding is assumed at the soil-pile interface.

The stiffness variation of the layered soil is expressed through a depth-dependent soil shear modulus $G_s(z)$ expressed using the familiar unit step function $H(\cdot)$

$$G_s(z) = G_{s1} + (G_{s2} - G_{s1}) H(z - h_1) \quad (1)$$

where

$$H(z - h_1) = \begin{cases} 0, & 0 \leq z \leq h_1 \\ 1, & h_1 \leq z \leq H \end{cases} \quad (2)$$

MODEL DEVELOPMENT

The analysis is decomposed into two modular problems: (i) analysis of the soil medium and (ii) analysis of the pile rod.

Mathematical Formulation for the Soil Medium

Each soil layer is treated separately as a single homogeneous layer, yet with different boundary conditions (i.e., at the surface, base and layer interface). In this light, the equilibrium equation in the vertical direction in cylindrical coordinates holds for each soil layer based on the solution of Mylonakis (2001) for a single homogeneous layer:

$$\frac{\partial(\tau_i r)}{\partial r} + r \frac{\partial \sigma_i}{\partial z} = 0 \quad (3)$$

where subscript i refers to the upper ($i = 1$) and the bottom layer ($i = 2$), respectively. Substituting the expressions for the vertical normal stresses σ

$$\sigma_i \simeq -\eta_{si}^2 G_{si}^* \frac{\partial u_i}{\partial z} \quad (4)$$

and vertical shear stresses τ

$$\tau_i \simeq -G_{si}^* \frac{\partial u_i}{\partial r} \quad (5)$$

Equation (3) can be written in terms of displacements u as (Mylonakis 2001):

$$\frac{\partial^2 u_i}{\partial r^2} + \frac{1}{r} \frac{\partial u_i}{\partial r} + \eta_{si}^2 \frac{\partial^2 u_i}{\partial z^2} = 0 \quad (6)$$

The dimensionless quantity η_s is a compressibility parameter (see Equation 4) and is expressed exclusively as a function of Poisson's ratio

$$\eta_{si} = \sqrt{\frac{2}{1 - \nu_{si}}} \quad (7)$$

Assumptions of a “Tajimi” Medium

In the realm of classical elasticity, stresses in Equations (4) and (5) are expressed as a function of the vertical and horizontal soil displacement components u_z and u_r , respectively (for

an axisymmetric problem such as the one considered in this work the tangential component u_θ is zero).

Fundamental to a “Tajimi” medium approach, as applied in this work, is the assumption that the vertical normal and vertical shear stresses obtained from Equations (4) and (5) are controlled explicitly only by the vertical soil displacement component u_z . This approximation is attractive, as it leads to a straightforward uncoupling of the governing elasticity equations, unlike the case of classical elasticity (Graff 1975). It results in one governing equation (Equation 6) instead of two (one for the vertical and the other for the horizontal equilibrium). In Equation (5) this is achieved through vanishing the term $\partial u_r / \partial z = 0$ and in Equation (4) by introducing the parameter η_s .

Equation (4) was first employed by Nogami and Novak (1976) for the dynamic analysis of a pile embedded in a homogeneous soil layer. In that work, however, the radial displacement of the medium was set equal to zero ($u_r = 0$), leading to an expression for η_s (Appendix I, Equation I.3) which is unrealistically sensitive to Poisson’s ratio.

In the present study, the assumption is less restrictive: while u_r is not explicitly present its influence on σ ’s is implicitly taken into account considering the non-zero radial strain ε_r (Mylonakis 2001, Anoyatis and Mylonakis 2012). This is achieved by setting the horizontal stress and tangential strain in soil $\sigma_r = 0$ and $\varepsilon_\theta = 0$, respectively. This set of assumptions relate ε_r to the vertical strain component ε_z (Appendix I, Equation I.4). The current approach captures better the condition of zero tangential displacement in the domain, while offers the additional advantage of producing a η_s (Appendix I, Equation I.6) which does not collapse for Poisson’s ratio $\nu_s = 0.5$ as in the Nogami and Novak solution.

Soil Analysis

The general solution to Equation (6) is obtained by solving it separately for each (homogeneous) soil layer. This is achieved by employing the method of separation of variables: soil displacement is expressed as a product of two spatial functions R and Z which are solely functions of the spatial variables r and z , respectively. Thus, substituting $u_i = R_i(r) Z_i(z)$ into Equation (6) yields:

$$\frac{1}{r} \frac{1}{R_i} \frac{dR_i}{dr} \left(r \frac{dR_i}{dr} \right) + \eta_{si}^2 \frac{1}{Z_i} \frac{d^2 Z_i}{dz^2} = 0 \quad (8)$$

To ensure that the equality holds at every r and z , Equation (8) is decomposed into the pair of ordinary differential equations:

$$\frac{1}{r} \frac{1}{R_i} \frac{dR_i}{dr} \left(r \frac{dR_i}{dr} \right) = q_i^2 \quad (9)$$

and

$$\frac{1}{Z_i} \frac{d^2 Z_i}{dz^2} = -a^2 \quad (10)$$

where variables q_i are related to a through the following equation:

$$q_i = a \eta_{si} \quad (11)$$

Equations (9) and (10) yield the following general solutions, respectively:

$$R_i = A_i I_0(q_i r) + B_i K_0(q_i r) \quad (12)$$

and

$$Z_i = C_i \sin az + D_i \cos az \quad (13)$$

Thus, soil displacement can now be expressed as:

$$u_i(r, z) = [A_i I_0(q_i r) + B_i K_0(q_i r)] [C_i \sin az + D_i \cos az] \quad (14)$$

In Equations (12) – (14), $I_0()$ and $K_0()$ are the modified Bessel functions of the zero order and the first and second kind, respectively. a is a positive real variable of dimensions 1/Length. A_i , B_i , C_i and D_i are integration constants to be determined from the conditions at the boundaries (i.e., soil-pile interface and the interface between the two soil layers).

To ensure bounded response at large radial distances from the pile ($r \rightarrow \infty$), constants A_i in Equation (14) must vanish. Accordingly, the solutions reduce to:

$$u_i = B_i K_0(q_i r) [C_i \sin az + D_i \cos az] \quad (15)$$

The condition of a free stress soil surface ($\frac{\partial u_1}{\partial z} \Big|_{z=0} = 0$, Equation 15) yields

$$u_1(r, z) = B_1 K_0(q_1 r) \cos az \quad (16)$$

where the constant D_1 has been embodied into a new constant B_1 .

The condition of the soil resting on a rigid base ($u_2(r, H) = 0$, Equation 15) yields for every r

$$C_2 \sin aH + D_2 \cos aH = 0 \quad (17)$$

The compatibility of displacements at the layer interface [i.e., $u_1(r, h_1^-) = u_2(r, h_1^+)$] requires

$$B_1 K_0(q_1 r) \cos ah_1 = K_0(q_2 r) [C_2 \sin ah_1 + D_2 \cos ah_1] \quad (18)$$

Likewise, the compatibility of stresses at the layer interface [$\sigma_1(r, h_1^-) = \sigma_2(r, h_1^+)$] requires

$$-\eta_{s1}^2 G_{s1} B_1 K_0(q_1 r) \sin ah_1 = \eta_{s2}^2 G_{s2} K_0(q_2 r) [C_2 \cos ah_1 - D_2 \sin ah_1] \quad (19)$$

Taking into account Equations (15) – (19), soil displacements for the upper (u_1) and lower layer (u_2) can be expressed as

$$u_1(r, z) = B_1 K_0(q_1 r) \cos az, \quad 0 \leq z \leq h_1 \quad (20)$$

and

$$u_2(r, z) = B_1 K_0(q_1 r) \frac{\cos a_m h_1}{\sin a_m h_2} \sin a_m (H - z), \quad h_1 \leq z \leq H \quad (21)$$

Combining Equations (17), (18) and (19) leads to the following characteristic equation

$$\cos a_m h_1 \cos a_m h_2 - \frac{\eta_{s1}^2}{\eta_{s2}^2} \frac{G_{s1}}{G_{s2}} \sin a_m h_1 \sin a_m h_2 = 0 \quad (22)$$

the solution of which yields the discrete eigenvalues a_m , $m = 1, 2, 3, \dots$. This indicates that the solution can be expressed in terms of infinite Fourier Series.

In light of the above developments, displacements u and shear stresses τ in the soil can be expressed in the following infinite series form:

$$u(r, z) = \sum_{m=1}^{\infty} B_m K_0(q_{1m} r) \Phi_m(z) \quad (23)$$

and

$$\tau(r, z) = \sum_{m=1}^{\infty} B_m q_{1m} K_1(q_{1m} r) G_s(z) \Phi_m(z) \quad (24)$$

Soil Modes and Orthogonality Identity

In Equations (23) and (24) Φ_m is the m -th eigenfunction (“static” mode) of the soil deposit and is expressed in a manner similar to $G_s(z)$ (Equation 1)

$$\Phi_m(z) = \Phi_{1m}(z) + [\Phi_{2m}(z) - \Phi_{1m}(z)] H(z - h_1) \quad (25)$$

where Φ_{1m} and Φ_{2m} are the m -th soil mode components. The subscripts “1” and “2” pertain to the first and the second layer, respectively, and are expressed as follows:

$$\Phi_m(z) = \begin{cases} \Phi_{1m} = \cos a_m z, & 0 \leq z \leq h_1 \\ \Phi_{2m} = \frac{\cos a_m h_1}{\sin a_m h_2} \sin a_m (H - z), & h_1 \leq z \leq H \end{cases} \quad (26)$$

Equations (26) are inspired from the analysis of a homogeneous soil, where soil modes are expressed solely via the simple trigonometric function $\cos a_m z$ (Mylonakis 2001). Herein, a modification has been applied to satisfy the boundary conditions of zero normal stress at the soil surface (i.e., $\sigma_z|_{z=0} = 0$), zero displacement at the soil base (i.e., $u|_{z=H} = 0$) and the compatibility of stresses and displacements at the layer interface. As in the homogeneous case, Φ_m satisfy the fundamental for the solution orthogonality condition. This is mathematically expressed via the familiar equation

$$\int_0^H G_s(z) \Phi_m(z) \Phi_k(z) dz = 0, \quad m \neq k \quad (27)$$

where m, k are positive integers (i.e., $m = 1, 2, \dots$ and $k = 1, 2, \dots$).

In Sturm-Liouville theory the shear modulus $G_s(z)$ in Equation (22) is usually referred to as the “weight function”. For the two-layered soil medium treated herein, this equation can be alternatively written in the following expanded form:

$$G_{s1} \int_0^{h_1} \Phi_{1m}(z) \Phi_{1k}(z) dz + G_{s2} \int_{h_2}^H \Phi_{2m}(z) \Phi_{2k}(z) dz = 0, \quad m \neq k \quad (28)$$

Note that Equation (27) is valid for existing solutions in homogeneous soils (Nogami and Novak 1976, Anoyatis and Mylonakis 2012). The orthogonality identity is then expressed in the simpler form $\int_0^H \Phi_m \Phi_k dz = 0$ (for $m \neq k$) as G_s is depth-independent and escapes the integration.

Pile Analysis

The equilibrium of vertical forces acting upon an arbitrary pile segment yields the governing differential equation of the pile (Mylonakis 2001, Anoyatis 2013):

$$- E_p A_p \frac{d^2 w}{dz^2} + \pi d \tau_0 = F(z) \quad (29)$$

$\tau_0 = \tau(d/2, z)$ is the vertical soil reaction at the pile periphery (Equation 24). $F(z)$ stand for the forces distributed along the pile body. These forces can be alternatively expressed through Fourier series:

$$F(z) = \sum_{n=1}^{\infty} F_n G_s(z) \Phi_n(z) \quad (30)$$

Using the orthogonality identity (Equation 27) coefficients F_n ($n = 1, 2, 3, \dots$) are determined as

$$F_n = \frac{\int_0^H P \delta(z) \Phi_n(z) dz}{\int_0^H G_s(z) \Phi_n^2(z) dz} = \frac{P}{\int_0^H G_s(z) \Phi_n^2(z) dz} \quad (31)$$

In Equation (31) $F(z)$ is equivalently expressed via the pile load P using the Dirac's Delta distribution symbol $\delta(z)$ as $F(z) = P \delta(z)$.

In the available solutions for homogeneous (Mylonakis 2001) and vertically inhomogeneous (Anoyatis et al. 2019) soils pile displacements w are expressed as a function of the soil modes Φ_m . Fundamental to the proposed solution is that pile modes are different to the soil modes. This is novel to the solution of layered “Tajimi” media, and allows w to be written in Fourier series as a function of the pile modes Y_k ($k = 1, 2, 3, \dots$)

$$w(z) = \sum_{k=1}^{\infty} C_k Y_k(z) \quad (32)$$

where C_k are the Fourier coefficients and Y_k is the k -th mode expressed as

$$Y_k(z) = \cos p_k z \quad (33)$$

In Equation (33) p_k are the pile eigenvalues

$$p_k = \frac{\pi}{2L} (2k - 1) \quad (34)$$

which for homogeneous soil (e.g., Nogami and Novak 1976, Mylonakis 2001, Anoyatis et al. 2013) are identical to the soil eigenvalues.

Substituting Equations (24) for $r = d/2$, (30) and (32) into (29) one obtains:

$$-E_p A_p \sum_{k=1}^{\infty} C_k Y_k''(z) + 2\pi \sum_{m=1}^{\infty} B_m s_{1m} K_1(s_{1m}) G_s(z) \Phi_m(z) = \sum_{n=1}^{\infty} F_n G_s(z) \Phi_n(z) \quad (35)$$

where s_{1m} is the dimensionless parameter

$$s_{1m} = q_{1m} d/2 \quad (36)$$

Key to the analysis presented herein is expressing the second derivative of the pile modes Y_k'' in terms of the soil modes Φ_m (Anoyatis 2013):

$$Y_k''(z) = \sum_{j=1}^{\infty} R_j G_s(z) \Phi_j(z) \quad (37)$$

where coefficients R_m are obtained as:

$$R_m = \frac{\int_0^H Y_k''(z) \Phi_m(z) dz}{\int_0^H G_s(z) \Phi_m^2(z) dz} = - \frac{\int_0^H Y_k'(z) \Phi_m'(z) dz}{\int_0^H G_s(z) \Phi_m^2(z) dz} \quad (38)$$

In light of this substitution Equation (35) is rewritten as:

$$\begin{aligned} -E_p A_p \sum_{k=1}^{\infty} C_k \left(\sum_{j=1}^{\infty} R_j G_s(z) \Phi_j(z) \right) + 2\pi \sum_{m=1}^{\infty} B_m s_{1m} K_1(s_{1m}) G_s(z) \Phi_m(z) \\ = \sum_{n=1}^{\infty} F_n G_s(z) \Phi_n(z) \end{aligned} \quad (39)$$

Multiplying all terms in Equation (39) by Φ_m and integrating over the soil thickness, one obtains the following expression

$$\begin{aligned}
& - E_p A_p \sum_{k=1}^{\infty} C_k R_m \left(\int_0^H G_s(z) \Phi_m^2(z) dz \right) + 2 \pi B_m s_{1m} K_1(s_{1m}) \left(\int_0^H G_s(z) \Phi_m^2(z) dz \right) \\
& = F_m \left(\int_0^H G_s(z) \Phi_m^2(z) dz \right)
\end{aligned} \tag{40}$$

Further substituting the Fourier coefficients F_m and R_m from Equations (31) and (38), can achieve the following form:

$$E_p A_p \sum_{k=1}^{\infty} C_k \left(\int_0^H Y'_k(z) \Phi'_m(z) dz \right) + 2 \pi B_m s_{1m} K_1(s_{1m}) \left(\int_0^H G_s(z) \Phi_m^2(z) dz \right) = P \tag{41}$$

Coefficients C_k and B_m are related through the following expression that emerges from the condition of perfect bonding along the soil-pile interface [i.e., $w(z) = u(d/2, z)$]

$$B_m = \frac{\sum_{k=1}^{\infty} C_k \left(\int_0^H G_s(z) Y_k(z) \Phi_m(z) dz \right)}{K_0(s_{1m}) \left(\int_0^H G_s(z) \Phi_m^2(z) dz \right)} \tag{42}$$

This allows writing Equation (41) as function of one unknown Fourier coefficient, C_k :

$$\begin{aligned}
& \sum_{k=1}^{\infty} \frac{C_k}{K_0(s_{1m})} \left\{ \left(\int_0^H Y'_k(z) \Phi'_m(z) dz \right) \right. \\
& \left. + \frac{2 \pi s_{1m}}{E_p A_p} K_1(s_{1m}) \left(\int_0^H G_s(z) Y_k(z) \Phi_m(z) dz \right) \right\} = \frac{P}{E_p A_p}
\end{aligned} \tag{43}$$

The above expression can be written in matrix form as follows:

$$\underbrace{\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1N} \\ T_{21} & T_{22} & \cdots & T_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ T_{N1} & T_{N2} & \cdots & T_{NN} \end{bmatrix}}_{m \times k} \times \underbrace{\begin{Bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{Bmatrix}}_{k \times 1} = \frac{P}{E_p A_p} \underbrace{\begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix}}_{m \times 1} \tag{44}$$

where the elements of the “ T ” matrix can be obtained as

$$T_{mk} = \frac{1}{K_0(s_{1m})} \left\{ \left(\int_0^H Y'_k(z) \Phi'_m(z) dz \right) + \frac{2 \pi s_{1m}}{E_p A_p} K_1(s_{1m}) \left(\int_0^H G_s(z) Y_k(z) \Phi_m(z) dz \right) \right\} \quad (45)$$

The solution can be obtained when “ T ” is a square matrix. From a mathematical point of view this allows inversion, while the physical meaning is that the same total number of modes for pile and soil should be used in the analysis (i.e., $k = 1, 2, \dots, N$ and $m = 1, 2, \dots, N$). The adequate number of modes N required to achieve convergence of the solution, and thus the dimensions of the matrix, are discussed in the next section.

The matrix formulation in Equation (44) suggests that, contrary to the homogeneous case where the coefficients B_m are obtained in closed-form, herein B_m ’s (or equivalently the C_k ’s) are coupled and should be determined as solutions to system of simultaneous algebraic equations instead. Closed-form solutions for the integrals included in Equation (45) are presented in Appendix II.

Upon calculation of C_k and B_m , for a given number of modes N , pile and soil displacements can be determined from the following expressions:

$$w(z) = \sum_{k=1}^N C_k Y_k(z) \quad (46)$$

$$u(r, z) = \sum_{m=1}^N B_m K_0(s_{1m}) \Phi_m(z) \quad (47)$$

In the special case of a homogeneous medium the soil modes are identical to the pile modes (i.e., $\Phi(z) = Y(z) = \cos a_m z$). In this case, the integrals in Equation (45) collapse to the Mylonakis’ (2001) solution (Appendix III) leading to $B_m = C_k$:

$$B_m = \frac{2 P}{E_p A_p H p_m^2} \left[K_0(s_{1m}) + \frac{2 \pi s_{1m} G_s^*}{E_p A_p p_m^2} K_1(s_{1m}) \right]^{-1} \quad (48)$$

In the ensuing and except specifically otherwise stated results are provided for Poisson's ratios $\nu_{s1} = \nu_{s2} = 0.4$ and $N = 10^3$ modes.

MODEL VERIFICATION

In Tables 1 and 2, predictions of the pile head stiffness are compared against results from rigorous finite element (FE) analyses conducted using the commercial code ANSYS (ANSYS-10.0). The pile Poisson's ratio is $\nu_p = 0.2$; 10^3 elements in axisymmetric mode were used in the finite element model.

In Table 1 a wide range of pile configurations is used to investigate the effect of number of modes on the accuracy of the model. As anticipated, increasing the number of modes, the agreement between the proposed model and the FE results improves. An increase beyond approximately 500 modes only slightly improves the solution. With increasing pile slenderness L/d it is evident that a higher number of modes is required to achieve convergence for all cases examined. In particular, for a long pile ($L/d = 90$), increasing the number of modes from 20 to 1000 reduces the deviation from FE to 2.83 % ($= 8.40 - 5.57$). For a short pile ($L/d = 30$) the effect of modes is less pronounced limiting the drop to 1.46 % ($= 5.55 - 4.09$). In general, the analytical model follows closely the FE solution with minimum and maximum deviations being 4.09 % and 5.57 % for the maximum number of modes ($N = 10^3$).

Note that the optimum number of modes additionally depends on parameters such as pile-soil stiffness ratio, stiffness contrast between the layers, depth, as well as excitation frequency in presence of dynamic loads (e.g., Anoyatis and Mylonakis 2012). Normally, $N = 500$ modes ensure a satisfactory performance of the model in the range of realistic values for parameters met

in foundation engineering. In the ensuing, 10^3 modes are employed to provide results. Deviations from rigorous solutions should then be attributed to the inherent limitations associated with the underlying assumptions of the proposed analysis, rather than the number of modes employed.

An alternative representation of the results presented in Table 1 is provided in Table 2. Pile head stiffness is normalized using a weighted average Young's modulus $E_{s,aver}$. This can be considered a more appropriate normalization for the case of a two-layer soil:

$$E_{s,aver} = \frac{h_1}{H} E_{s1} + \frac{h_2}{H} E_{s2} \quad (49)$$

Note that for the soil-pile configurations included in Table 1, $E_p/E_{s,aver} = 333.33$.

Additional comparisons for normalized pile head stiffness are presented in Figs. 2 and 3. $K/(E_p d)$ and $K/(E_{s,aver} d)$ are plotted against pile slenderness L/d for selected values of the layer stiffness contrast E_{s2}/E_{s1} . Overall, the analytical predictions are in good agreement with the rigorous numerical results for all cases examined. As anticipated, the agreement improves with increasing E_p/E_{s1} (i.e., softer soils).

In Fig. 2, $K/(E_p d)$ decreases with pile slenderness. The decrease is stronger for pile slenderness ratios in the range $25 \leq L/d \leq 60$ and for high values of stiffness contrast between the soil layers. This effect is more pronounced for $E_{s2}/E_{s1} = 50$, $E_p/E_{s1} = 100$ and $L/d = 25$ (Fig. 2a). For a given stiffness of the surface layer (E_{s1}), an increase in stiffness of the bottom one (E_{s2}) leads to higher pile head stiffness. Similar trends are observed in Fig. 2 (b). However, the phenomena discussed earlier are suppressed (e.g., decrease in stiffness with pile slenderness is smoother and takes place along the entire range of L/d ratios examined).

Fig. 2 presents additional comparisons of pile head stiffness with results obtained from the Winkler solution of Mylonakis and Gazetas (1998). In that work pile head stiffness is expressed in closed-form as shown below

$$K = E_p A_p \lambda_1 \frac{(\lambda_1/\lambda_2) \tanh(\lambda_1 h_1) \tanh(\lambda_2 h_2) + 1}{(\lambda_1/\lambda_2) \tanh(\lambda_2 h_2) + \tanh(\lambda_1 h_1)} \quad (50)$$

where subscripts “1” and “2” refer to the upper and bottom layer, respectively. Quantity λ refers to the familiar Winkler parameter $\lambda_i = \sqrt{(k_i)/(E_p A_p)}$, $i = 1, 2$. The soil surrounding the pile is modelled by distributed springs along the pile periphery described by the Winkler moduli $k_1 = \delta G_{s1}$ and $k_2 = \delta G_{s2}$ for each layer. δ is a dimensionless parameter and for all results presented in this work is expressed via the expression proposed by Randolph and Wroth (1978) i.e., $\delta = 2 \pi / \ln \left(2 r_m / d \right)$. r_m is commonly termed as “magic radius” and refers to the horizontal distance from the pile at which the soil settlement is vanishingly small. For the results presented herein is taken as $r_m = 25 d$.

While the Winkler solution is not working well for stiff soils (Fig. 2a), it improves significantly for softer soils (Fig. 2b). Its best performance is shown for a homogeneous soil. All methods practically coincide for homogeneous soil and very long piles in layered soils independent of layer stiffness contrast. It is evident that the Winkler solution gradually deteriorates with increasing layer stiffness contrast. Maximum deviation is observed for shorter piles (i.e., $L/d = 30$) and high layer stiffness contrast ratios (i.e., $E_{s2}/E_{s1} = 50$). Within this range the proposed model yields more accurate results in comparison with FE analysis.

In Fig. 3, pile head stiffness results are alternatively normalized by the average soil deposit stiffness (Equation 49). This alternative representation reverses the trend: $K/(E_{s,aver} d)$ decreases with increasing E_{s2}/E_{s1} .

NUMERICAL RESULTS

In Fig. 4 stiffness results are presented as a function of layer stiffness contrast E_{s2}/E_{s1} for a relatively short ($L/d = 25$) and a long ($L/d = 50$) pile, four selected values of the pile-soil stiffness contrast (E_p/E_{s1}) and various depths of the layer interface (h_1/H). An increasing trend is observed for all curves. This increase is stronger for stiff soil (e.g., $E_p/E_{s1} = 100$) and shallow layer interfaces (e.g., $h_1/H = 0.25$). This effect diminishes with increasing E_p/E_{s1} and is more pronounced for short piles (Fig. 4 (a)). For very soft soil ($E_p/E_{s1} = 5000$), the variation of pile head stiffness with E_{s2}/E_{s1} is practically linear. As anticipated, for deep interfaces ($h_1/H = 0.90$), pile stiffness seems to be practically unaffected by E_{s2}/E_{s1} .

Besides pile head stiffness, it is of interest to investigate how load is transferred along the pile length. For this purpose, results for the following normalized parameters against depth (z/L) are shown in Figs. 5 to 7: pile displacement (normalized by its value at the head $w(z)/w_0$), axial stresses $[\sigma(z) A_p]/P$ along the pile and vertical soil reaction (side friction) at the soil-pile interface. Each of the Figs. 5 – 7 includes four subfigures which investigate the effects of layer stiffness contrast (a), pile-soil stiffness contrast (b), pile slenderness (c) and depth of layer interface (d).

Fig. 5 (a) shows that piles with slenderness $L/d = 25$ embedded in a homogeneous soil ($E_{s2}/E_{s1} = 1$), exhibit a practically linear variation of displacement with depth, which indicates a column-like behaviour. This behaviour changes with increasing layer stiffness contrast. Beyond the layer interface linearity vanishes and displacements die out exponentially. This is reasonably more pronounced for high values of E_{s2}/E_{s1} .

In Fig. 5 (b) it is evident that with increasing pile-soil stiffness contrast, for a specific slenderness ratio and layer stiffness contrast, the effect of the interface gradually diminishes for softer soils and practically vanishes for $E_p/E_{s1} = 5000$.

The effect of pile slenderness on the attenuation of the displacement with depth is investigated in Fig. 5(c). More slender piles lead to faster attenuation of pile displacement with depth. For very long piles ($L/d = 100$), a considerable drop in displacement is observed at the layer interface, which remains significant up to approximately the mid-length of the pile. On the other hand, for very short piles ($L/d = 10$) it is evident that the entire length contributes equally to the attenuation of displacement.

Additional results on the effect of the interface depth are presented in Fig. 5 (d). It is evident that the interface marks the point beyond which a stronger attenuation for pile displacement (settlement) occurs. This is anticipated as the bottom layer is stiffer than the upper for all cases examined.

Similar to Fig. 5, Figs. 6 and 7 present results for the profiles of normalized axial pile stresses and vertical soil reactions, respectively. Note that results shown in Fig. 6 are obtained using the following indirect equation to calculate axial stresses, which arises from the pile equilibrium (Equation 29):

$$\sigma^p(z) = \left(P - \pi d \int_0^z \tau(d/2, z) dz \right) / A_p \quad (51)$$

This equation is advantageous over the straightforward expression $\sigma^p = E_p w'(z)$. It reduces undulations in their profile and produces smooth curves with depth even when a small number of modes (e.g., $N = 10$) is used (Anoyatis 2013).

In Fig. 6 (a) all curves are compared to the one corresponding to homogeneous soil ($E_{s2}/E_{s1} = 1$), which decreases monotonically with depth. For $E_{s2}/E_{s1} > 1$ a change in gradient is observed at the layer interface. As anticipated, this is more pronounced for strong stiffness contrast between the soil layers. Lower values of E_{s2}/E_{s1} correspond to slightly stronger attenuation of stresses for depths up to the layer interface. Beyond that depth, the trend is reversed. The effect of layer stiffness contrast becomes more pronounced.

As shown in Fig. 6 (d), the shallower the interface, the stronger the effect on the variation of stresses. Further, stronger displacement attenuation is observed for lower E_p/E_{s1} values (Fig. 6 (b)) and higher L/d ratios (Fig. 6 (c)). Note that for $E_p/E_{s1} = 100$ and $L/d = 100$ attenuation is practically unaffected for depths greater than $0.70 L$.

Similar trends are observed in the behaviour of vertical shear stresses at the pile-soil interface, as shown in Fig. 7. For homogeneous soils ($E_{s2}/E_{s1} = 1$) peak values always develop at the level of the soil surface. This is attributed to the development of a boundary layer phenomenon (Anoyatis and Mylonakis 2012): soil reaction should obtain a minimum and a maximum value essentially at that same point. This is because the applied load is resisted at a maximum rate near the pile head, while at the same time shear traction should satisfy the condition of a free-stress soil surface. In presence of two soil layers, an additional boundary layer develops at the layer interface. The increase in soil reaction at this location cannot be overstated and is attributed to the discontinuity in soil shear modulus, expressed through the step function $\mathbf{H}()$ (Equation 2) in the solution. Specifically, this is a boundary layer effect (edge layer), encountered in various problems of elasticity theory associated with the presence of “corners” in geometry and/or discontinuities in material properties. Similar results have been presented by Poulos and

Davis (1968) and Banerjee and Butterfield (1971) and in other classical works. This is an inherent weakness of the model due to its “approximate” continuum nature.

WINKLER MODULUS

The depth-dependent Winkler modulus k can be obtained by dividing the vertical soil reaction per unit pile length (Equation 24) with the corresponding settlement at the pile-soil interface (Equation 47 for $r = d/2$) i.e.,

$$k(z) = 2 \pi G_s(z) \frac{\sum_m^\infty B_m s_{1m} K_1(s_{1m}) \Phi_m(z)}{\sum_m^\infty B_m K_0(s_{1m}) \Phi_m(z)} \quad (52)$$

The behavior of the Winkler modulus is investigated in Fig. 8. The monotonic decrease with depth observed for a homogeneous soil ($E_{s2}/E_{s1} = 1$) is altered due to the presence of the layer interface.

Fig. 8 (a) presents the effect of layer stiffness contrast on the Winkler springs for a relatively stiff pile ($E_p/E_{s1} = 1000$) with slenderness ratio $L/d = 25$. For depths down to the interface ($0 \leq z < h_1$), higher values of E_{s2}/E_{s1} correspond to higher k values. Below the interface though, no distinct trend is observed. However, curves illustrating low layer stiffness contrasts exhibit a trend similar to the homogeneous soil. As anticipated, this tendency becomes stronger moving downwards away from the interface.

The effect of pile-soil stiffness contrast on k for a shallow interface is examined in Fig. 8 (b). Higher values of E_p/E_{s1} correspond to lower values of k above the interface. This trend is still valid “just” below the interface ($\sim 0.30 H$) and is reversed below that depth.

The effect of pile slenderness on the variation of Winkler modulus is examined in Fig. 8 (c). The modulus is clearly decreasing with increasing pile slenderness. The presence of the interface does not affect this tendency. Finally, the layer interface is responsible for reversing the monotonic

decrease of k which is observed above the interface. For a given pile-soil configuration, as shown in Fig. 8 (d), the interface depth seems to affect significantly k values.

As a final remark, undulations in the Winkler modulus observed at the rigid base are anticipated, since at that level both soil and pile displacements are equal to zero. Evidently, this affects Equation (52), which exhibits an asymptotic behavior at $z = H$. From all subfigures in Fig. 8, it is obvious that the undulations are stronger for stiffer soil ($E_p/E_{s1} = 100$, Fig. 8 (b)) and longer piles ($L/d = 100$, Fig. 8 (c)). Such undulations can be mitigated by increasing the number of modes.

CONCLUSIONS

This study reports on the derivation of an analytical elastic solution of the Tajimi-type for the soil medium to investigate the behavior of axially-loaded end-bearing piles in a two-layer soil deposit. The proposed model yields closed-form solutions for displacement and stresses in the soil and pile. In particular, the pile head stiffness, pile settlement, axial pile stresses, vertical soil displacement, vertical reaction at the soil-pile interface and depth-dependent Winkler moduli, are obtained. The proposed predictions for pile head stiffness were verified against results from rigorous finite-element analyses.

The main conclusions of this study, focusing on new developments, may be summarized as follows:

1. Considering a set of static soil “modes” Φ (eigenfunctions) different to corresponding pile “modes” Y is fundamental to the solution. This is novel to analytical models of the Tajimi type and allows the solution of layered soil media.

2. The second novelty lies in the expansion of the second derivative of the *pile modes* in the equilibrium equation of the pile using Fourier series in terms of the *soil modes* (Equation 37). This is key to formulating the analytical solution.
3. Unlike the traditional Tajimi models for homogeneous soil, the Fourier coefficients B_m pertaining to soil response cannot be obtained independently. Solving a set of simultaneous algebraic equations is required, instead. This is also the case for the coefficients C_k pertaining to pile response (which are coupled with the B_m 's).
4. In addition to the soil surface, a boundary-layer phenomenon develops at the layer interface. Its effect on pile displacement and vertical soil reaction at the soil-pile interface, as well as on Winkler modulus, cannot be overstated and is attributed to the abrupt discontinuity (“jump”) of the soil shear modulus at that depth.
5. A large number of modes (10^2 to 10^3) must be employed to capture pile-soil interaction effects in a satisfactory manner. This is in contrast to structural analysis where the first mode often governs the response. Comparisons with finite element results in terms of pile head stiffness showed very good agreement (maximum discrepancies of less than 5%) using 500 modes. More modes are suggested to be used when calculating Winkler modulus (to avoid undulations especially close to the pile base). Overall, the more slender the pile and the stiffer the soil, the higher the number of modes to be used in the analysis.
6. Axial pile stresses were calculated indirectly using the pile equilibrium equation by integrating the side friction along the pile. This alternative approach (Equation 50) avoids undulations associated with the differentiation of pile displacement, even when a small number of modes is employed.

7. For soft soils ($E_p/E_{s1} = 1000$) and a relatively shallow layer interface ($h_1/H = 0.25$), the Winkler modulus k attains values in the range $2 \lesssim k/G_{s1} \lesssim 3$ above depths $z = h_1$. These values pertain to piles of intermediate length ($L/d = 25$) and stiffness contrast between soil layers ($1 \leq E_{s2}/E_{s1} \leq 10$). For depths $z > h_1$, k range of values broadens: $0.5 \leq k/G_{s2} \lesssim 2.5$.
8. The shorter the pile (e.g., $L/d = 10$) and the stiffer the soil (e.g., $E_p/E_{s1} = 100$) the more pronounced the boundary layer phenomena (i.e., at the surface and the layer interface), the more “curved” the representation of k values.

The applicability of the proposed solution can be strengthened by extending the analysis to account for a multi-layer soil deposit. Such model with l soil layers ($l = 2, 3, 4, \dots$) should essentially repeat $(l - 1)$ times the compatibility equations of stresses and displacements between the l soil layers and between each layer and the corresponding pile segment. Evidently the more layers used the “longer” the solution. This extension is mainly of computational interest and all fundamental principles and techniques are included in the current work. However, detailed guidance along with equations on how to implement a multi-layer approach are included in Appendix IV.

As a final remark note it is fair to mention that the applicability of the current solution is limited by the assumption of the elastic soil medium and perfect bonding at the pile-soil interface. With reference to the latter, the extension of the “Tajimi” solutions to account for non-linearity in non-homogeneous soils is possible (Akiyoshi 1982) for a homogeneous soil layer. However, it lies beyond the scope of this work. In short, the displacements of end-bearing piles are sufficiently small to justify this assumption. On the other hand, pile-soil interface could indeed be highly nonlinear in floating piles.

APPENDIX I. TAJIMI'S ASSUMPTION AND COMPRESSIBILITY COEFFICIENT

Vertical normal stress σ_z in an elastic medium using cylindrical coordinates

$$\sigma_z/G_s = 2\varepsilon_z + \frac{2\nu}{1-2\nu} (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) \quad (53)$$

Radial normal stress σ_r in an elastic medium using cylindrical coordinates

$$\sigma_r/G_s = 2\varepsilon_r + \frac{2\nu}{1-2\nu} (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) \quad (54)$$

where ε_r , ε_θ and ε_z are the normal strains in directions r , θ and z respectively.

In the Nogami and Novak (1976) approach $\varepsilon_r = 0$ and $\varepsilon_\theta = 0$ (i.e., $u_r = 0$) and thus Equation (53) yields:

$$\sigma_z/G_s = \frac{2(1-\nu)}{1-2\nu} \varepsilon_z = \eta_s^2 \frac{\partial u_z}{\partial z} \quad (55)$$

where $\eta_s^2 = 2(1-\nu)/(1-2\nu)$.

In this work applying the assumptions of $\sigma_r = 0$ and $\varepsilon_\theta = 0$ Equation (54) yields:

$$\varepsilon_r = -\frac{\nu}{1-\nu} \varepsilon_z \quad (56)$$

Under the assumption of $\varepsilon_\theta = 0$ Equation (53) is re-written as:

$$\sigma_z/G_s = \frac{2(1-\nu)}{1-2\nu} \varepsilon_z + \frac{2\nu}{1-2\nu} \varepsilon_r \quad (57)$$

which includes the additional term related to ε_r and is not present in Equation (55). Thus, Equation (53) taking into account Equation (56) is expressed as:

$$\sigma_z/G_s = \frac{2}{1-\nu} \varepsilon_z = \eta_s^2 \frac{\partial u_z}{\partial z} \quad (58)$$

where $\eta_s^2 = 2/(1-\nu)$.

APPENDIX II. CLOSED-FORM SOLUTIONS FOR INTEGRALS

Solutions for integrals used in the derivations:

$$\int_0^{h_1} Y_k(z) \Phi_{1m}(z) dz = \frac{a_m \cos b_k h_1 \sin a_m h_1 - b_k \cos a_m h_1 \sin b_k h_1}{a_m^2 - b_k^2} \quad (59)$$

$$\begin{aligned} \int_{h_1}^H Y_k(z) \Phi_{2m}(z) dz \\ = \frac{\cos a_m h_1}{\sin a_m h_2} \frac{b_k \sin a_m h_2 \sin b_k h_1 - a_m \cos a_m h_2 \cos b_k h_1}{a_m^2 - b_k^2} \end{aligned} \quad (60)$$

$$\int_0^H Y_k(z) \Phi_m(z) dz = - \frac{a_m \cos b_k h_1 \cos a_m H}{\sin a_m h_2 (a_m^2 - b_k^2)} \quad (61)$$

$$\begin{aligned} \int_0^H G_s(z) Y_k(z) \Phi_m(z) dz \\ = G_{s1} \frac{b_k \cos a_m h_1 \sin b_k h_1 \left[\frac{G_{s2}}{G_{s1}} - 1 \right] + a_m \cos b_k h_1 \sin a_m h_1 \left[1 - \left(\frac{\eta_{s1}}{\eta_{s2}} \right)^2 \right]}{a_m^2 - b_k^2} \end{aligned} \quad (62)$$

$$\begin{aligned} \int_0^{h_1} Y'_k(z) \Phi_{1m}(z) dz \\ = a_m b_k \frac{b_k \cos b_k h_1 \sin a_m h_1 - a_m \cos a_m h_1 \sin b_k h_1}{a_m^2 - b_k^2} \end{aligned} \quad (63)$$

$$\begin{aligned}
& \int_{h_1}^H Y'_k(z) \Phi_{2m}(z) dz \\
&= a_m b_k \frac{\cos a_m h_1}{\sin a_m h_2} \frac{a_m \sin a_m h_2 \sin b_k h_1 - b_k \cos a_m h_2 \cos b_k h_1}{a_m^2 - b_k^2}
\end{aligned} \tag{64}$$

$$\int_0^H Y'_k(z) \Phi'_m(z) dz = - \frac{a_m b_k^2 \cos b_k h_1 \cos a_m H}{\sin a_m h_2 (a_m^2 - b_k^2)} \tag{65}$$

$$\int_0^{h_1} \Phi_{1m}^2(z) dz = \frac{2 a_m h_1 + \sin 2 a_m h_1}{4 a_m} \tag{66}$$

$$\int_{h_1}^H \Phi_{2m}^2(z) dz = \left(\frac{\cos a_m h_1}{\sin a_m h_2} \right)^2 \frac{2 a_m h_2 + \sin 2 a_m h_2}{4 a_m} \tag{67}$$

APPENDIX III. INTEGRALS FOR A HOMOGENEOUS SOIL

Integrals from the proposed model which reduce to simple expressions describing a homogeneous soil when soil and pile are described by the same modes:

$$\int_0^H Y'_k(z) \Phi'_m(z) dz \rightarrow \int_0^H (Y'_k(z))^2 dz = \frac{H}{2} p_k^2 \tag{68}$$

$$\int_0^H Y_k(z) \Phi_m(z) dz \rightarrow \int_0^H Y_k^2(z) dz = \frac{H}{2} \tag{69}$$

$$\int_0^H G_s(z) Y_k(z) \Phi_m(z) dz \rightarrow G_s \int_0^H Y_k^2(z) dz = \frac{H}{2} G_s \quad (70)$$

APPENDIX IV. EXTENSION FOR A MULTI-LAYER SOIL

Equations (1) and (2) are generalised to express the variation of the shear modulus $G_s(z)$ for a layered deposit with l soil layers

$$G_s(z) = G_{s1} + (G_{s2} - G_{s1}) \mathbf{H}(z - H_1) + \dots + (G_{sl} - G_{s(l-1)}) \mathbf{H}(z - H_{(l-1)}) \quad (71)$$

where the step function is now written as:

$$\mathbf{H}(z - H_{(l-1)}) = \begin{cases} 0, & 0 \leq z < H_{(l-1)} \\ 1, & H_{(l-1)} < z \leq H \end{cases} \quad (72)$$

l is a positive integer for which $l \geq 2$. $H_{(l-1)}$ is a new parameter which denotes the distance of the bottom of the $(l - 1)^{th}$ layer from the surface of the deposit and is expressed as:

$$H_{(l)} = h_1 + h_2 + \dots + h_{(l)} \quad (73)$$

where $h_{(l)}$ denotes the thickness of the l^{th} soil layer.

Equation (25) for the m^{th} eigenfunction Φ_m of the soil deposit is written in the following generalised form

$$\Phi_m(z) = \Phi_{1m} + (\Phi_{2m} - \Phi_{1m}) \mathbf{H}(z - H_1) + \dots + (\Phi_{lm} - \Phi_{(l-1)m}) \mathbf{H}(z - H_{(l-1)}) \quad (74)$$

where $\Phi_{(l)m}$ is the m^{th} eigenfunction associated with the l^{th} soil layer.

Based on Equation (16) the displacement of the 1st layer will always be expressed as follows

$$u_1(r, z) = B_1 K_0(q_1 r) \cos az, \quad 0 \leq z \leq H_1 \quad (75)$$

due to the condition of vanishing normal stresses at the soil surface. The general solution for all intermediate layers as well as for the bottom layer is retrieved from Equation (15)

$$u_{(l)} = K_0(q_{(l)} r) [C_{(l)} \sin az + D_{(l)} \cos az] \quad (76)$$

where $q_{(l)} = a \eta_{s(l)}$.

The eigenvalues a can be calculated considering the compatibility of displacements and stresses at the layer interfaces and the boundary conditions at the surface and bottom of the deposit. At the bottom $u_{(l)}(r, H) = 0$. Thus, $D_{(l)}$ can be expressed as a function of $C_{(l)}$:

$$D_{(l)} = - \frac{\sin aH}{\cos aH} C_{(l)} \quad (77)$$

Compatibility of displacements at the layer 1 – layer 2 interface (i.e., $u_1(r, H_1^-) = u_2(r, H_1^+)$) yields:

$$\frac{K_0(q_1 r)}{K_0(q_2 r)} = \frac{C_2 \sin aH_1 + D_2 \cos aH_1}{B_1 \cos aH_1} \quad (78)$$

Compatibility of displacements at the interface between intermediate layers (i.e., $u_{(l-1)}(r, H_{(l-1)}^-) = u_{(l)}(r, H_{(l-1)}^+)$) yields:

$$\frac{K_0(q_{(l-1)} r)}{K_0(q_{(l)} r)} = \frac{C_{(l)} \sin aH_{(l)} + D_{(l)} \cos aH_{(l)}}{C_{(l-1)} \sin aH_{(l-1)} + D_{(l-1)} \cos aH_{(l-1)}} \quad (79)$$

From Equation (19) compatibility of stresses at the layer 1 – layer 2 interface (i.e., $\sigma_1(r, H_1^-) = \sigma_2(r, H_1^+)$) yields:

$$(D_2 \sin aH_1 - C_2 \cos aH_1) - \frac{\eta_{s1}^2}{\eta_{s2}^2} \frac{G_{s1}}{G_{s2}} \frac{K_0(q_1 r)}{K_0(q_2 r)} \sin aH_1 B_1 = 0 \quad (80)$$

from which D_2 can be expressed as a function of C_2

$$D_2 = \frac{\cos^2 aH_1 + \frac{\eta_{s1}^2}{\eta_{s2}^2} \frac{G_{s1}}{G_{s2}} \sin^2 aH_1}{\left(1 - \frac{\eta_{s1}^2}{\eta_{s2}^2} \frac{G_{s1}}{G_{s2}}\right) \cos aH_1 \cos aH_1} C_2 \quad (81)$$

Compatibility of stresses at the interface between intermediate layers (i.e., $\sigma_{(l-1)}(r, H_{(l-1)}^-) = \sigma_{(l)}(r, H_{(l-1)}^+)$) yields:

$$\left[C_{(l)} \cos aH_{(l)} - D_{(l)} \sin aH_{(l)} \right] - \frac{\eta_{s(l-1)}^2}{\eta_{s(l)}^2} \frac{G_{s(l-1)}}{G_{s(l)}} \frac{K_0(q_{(l-1)}r)}{K_0(q_{(l)}r)} = 0 \quad (82)$$

which can be further simplified by substituting Equation (79).

Finally, applying the additional conditions of perfect bonding between each layer and the associated pile segment one can calculate all constants (i.e., $B_1, C_{(l)}, D_{(l)}$, for $l \geq 2$).

NOTATION

The following symbols are used in this paper:

- A_p = pile cross-sectional area ($= \pi d^2/4$);
- A, B, C, D = integration constant;
- B_m = Fourier coefficient pertaining to soil;
- C_k = Fourier coefficient pertaining to pile;
- d = pile diameter;
- E_p = pile Young's modulus;
- E_{s1}, E_{s2} = soil Young's moduli for layer 1 and layer 2, respectively;
- F_n = Fourier coefficient pertaining to body forces;
- G_{s1}, G_{s2} = soil shear moduli for layer 1 and layer 2, respectively;

H = soil deposit thickness ($= L$);
 $\mathbf{H}$ = unit step (Heaviside) function;
 h_1 = upper layer thickness;
 h_2 = lower layer thickness;
 $I_0()$ = modified Bessel function of zero order and second kind;
 $I_1()$ = modified Bessel function of first order and second kind;
 $k(z)$ = depth-dependent Winkler modulus;
 $K_0()$ = modified Bessel function of zero order and first kind;
 $K_1()$ = modified Bessel function of first order and first kind;
 L = pile length ($= H$);
 m, k, n, j = real positive integer;
 P = pile head load;
 p_k = pile eigenvalues;
 q_{1m}, q_{2m} = soil parameter related to eigenvalues;
 R_j, S_i = Fourier coefficients pertaining to series expansion of Y_k'' and Y_k ;
 r = radial coordinate;
 s_{1m}, s_{2m} = dimensionless parameters ($= q_m d/2$);
 Y = pile eigenfunctions (modes);
 u = vertical soil displacement;
 w = vertical pile displacement;
 z = vertical coordinate;
 a_m = eigenvalues;

η_{s1}, η_{s2} = soil compressibility parameter;

ν_{s1}, ν_{s2} = soil Poisson's ratio;

σ = soil vertical normal stress;

σ^p = axial stress acting vertically on pile cross section;

τ = soil vertical shear stress; and

$\Phi, \Phi_m, \Phi_1, \Phi_2$ = soil eigenfunctions (modes);

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Table 1. Comparisons of the proposed pile head stiffness, for selected number of modes, with results from finite element analysis; $E_p/E_{s1} = 1000$, $E_{s2}/E_{s1} = 5$, $h_1/H = 0.5$, $\nu_{s1} = \nu_{s2} = 0.4$

$\frac{L}{d}$	pile head stiffness $K/(E_p d)$ (10^{-2})								
	FE	proposed model (P)				Deviation (%)			
	(F)	Number of modes N				$\frac{P - F}{F} \times 100 \%$			
	ANSYS								
		20	100	500	1000	20	100	500	1000
30	3.498	3.692	3.650	3.642	3.641	5.55	4.35	4.12	4.09
36	3.150	3.333	3.292	3.284	3.283	5.81	4.51	4.25	4.22
45	2.815	2.991	2.950	2.942	2.941	6.25	4.80	4.51	4.48
60	2.495	2.668	2.625	2.617	2.616	6.93	5.21	4.89	4.85
72	2.341	2.517	2.471	2.462	2.461	7.52	5.68	5.17	5.13
90	2.190	2.374	2.323	2.313	2.312	8.40	6.07	5.62	5.57

Table 2. Comparisons of the proposed pile head stiffness using an alternative representation, for selected number of modes, with results from finite element analysis; $E_p/E_{s1} = 1000$, $E_{s2}/E_{s1} = 5$, $h_1/H = 0.5$, $\nu_{s1} = \nu_{s2} = 0.4$

$\frac{L}{d}$	pile head stiffness $K/(E_{s,aver} d)$				
	FE (F)	proposed model (P)			
		Number of modes N			
		ANSYS	20	100	500
30	11.66	12.31	12.17	12.140	12.136
36	10.50	11.11	10.97	10.946	10.943
45	9.38	9.97	9.83	9.806	9.803
60	8.32	8.89	8.75	8.723	8.720
72	7.80	8.39	8.236	8.206	8.203
90	7.30	7.913	7.74	7.710	7.706