Rallabandi, B., Eggers, J. G., Herrada, M. A., & Stone, H. (2021). Motion of a tightly fitting axisymmetric object through a lubricated elastic tube. *Journal of Fluid Mechanics*, 926, 721-761. [A27].
<https://doi.org/10.1017/jfm.2021.712>

Peer reviewed version

License (if available):
CC BY-NC-ND

Link to published version (if available):
[10.1017/jfm.2021.712](https://doi.org/10.1017/jfm.2021.712)

[Link to publication record in Explore Bristol Research](#)
PDF-document

This is the author accepted manuscript (AAM). The final published version (version of record) is available online via Cambridge University Press at 10.1017/jfm.2021.712. Please refer to any applicable terms of use of the publisher.

University of Bristol - Explore Bristol Research

General rights

This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available:
<http://www.bristol.ac.uk/red/research-policy/pure/user-guides/ebr-terms/>

Force on an axisymmetric intruder moving through a lubricated elastic tube

Bhargav Rallabandi^{1†}, Jens Eggers² and Howard A. Stone³

¹Department of Mechanical Engineering, University of California, Riverside, California 92521, USA,

²School of Mathematics, University of Bristol, Fry Building, Woodland Road, Bristol BS8 1UG, United Kingdom,

³Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, NJ 08544, USA

(Received xx; revised xx; accepted xx)

We consider the translation of a rigid, axisymmetric, tightly-fitting object through a fluid-filled, cylindrical, elastic tube under applied axial and normal stresses. The intruding object is assumed to be slender and greater in size than the nominally undeformed tube radius, forcing solid-solid contact in the absence of relative motion between the surfaces. The motion of the object establishes a thin liquid film that lubricates this contact. We show that for small translation speeds, the force on the intruding object depends on the slope of its surface at the entrance region to the thin fluid film, and scales as the square root of the relative speed. As a consequence, asymmetric intruders experience a lower force when traveling narrow-end-first through soft tubes. We then analyze the effect of the axial tension in the tube, which is caused by the friction between the intruding object and tube itself. In the limit of small speeds, this tension leads to a reduction in the drag on the intruder, with this effect being reversed at larger speeds. This correction due to tension disappears in the limit of small deformation relative to the tube radius.

1. Introduction

The interaction between fluid flow and deformable boundaries is a topic with a wide range of applications including in blood flow, geophysics, biophysics and microrheometry. Relative motion between two surfaces separated by a fluid establishes a thin fluid film that is capable of supporting lift forces (Sekimoto & Leibler 1993; Skotheim & Mahadevan 2004; Szeri 2010; Urzay *et al.* 2007) and torques (Rallabandi *et al.* 2017; Saintyves *et al.* 2020) as well as reducing drag (Dowson & Higginson 1959, 1966; Saintyves *et al.* 2016). While many recent studies consider the case of nearly planar walls (Essink *et al.* 2020; Snoeijer *et al.* 2013), problems such as the motion of eggs through the oviduct (Bradfield 1951; Salamon & Kent 2014), the flow of red blood cells or vesicles through very narrow capillaries (Barakat & Shaqfeh 2018a; Freund 2014; Secomb *et al.* 1986; Vann & Fitzgerald 1982), or medical procedures such as vascular intervention Vurgaft *et al.* (2019) require an elastic boundary whose reference state is cylindrical or more generally curved. Introducing a solid object (which in the following we call the intruder) too large to fit into the undeformed cylinder produces a hoop stress that holds the intruder in place. The “dry” version of this problem (Rallabandi *et al.* 2019) revealed the interplay between hoop stresses and longitudinal stretching, which is characteristic of the cylindrical geometry. In the present paper we take the tube to be filled with fluid, and apply a force to the

† Email address for correspondence: bhargav@engr.ucr.edu

intruder (as if pulled by a piece of string), so that thin lubricating fluid layers reduce the friction between the tube and the intruder. Hoop stresses due to the curved boundary, the tension of the membrane, and viscous stresses are coupled with the pressure of the fluid flow.

To our knowledge, Lighthill (1968) was the first to treat a variant of this problem, considering a (possibly elastic) object pushed through an elastic tube by a pressure gradient. Lighthill (1968) made a parabolic approximation for the intruder shape, and only considered hoop stresses in his analysis. We will see below that even in the limit of low speeds, this is not possible, and stretching of the membrane always comes into play. In subsequent publications (Fitz-Gerald 1969; Secomb *et al.* 1986; Vann & Fitz-Gerald 1982), the pressure-driven model was refined further to account for the specifics of the motion of red blood cells through capillaries, including the effects of asymmetry, its detailed elastic response, and the tube porosity. The pressure-driven motion of close-fitting vesicles in tubes has been analyzed using both asymptotic analysis (Barakat & Shaqfeh 2018*a*) and numerical simulations (Barakat & Shaqfeh 2018*b*). By contrast, Chatkoff (1975) considered a different driving mechanism: the motion of ovoids through tubes deforming peristaltically. Takagi & Balmforth (2011) analyzed the peristaltic pumping of a small (i.e., narrower than the tube) object suspended in a fluid-filled tube. Vurgaft *et al.* (2019) consider the flow in an elastic tube due to the insertion of a narrower rigid cylinder.

We focus on the drag force required to move an axisymmetric intruder through an elastic tube. Our formulation of the problem accounts for both hoop stresses and the membrane stresses in the tube through a Föppl–von Karman-like description of the tube elasticity (Audoly & Pomeau 2010; Landau & Lifshitz 1984), which is coupled to the lubrication pressure through the fluid film. We analyze this framework for intruders with arbitrary axisymmetric shape. Performing a systematic asymptotic analysis in the limit of small velocities, we find analytically that the force only depends on the slope of the intruder at the leading edge and is insensitive to other details of its shape. Numerical solutions of the lubrication equations allow us to obtain a quantitative description of the force over the entire parameter space of relative velocity and tube deformation, and over a wide range of intruder shapes. We demonstrate good agreement with theoretical calculations.

In Sec. 2 below, we introduce the problem and set up its mathematical structure, starting with the simplest description of the tube in which it is deformed solely due to the fluid pressure, and only the hoop stress in the membrane is taken into account. As we will see, this approximation is valid in the limit of small deformations (compared with the undeformed tube radius), but works quite well for moderate deformations. In Sec. 3, we solve the resulting model for arbitrary, but slender, intruder shapes in the singular limit of small speeds, the reference speed corresponding to a situation where the thickness of the lubricating film is comparable with the “dry” elastic deformation of the tube. In Section 4 these results are validated and extended to moderate speeds with the help of numerical solutions of the lubrication equations, valid for slender intruders. In particular, we show that asymmetric intruders experience a lower resistance if they enter with their narrow end. We then revisit the elastic response of the tube in section 5, including the effects of both membrane tension and hoop stresses to build a more comprehensive theory of the coupled fluid-elastic problem. We find that membrane tension has to be included, even in the limit of small speeds; however, corrections to the drag are generally small, and vanish for small deformations. Our findings are compared in detail to numerical simulations of the lubrication equations. We conclude with a discussion in Sec. 6.

FIGURE 1. Sketch of the cross-section of the setup showing an axisymmetric intruder translating through an elastic tube (red solid line) filled with liquid, as seen in a reference frame attached to the intruder, which moves from right to left. The radial excess of the intruder beyond the undeformed tube (dashed line) is characterized by $\delta(z)$ and determines a “dry contact” region of length ℓ , as indicated in (a). Relative motion between the intruder and the tube is supported by a fluid film of thickness $h(z)$, as shown in (b), which is to be determined.

2. Problem setup

We consider the motion of a rigid axisymmetric intruder through a fluid-filled cylindrical elastic tube, as illustrated in Fig. 1, for the case that the particle translates from right to left with its narrower end first. In its undeformed (stress-free) state, the tube has a radius a and walls of thickness $b \ll a$. The tube is assumed to be made of a linearly elastic material with Young’s modulus E and Poisson ratio ν . An axisymmetric intruder with radius $R(z)$ moves through the tube at a prescribed steady speed V due to the application of a force F . Our objective is to calculate the thickness $h(z)$ of the lubricating film, and thus the force necessary to translate the intruder through the tube.

We restrict our attention to intruders whose size exceeds the radius of the tube, and define the “radial excess” of the intruder relative to the tube by $\delta(z) = R(z) - a$, assuming it has a single maximum δ_m (see Fig. 1). In the absence of any motion, and for negligible bending rigidity, the intruder contacts and deforms the tube in the region where $\delta(z) > 0$, which defines the “dry” contact region (Rallabandi *et al.* 2019). Since $\delta(z)$ has a single maximum, in the absence of motion there is a single contiguous region of dry contact of the intruder with the tube, with length ℓ as shown in Fig. 1(a). The motion of the intruder in the presence of a lubricating fluid breaks solid-solid contact by dynamically establishing a fluid film between the intruder and the deformed tube; Fig. 1(b).

As shown in Fig. 1, it is convenient to analyze the problem in a reference frame attached to the intruder, so that the tube wall now has a rightward speed V , and the nominal “dry” contact lines are fixed at $z = 0$ and $z = \ell$. The relative motion establishes a thin fluid film of thickness $h(z)$. Thus, the local radial deformation of the tube wall is $u(z) = \delta(z) + h(z)$. The shape of the thin film $h(z)$ is unknown a priori and must be determined as a part of the fluid-elastic problem.

We neglect the inertia of both the fluid and the intruder and use the lubrication approximation to describe the flow in the film. For this approximation to be valid throughout, we have to assume that the intruder has a slender shape: $\delta_m/\ell \ll 1$. For a calculation of the force alone, even this assumption can be dropped in some limits; we will discuss later the self-consistency of our approximations. As usual, in lubrication the flow profile in the film is quadratic in the transverse coordinate, and the pressure is constant across the film (Batchelor 1967). Then at steady state, $h(z)$ satisfies

$$\frac{Vh}{2} - \frac{h^3}{12\mu} \frac{dp}{dz} = q \equiv \frac{Vh^*}{2}, \quad (2.1)$$

where $p(z)$ is the fluid pressure, μ fluid viscosity and q is the volumetric flux per circumferential length of the tube. For convenience, we have introduced $h^* \equiv 2q/V$ as the effective gap thickness that can support the flux q in axial shear flow between locally cylindrical surfaces; its value is determined as part of the solution. The fluid pressure $p(z)$ deforms the elastic tube surface. We will see in Section 5 that for small deformations we can neglect axial stresses (e.g., viscous stresses in the flow and tangential elastic stresses in the membrane) and obtain

$$p(z) = \frac{Eb}{a^2}u(z), \quad (2.2)$$

which is the result for a cylindrical tube, uniformly stretched in the radial direction (Audoly & Pomeau 2010). The quantity on the right of (2.2) is proportional to the hoop stress, characteristic of the cylindrical geometry (Rallabandi *et al.* 2019). The bending resistance of the tube scales as b^3 and is neglected in (2.2) under the assumption that the tube walls are thin ($b \ll \delta_m$).

Substituting (2.2) into (2.1) and noting that $u(z) = h(z) + \delta(z)$, results in an ordinary differential equation for $h(z)$ (given $\delta(z)$):

$$\frac{dh}{dz} - \frac{6\mu Va^2}{Eb} \frac{(h - h^*)}{h^3} + \frac{d\delta}{dz} = 0. \quad (2.3)$$

We solve this equation subject to the condition $u(z) = h(z) + \delta(z) \rightarrow 0$ as $z \rightarrow \pm\infty$, corresponding to vanishing radial deformation $u(z)$ (no pressure) far away from the intruder. We note that if the tube is closed at either end the fluid mechanics of the problem are qualitatively modified since all the fluid is now required to pass through the gap.

We rescale the equations in a way that is suited to the singular limit of small velocities. It is natural to define a dimensionless axial coordinate $Z = z/\ell$, and a dimensionless intruder shape $\Delta(Z) = \delta(z)/\delta_m$. This means that the geometrical contact area is between $Z = 0$ and $Z = 1$, where $\Delta = 0$ by definition of $\delta(z)$. For small velocities of the intruder, the film thickness h is also small, so that the second term in (2.3) is large and must balance the third term over the contact region where $Z = O(1)$, yielding the film thickness scale $h_s = (6\mu Va^2\ell/(Eb\delta_m))^{1/2}$. Defining $H(Z) = h(z)/h_s$ and $H^* = h^*/h_s$, we obtain

$$\lambda^{1/2} \frac{dH}{dZ} - \frac{(H - H^*)}{H^3} + S(Z) = 0, \quad \text{where} \quad \lambda = \frac{6\mu Va^2\ell}{Eb\delta_m^3} \quad (2.4)$$

is a characteristic ratio of viscous to elastic stresses and $S(Z) = d\Delta/dZ$ is the dimensionless slope of the intruder surface, which is by definition $O(1)$. The parameter λ can also be interpreted as a rescaled velocity and has counterparts in the analysis of lubricated Hertzian contacts (Essink *et al.* 2020; Snoeijer *et al.* 2013) and plays a role analogous to the capillary number in the seminal work of Bretherton (1961). The characteristic film thickness scale is $h_s = \lambda^{1/2}\delta_m$, and so the dimensionless deformation of the tube is $U(Z) = u(z)/\delta_m = \Delta(Z) + \lambda^{1/2}H(Z)$. The yet-unknown constant $H^* = h^*/h_s$ sets the flux through the thin film and must be determined as part of the solution. Thus, we have reduced the parameters of the problem to a single dimensionless quantity λ (apart from the shape of the intruder, encoded by the slope function $S(Z)$).

To calculate the shear stress and ultimately the force on the tube, we denote the fluid-facing normal to the tube surface by $\mathbf{n} \approx -\mathbf{e}_r + \mathbf{e}_z \frac{du}{dz}$, where $\mathbf{e}_r$ and $\mathbf{e}_z$ are unit vectors in the r and z directions (Fig. 1). We also introduce the axial fluid velocity $v(r, z)$ and the dimensionless pressure $P(Z) = p(z)/(Eb\delta_m/a^2) = U(Z) = \Delta + \lambda^{1/2}H$. Using (2.4)

and recalling (2.2), the axial stress on the tube is

$$\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{e}_z \Big|_{r=a+u(z)} = -p \frac{\partial u}{\partial z} - \mu \frac{\partial v}{\partial r} \Big|_{r=a+u(z)} \quad (2.5a)$$

$$= -\frac{Eb\delta_m^2}{a^2\ell} \left\{ P \frac{\partial U}{\partial Z} + \lambda^{1/2} \left(\frac{1}{6H} + \frac{H}{2} \frac{dP}{dZ} \right) \right\} \quad (2.5b)$$

$$= -\frac{Eb\delta_m^2}{a^2\ell} \left\{ \frac{1}{2} \frac{d(U^2)}{dZ} + \lambda^{1/2} \left(\frac{4H - 3H^*}{6H^2} \right) \right\}. \quad (2.5c)$$

The hydrodynamic force F necessary to hold the tube stationary relative to the lab is found by integrating (2.5c) over the tube surface. As noted earlier inertia is negligible so this force is equal in magnitude and opposite in sign to the force that must be applied in order to move the intruder. The first term in (2.5c) is an exact derivative and thus, after integration, does not contribute to the net force. As a result, the dimensionless force on the intruder, measured in units of a characteristic force scale $F_s = 2\pi Eb\delta_m^2/a$, is

$$\mathcal{F} = \frac{F}{F_s} = \lambda^{1/2} \int_{-\infty}^{\infty} \frac{4H - 3H^*}{6H^2} dZ. \quad (2.6)$$

In this formulation, we have assumed that the main contribution to the force comes from the neighborhood of the intruder, and have neglected the resistance coming from the Poiseuille flow far from it. This will certainly be true if the speed is small, since the force in (2.6) is proportional to $\sqrt{\lambda}$, while the far-field contribution is proportional to the speed itself.

This completes the general setup. To compute the force, and the integral (2.6), we need to find $H(Z)$ from integrating (2.4) or its dimensional form (2.3), which we will do numerically in Section 4. To find an analytical approximation, we consider the limit of small speeds, i.e., $\lambda \ll 1$.

3. Asymptotic solution for $\lambda^{1/2} \ll 1$

To find an analytical solution of (2.4), we consider the limit $\lambda^{1/2} \ll 1$, for which the problem splits into three parts. First, the “dry” contact region $0 < Z < 1$ along the length of the intruder (the “outer” region in the language of matched asymptotics), and two “inner” regions of width $\lambda^{1/2}$, located around the leftmost contact point $Z = 0$ (the entry region) and the rightmost point $Z = 1$ (the exit region), respectively.

3.1. Contact region (outer solution)

In the outer region $Z \in [0, 1]$ the gradient term in (2.4), which multiplies $\lambda^{1/2}$, can be neglected, and so the film thickness $H(Z)$ is the solution to the cubic equation

$$\frac{H - H^*}{H^3} = S(Z). \quad (3.1)$$

Observe that $H = H^*$ when $S = 0$; thus, H^* is identified with the film thickness at the maximum radial extent of the intruder. Solutions to (3.1) are of the form $H(Z) = f(S(Z); H^*)$ and require the solution of a third-order algebraic equation, as illustrated in Fig. 2(a). Roots have to be chosen such that $H(Z)$ remains continuous, with $H = H^*$ at the maximum; physically meaningful solutions can only be found for a finite range of combinations of $S(Z)$ and H^* .

To illustrate this point, it is convenient to think of H^* as the dependent variable; rearranging (3.1) gives

FIGURE 2. (a) The solution $H(Z)$ of (3.1) for a quadratic intruder shape, $\Delta = 4Z(1 - Z)$; $H = H^*$ for the maximum of Δ , at $Z = 1/2$. (b) The local film thickness H as a function of H^* for $S = S_0 > 0$ (red) and $S = S_1 < 0$ (black), as constrained by the outer solution (3.1). The dashed line $H = H^*$ corresponds to $S = 0$. The space of possible solutions sweeps from the red curve to the black curve across the intruder as the slope changes from S_0 to S_1 . Continuity with the asymptote $H(S \rightarrow 0) \rightarrow H^*$ implies that the upper branch (crosses) is unphysical. Thus, the outer solution restricts $H \leq 1/\sqrt{3S_0}$ and $H^* < 2/(3\sqrt{3S_0})$.

$$H^* = H(1 - H^2 S), \quad (3.2)$$

where we recall that $H(Z)$ and $S(Z)$ are functions of Z , but H^* is a constant that characterizes the entire solution. As shown in Fig. 2(b), (3.2) describes two types of curves depending on the sign of S . At the maximum, the two solutions have to be joined together continuously. For regions where $S < 0$, H^* grows monotonically with H , while for any value $S > 0$, H^* is only positive for values of H in the range $0 < H \leq 1/\sqrt{3S}$, attaining a maximum value of $2/(3\sqrt{3S})$ when $H = 1/\sqrt{3S}$. Because S depends on Z and H^* is a constant, the maximum permissible value of H^* is set by the maximum positive value of the slope S over the dry contact region. If the intruder shape is convex, this maximum slope occurs at $Z = 0$, and we obtain the upper bound (see Fig. 2)

$$H^* \leq \frac{2}{3\sqrt{3S_0}} \quad \text{where} \quad S_0 \equiv S(Z = 0), \quad (3.3)$$

which must be satisfied for (3.2) to admit solutions throughout the “outer” region $0 < Z < 1$.

Furthermore, since the local thickness $H(Z)$ must approach H^* as $S(Z) \rightarrow 0$, only the solution branch $H < 1/\sqrt{3S}$ is physically meaningful (Fig. 2b), leading to the upper bound

$$H(Z) \leq \frac{1}{\sqrt{3S_0}}. \quad (3.4)$$

Thus, for convex intruder shapes the slope at the entry region S_0 limits the space of possible solutions. This conclusion does not involve any detailed analysis of the entry region, but is rather a consequence of the outer problem. We now show that the structure of the entry region picks a unique value of H^* .

3.2. Boundary layers at entrance and exit (inner solutions)

Within the contact region, the film thickness is of order $h_s = \lambda^{1/2}\delta_m$ as noted in Sec. 2, while the typical length scale is ℓ , and thus gradients in the film thickness are small in the limit of small λ . However, at the edges of the contact region, located at $Z = 0$ and $Z = 1$, the film profile has to adapt to the shape of a tube of constant radius, and axial derivatives in (2.3) or (2.4) can no longer be neglected. In the scaling of (2.4), since S and H are of order one, balancing the first and the last term we find that boundary layers are of dimensionless size $\lambda^{1/2}$. This means that both the film thickness and the width of the boundary layers scale like $\lambda^{1/2}$, and so the lubrication approximation is not automatically satisfied for small λ . However within these thin boundary layers, $dh/dz = O(\delta_m/\ell)$, so the lubrication approximation remains valid for a slender intruder.

We first analyze the boundary layer near the leading edge ($Z = 0$) which, as we will show, uniquely determines the dimensionless flux H^* when combined with (3.1). This feature is analogous to many other, similar problems, such as the Bretherton problem for the motion of long bubbles in tubes (Bretherton 1961), or the motion of an elastic object translating next to an elastic wall (Snoeijer *et al.* 2013). The scaling of the film thickness remains unchanged from the outer problem, so we avoid introducing new “inner” variables for H . Rescaling $Z = \lambda^{1/2}\zeta$, we obtain the similarity equation

$$\frac{dH}{d\zeta} - \frac{(H - H^*)}{H^3} + S_0 = 0 + O(\lambda^{1/2}), \quad (3.5)$$

since the slope $S_0 = S(0)$ is approximately constant across the boundary layer.

This equation has to satisfy matching conditions for $\zeta \rightarrow \pm\infty$: towards the film ($\zeta \rightarrow \infty$), where the film thickness is varying slowly, H has to match the film thickness of the outer solution, and thus $H(\zeta \rightarrow \infty) = H_0 = f(S_0; H^*)$; cf. Fig. 3(a). Towards the tube of constant radius, the film thickness increases linearly with the slope of the intruder: $H'(\zeta \rightarrow -\infty) = -S_0$ (Fig. 3a). This latter condition is satisfied automatically as H increases towards $\zeta \rightarrow -\infty$, since the second term of (3.5) becomes smaller.

We now show that the condition as $\zeta \rightarrow \infty$ selects a unique solution to (3.5): substituting $H(\zeta) \approx H_0 + G(\zeta)$ into (3.5) and linearizing in G we find

$$\frac{dG}{d\zeta} = \frac{1}{H_0} \left(\frac{1}{H_0^2} - 3S_0 \right) G \implies G(\zeta) \propto \exp \left\{ \frac{1}{H_0} \left(\frac{1}{H_0^2} - 3S_0 \right) \zeta \right\}. \quad (3.6)$$

For the perturbation $G(\zeta)$ to remain small as $\zeta \gg 1$ requires that $H_0 \geq 1/\sqrt{3S_0}$. This condition on H_0 is only consistent with the previously obtained upper bound (3.4) if H_0 is exactly equal to $1/\sqrt{3S_0}$. Thus, using (3.1) we obtain

$$H^* = \frac{2}{3\sqrt{3S_0}}, \quad (3.7)$$

which shows that the boundary layer solution selects the largest possible flux H^* that permits solutions of the outer problem (3.1). Solving (3.2) for H , the film thickness at the entrance becomes

$$H_0 = \frac{1}{\sqrt{3S_0}}. \quad (3.8)$$

Since this is the film thickness in the vertical direction, the thickness perpendicular to the surface of the solid intruder is $H_0/\sqrt{1 + S_0^2}$.

FIGURE 3. The self-similar profiles (a) at the entry to the film, located at $Z = 0$, and (b) at the back of the film, located at $Z = 1$. Towards the left and right in (a) and (b), respectively, the profiles asymptote to the undeformed tube, represented by the dashed lines.

Separating variables, and using (3.7), (3.5) can be solved as

$$\zeta = -S_0^{-3/2} \int^{S_0^{1/2}H} \frac{G^3 dG}{(G + 2/\sqrt{3}) (G - 1/\sqrt{3})^2} \quad (3.9a)$$

$$= -\frac{H}{S_0} + \frac{1}{9S_0^2(H - H_0)} + \frac{8}{9\sqrt{3}S_0^{3/2}} \ln \left(\frac{H + 2H_0}{H - H_0} \right), \quad (3.9b)$$

which contains S_0 as the only parameter. The constant of integration has been chosen such that $H = -S_0\zeta + O(1/\zeta)$ for $\zeta \rightarrow -\infty$, which is the profile of the undeformed tube. This function is plotted in Fig. 3(a). Clearly, (3.9a) has the required linear growth for $\zeta \rightarrow -\infty$ (first term on the right). On the other hand, as H approaches H_0 for $\zeta \rightarrow \infty$, the second term on the right of (3.9a) dominates, and the profile approaches its limiting thickness like a power law:

$$H \approx H_0 + \frac{1}{9S_0^2\zeta}. \quad (3.10)$$

A boundary layer of size $\lambda^{1/2}$ also exists near the exit $Z = 1$, where we denote the slope as $S_1 = S(Z = 1) < 0$. However, H^* is now fixed, having been selected at the entrance as usual (see also Bretherton (1961); Snoeijer *et al.* (2013)). Introducing the similarity variable $\xi = (Z - 1)/\lambda^{1/2}$, (3.5) remains the same (with the substitution $S_0 \mapsto S_1$), but the matching conditions are reversed: $H(\xi \rightarrow -\infty) = H_1 = f(S_1; H^*)$ toward a film on the left, and $H'(\xi \rightarrow \infty) = S_1$ toward the tube on the right. Linearizing around a constant film thickness H_1 as before, we find

$$H(\xi) = H_1 + A \exp \left\{ \frac{1}{H_1} \left(\frac{1}{H_1^2} - 3S_1 \right) \xi \right\} \quad \text{as } \xi \rightarrow -\infty. \quad (3.11)$$

Now the prefactor in the exponential is always positive, so for $\xi \rightarrow -\infty$ the solution converges exponentially onto the correct boundary condition (the outer problem), as seen in Fig. 3(b), regardless of H^* . Explicit solutions to the similarity solution in the back,

$$\xi = \int^H \frac{G^3 dG}{G - H^* - S_1 G^3}, \quad (3.12)$$

can be found in principle using solutions of the cubic equation in the denominator, but the

result is uninspiring. The result now depends on the intruder shape at both the entrance and the exit through the values of S_0 and S_1 separately. This time $H = -S_1\zeta + O(1/\zeta)$ for $\zeta \rightarrow \infty$, approaching the tube.

3.3. Force on the intruder

The formula (2.6) can be simplified in the limit of $\lambda^{1/2} \ll 1$, by observing that the main contribution to the force integral is from the “contact” region $0 < Z < 1$, with the contribution from boundary layers being $O(\lambda^{1/2})$ smaller. Inside of the contact region, we can find $H(Z)$ from solving the cubic equation (3.1), where $S(Z)$ is given by the shape of the intruder:

$$\mathcal{F} = \lambda^{1/2} \int_0^1 \frac{4H - 3H^*}{6H^2} dZ + O(\lambda). \quad (3.13)$$

Note that at low speeds contributions to the force only come from the contact region, where the lubrication approximation is valid without assuming a slender intruder.

In general, the integral (3.13) has to be computed numerically. However, for slender shapes (small S), it follows that $H \approx H^*$ (the profile $H(Z)$ is approximated by the dashed line in Fig. 2(a)), and the integral becomes trivial. Using (3.7), we obtain

$$\mathcal{F} \simeq \frac{\sqrt{3}}{4} (\lambda S_0)^{1/2} \quad \text{for } \lambda^{1/2} \ll 1. \quad (3.14)$$

This surprisingly simple expression predicts that the dimensionless force depends solely on the slope of the intruder surface at the entrance region. We emphasize, however, that the stresses that contribute to this force are distributed over the entire contact region ($0 < Z < 1$) and is thus the force is dominated by the outer problem as discussed above. The dependence of $\mathcal{F}$ on local geometric information S_0 despite this non-local distribution of stress is due to the matching between the outer solution and the inner solution at the entrance, which sets the fluid flux $H^* \propto S_0^{-1/2}$ throughout film; see (3.7). The approximation (3.14) is valid for convex shapes for small λ but places no other restriction on the details of the intruder shape. We will show below that (3.14) is an excellent approximation for a wide range of shapes. In dimensional terms, the force on the intruder is

$$F \simeq \frac{3\sqrt{2}}{2} \pi (Eb\mu V)^{1/2} \ell \left(\frac{d\delta}{dz} \right)^{1/2} \Big|_{z=0}. \quad (3.15)$$

The slope term in (3.15) scales as $(\delta_m/\ell)^{1/2}$, so the force has a scale $(Eb\mu V \delta_m \ell)^{1/2}$. An interesting consequence of (3.14) is that fore-aft asymmetric objects experience a smaller drag force when they translate narrow-end-first (small S_0) when compared with the same object translating wide-end-first (large S_0). These conclusions are in quantitative agreement with numerical results as we will show below.

4. Comparison with numerical solutions

We now test the quality of our approximations by comparing to numerical simulations, based on the lubrication description. This means our results are valid in a strict sense only for slender intruders, for which $\delta_m \ll \ell$.

For a prescribed intruder shape $\Delta(Z)$ and value of the parameter λ , we solve (2.3) using a shooting procedure. Starting with a guess for H^* , we integrate (2.3) starting from large positive z , with the boundary condition of zero displacement $u(z)$ (or zero pressure), towards large negative z . We repeat this procedure until we converge on a

FIGURE 4. The film shape for an ellipsoidal intruder (a) Numerical results for the normalized radial displacement $U(Z)$ of the tube for a paraboloidal intruder ($\Delta = 4Z(1 - Z)$), translating to the left, with $\lambda = \{0.1, 1, 4\}$. The tube deformation relaxes over a length scale that increases with λ . (b,c) Profiles of film thickness H in the boundary layers near the entry region (b) and exit region (c), where numerical solutions (symbols) and theory (solid lines) are shown. Numerical solutions are computed for $\lambda = \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1\}$. Observe that the film thickness in the dry contact region $0 < Z < 1$ is relatively insensitive to λ .

unique value of H^* that results in zero tube displacement to the far left of the intruder (Fig. 1).

4.1. Symmetric intruders

We first focus on the generic slender intruder with both axisymmetry and fore-aft symmetry (e.g., an ellipsoid), which can be approximated by the quadratic function $\Delta(Z) = 4Z(1 - Z)$. In this case $S_0 = \Delta'(0) = 4$. Denoting the axial radius of curvature of the intruder by R_z (for a sphere $R_z = a + \delta_m$), the dry contact length is $\ell = 2\sqrt{2R_z\delta_m}$.

We first study the effect of λ on the shape of the liquid film. Figure 4(a) shows numerical results for the normalized displacement of the tube $U(Z) = u(z)/\delta_m = \Delta(Z) + \lambda^{1/2}H(Z)$ for this shape. The tube deformation relaxes over a length scale that increases with increasing λ corresponding, for example, to faster translation speeds or softer tubes. As $\lambda \rightarrow 0$, the solution adopts a boundary-layer structure with a dimensional axial width $\lambda^{1/2}\ell$, as expected from Sec. 3; solutions for the film near the entrance ($Z = 0$) and exit ($Z = 1$) regions are shown as functions of the rescaled boundary layer coordinates in Fig. 4(b) and Fig. 4(c), respectively. Numerical solutions are found to collapse onto the

FIGURE 5. (a) Rescaled flux H^* versus λ for the generic symmetric intruder $\Delta = 4Z(1 - Z)$, showing the asymptotic behavior $H^* = 1/(3\sqrt{3})$ at small λ (solid line) and the scaling behavior $H^* \propto \lambda^{-1/10}$ at large λ . Observe that the numerical value of H^* is relatively insensitive to λ up to $\lambda \simeq 10^4$. (b) Rescaled force $\mathcal{F}$ as a function of λ , showing the small- λ asymptotic results (3.14) (solid line), and the scaling laws $\mathcal{F} \propto \lambda^{2/3}$ and $\mathcal{F} \propto \lambda^{4/5}$ for intermediate and large λ , respectively.

theoretical predictions (3.9a) and (3.12) for values of $\lambda \lesssim 1$. The qualitative structure of the entry and exit layers remains unchanged even when $\lambda \gtrsim 1$.

Next, we test the dependence of the solution on λ more systematically, and over a very wide range. As seen in Fig. 5(a), the dimensionless fluid film thickness H^* (thickness at the maximum of the intruder), is relatively insensitive to λ , even over several orders of magnitude. The asymptotic result $H^* = 1/(3\sqrt{3})$ from (3.7) for $\lambda \ll 1$ (note that $S_0 = \frac{d\Delta}{dZ}(0) = 4$ for the chosen intruder shape) is indicated by the horizontal line. For $\lambda \gtrsim 1$, we expect H^* to still be $O(1)$.

However for $\lambda \gg 1$, both the axial length scale and the film thickness become much greater than the geometric scales imposed by the intruder, a trend which is already apparent in Fig. 4(a). At large axial distances from a paraboloidal intruder the surface slope grows as $S(Z) = \Delta'(Z) = O(Z)$. Requiring a balance of all three terms of (2.4) far away from the intruder ($\lambda^{1/2}dH/dZ = O(H^{-2}) = O(Z)$) reveals a film thickness $H = O(\lambda^{-1/10})$ with variations over a characteristic axial scale $Z = O(\lambda^{1/5})$ [i.e., $h = O(\delta_m \lambda^{2/5})$, $z = O(\ell \lambda^{1/5})$].

This scaling law for H is reflected in the behavior of H^* for $\lambda \gtrsim 10^2$, indicated by the dashed line in Fig. 5(a). It should however be noted that in this limit $dh/dz = O(\lambda^{1/5}\delta_m/\ell)$, which is assumed small in the theory, so the intruder must be sufficiently slender for the thin-film and the hoop-stress approximations to remain valid at large λ . Note also that paraboloidal approximation of the surface shape is valid over axial length scales $Z \lesssim R_z/\ell$, so the above scaling relations for large λ are valid if $\lambda^{1/5} \lesssim R_z/\ell$.

Using (3.15), the force on a symmetric intruder is approximately

$$\mathcal{F} \simeq (3\lambda/4)^{1/2} \quad \text{or} \quad F \simeq 6\pi (\mu V E b)^{1/2} (2R_z)^{1/4} \delta_m^{3/4} \quad \text{for} \quad \lambda \ll 1. \quad (4.1)$$

The dimensional result depends only weakly on the axial radius of curvature of the intruder R_z and does not explicitly depend on the tube radius a , although the two approximately coincide for spherical intruders with $\delta_m \ll a$. Fig. 5(b) shows numerical results (symbols) for the dimensionless force on the intruder, with the prediction of (4.1) indicated as a solid line, showing good agreement for $\lambda \lesssim 0.1$.

At intermediate values of $\lambda \gtrsim 1$, the axial length is still set by the intruder geometry,

λ range	H^*	Z	$\mathcal{F}$	F	$\max (dh/dz)$
$\lambda \ll 1$	$1/\sqrt{3}$	$O(1), O(\lambda^{1/2})$	$(3\lambda/4)^{1/2}$	$6\pi (\mu V Eb)^{1/2} (2R_z \delta_m^3)^{1/4}$	$O(\delta_m/\ell)$
$\lambda \gtrsim 1$	$O(1)$	$O(1)$	$\lambda^{2/3}$	$(\mu V)^{2/3} (Eb)^{1/3} (R_z a \delta_m)^{1/3}$	$O(\delta_m/\ell)$
$\lambda \gg 1$	$O(\lambda^{-1/10})$	$O(\lambda^{1/5})$	$\lambda^{4/5}$	$(\mu V)^{4/5} (Eb)^{1/5} (R_z^2 a^3)^{1/5}$	$O(\lambda^{1/5} \delta_m/\ell)$

TABLE 1. Summary of scaling laws in different ranges of λ (see equation (2.4)) for a generic fore-aft-symmetric intruder with axial curvature $1/R_z$ and a maximum radius $a + \delta_m$. The shape of such an intruder is $\Delta(Z) = 4Z(1 - Z)$. The results are valid for $\delta_m \ll \ell$, where $\ell = 2\sqrt{2\delta_m R_z}$. Prefactors for $\lambda \ll 1$ are provided by the asymptotic theory.

while the film thickness becomes comparable to δ_m (recall that the characteristic film thickness is $h_s = \lambda^{1/2} \delta_m$). A balance between the first two terms in (2.3) with $Z = O(1)$ determines $H = O(\lambda^{-1/6})$. Then, the force integral (2.6) yields the scaling estimate

$$\mathcal{F} \propto \lambda^{2/3} \quad \text{or} \quad F \propto (Eb \delta_m R_z a)^{1/3} (\mu V)^{2/3}.$$

This approximate scaling law at intermediate λ is indicated in Fig. 5(b) and accurately represents the numerical results in the range $0.1 \lesssim \lambda \lesssim 10$. For $\lambda \gg 1$, the characteristic scales $Z = O(\lambda^{1/2})$ and $H = O(\lambda^{-1/10})$, obtained earlier, yield the scaling law $\mathcal{F} \propto \lambda^{4/5}$ or $F \propto (\mu V)^{4/5} (Eb R_z^2 a^3)^{1/5}$ when substituted into (2.6). This prediction is consistent with numerical calculations for $\lambda \gtrsim 10^2$, as indicated in Fig. 5(b). The forgoing results are summarized in Table 1.

As indicated in Table 1, the necessary condition for the thin-film approximation ($dh/dz \ll 1$) to apply is that the intruder is slender: $\delta_m \ll \ell$ for moderate λ , and the slightly stricter condition $\lambda^{1/5} \delta_m \ll \ell$ for $\lambda \gtrsim 10^2$. For small λ , dh/dz scales as $\lambda^{1/2} \delta_m/\ell$ in the outer region and as δ_m/ℓ in the entry and exit regions. This means that the present thin-film analysis of the outer problem for $\lambda^{1/2} \ll 1$ remains valid even for non-slender intruders even though this approximation is inadequate in the entry and exit regions. Because the force is dominated by the outer problem (see Sec. 3.3), we speculate that $\mathcal{F} \propto \lambda^{1/2}$ may persist for non-slender intruders at small λ , although the modified flow in the entry region may select a different H^* , and therefore a different prefactor for $\mathcal{F}$. We note the sub-linear power-law dependence of the force on the velocity is qualitatively consistent with the experimental results of Tani *et al.* (2017).

4.2. Effect of fore-aft asymmetry

The result (3.14) for the hydrodynamic force on the intruder predicts a particularly simple dependence on its shape, including the effect of asymmetry. For example, one might ask which orientation of an egg traveling down an oviduct (Bradfield 1951) might offer the least resistance? Equation (3.14) clearly favors a small value of S_0 , which occurs if the intruder enters with its narrow side first.

We will now test these predictions using our numerical simulations, by introducing a family of intruder shapes whose degree of fore-aft asymmetry can be varied systematically.

FIGURE 6. (a) Dimensionless force $\mathcal{F}$ versus λ for objects of different shapes defined by (4.2). The drag force for pointy-end-first motion (here, $Z_m = 0.1$) is greatest, followed by a symmetric object ($Z_m = 0.5$) and then wide-end-first motion has the highest drag (here, $Z_m = 0.9$). Note that the two asymmetric cases correspond to the same physical intruder, but with their orientations flipped relative to the direction of motion. (b) Rescaled force $\mathcal{F}/\lambda^{1/2}$ versus the slope at the entry region $S_0 = S(Z = 0)$ for different values of λ for intruders prescribed by (4.2). The solid line is the prediction of (3.14), $\mathcal{F} \simeq (3\lambda S_0/16)^{1/2}$. The inset plots the same data on a logarithmic scale, showing that the scaling relation $\mathcal{F} \propto S_0^{1/2}$ holds approximately even when $\lambda = O(1)$.

For simplicity, we construct intruder shapes using a family of piecewise functions

$$\Delta(Z) = g(Z; Z_m) \equiv \begin{cases} 1 - \left(\frac{Z-Z_m}{-Z_m}\right)^2 & Z \leq Z_m \\ 1 - \left(\frac{Z-Z_m}{1-Z_m}\right)^2 & Z > Z_m, \end{cases} \quad (4.2)$$

parametrized by $0 < Z_m < 1$. Defined this way, $\Delta(Z)$ is convex with a maximum value of unity at $Z = Z_m$, and is continuous in value and slope throughout. For $Z_m = 1/2$ one obtains the previous parabolic shape, whereas shapes with $Z_m < 1/2$ are blunt on the left, which is the part that goes first. Conversely, shapes with $Z_m > 1/2$ are increasingly pointed toward the leading edge. Slopes are constant with $S_0 = 2/Z_m$ and $S_1 = -2/(1 - Z_m)$.

We vary Z_m between 0 and 1 to smoothly change the shape of the intruder and, consequently, the slope S_0 . We solve the problem numerically for $H(Z)$ and then compute the force on the intruder using (2.6). As predicted by (3.14), the force is indeed smaller for intruders with smaller slope S_0 at the entrance of the thin film contact region, as indicated in Fig. 6(a). This qualitative prediction remains valid over the entire range of λ . Thus, asymmetric intruders traveling narrow-end-first encounter a lower resistance than the same object traveling wide-end-first. This fore-aft symmetry breaking of the mobility of the object in spite of negligible inertia is a result of nonlinearities due to the coupling between the fluid flow and the tube deformation.

To test the dependence of $\mathcal{F}$ on S_0 , we vary the shape of the intruder through the parameter Z_m . Figure 6(b) shows the force as a function of the entry slope S_0 of the intruder. Symbols represent numerical results, while the solid curve is the prediction of (3.14) valid for small λ . We note that the force increases relative to this prediction for larger λ (cf. Fig. 5b), although the scaling $\mathcal{F} \propto S_0^{1/2}$ appears to be relatively robust over a wide range of λ (inset of Fig. 6(b)).

5. Effect of membrane tension on the force

In the previous sections, we assumed that the pressure in the film is generated exclusively by the hoop stress, which comes from stretching the tube in the radial direction. This results in (2.2), according to which the pressure is proportional to the radial deformation. However, viscous stresses in a thin film also generate shear forces on the tube, which normally are of lower order in the lubrication approximation. However, we will see that in the present problem shear generates a tension in the elastic tube, which remains finite even in the limit of vanishing speeds. As a result, it produces a correction to the propulsion force (3.14), even for slender shapes. However, our analysis shows that (3.14) is still valid in the limit of small deformations. We compute the corrections coming from the tension to leading order in the deformation.

5.1. General theory for cylindrical elastic shells

We treat the elastic problem within the axisymmetric elastic shell theory as described by Audoly & Pomeau (2010), which we specialize to cylindrical shells (i.e., tubes). In Rallabandi *et al.* (2019), we treat the “dry” version of the present problem, in which the tube is stretched by elastic contacts alone.

As before, we denote the undeformed radius of the tube’s mid-surface by a and the thickness of its walls by b . It is convenient to introduce a cylindrical coordinate system (r, θ, z) and define an arclength in the axial direction s , so that any point on the undeformed cylinder surface has a position $\mathbf{r}(s) = (r = a, z = s)$. Note that s is the arclength in the *undeformed* configuration (i.e., s labels material points), which is a subtle but important point for large axial deformations, although we will ultimately ignore this distinction here.

Applying the surface tractions (forces per area) $\mathbf{f}(s) = \{f_r(s), f_z(s)\}$ and line force densities (force per length) $\boldsymbol{\gamma}^{(i)} = (\gamma_r^{(i)}, \gamma_z^{(i)})$, $i \in \{1, 2\}$ along the circumference of the tube at the free ends $s = s_i$, $i \in \{1, 2\}$ deforms the tube. We define an twistless, axisymmetric deformation field $\mathbf{u}(s) = (u_r(s), u_z(s))$ associated with the mapping $\mathbf{r}(s) \mapsto \mathbf{r}(s) + \mathbf{u}(s)$. Note that the above description is a Lagrangian one that tracks material points labeled by s . The distinction between Eulerian and Lagrangian descriptions is small if axial gradients of the axial displacement are small, i.e., $u'_z(s) \ll 1$ where the primes denotes a derivative with respect to s .

We denote the local tangent to the surface by $\mathbf{t}(s) = (t_r(s), t_z(s))$ and the membrane strains in the s and θ directions, respectively, by ε_s and ε_θ . In the spirit of the approximations in the Föppl–von Karman equations (small strains, moderate rotations; see Audoly & Pomeau (2010); Landau & Lifshitz (1984)), we can write, for cylindrical tubes,

$$\varepsilon_s = u'_z + \frac{1}{2}(u'_r)^2, \quad \varepsilon_\theta = \frac{u_r}{a}, \quad (5.1a)$$

$$t_r \approx u'_r \quad \text{and} \quad t_z \approx 1, \quad (5.1b)$$

where we assume $u'_r \ll 1$.

The strain field produces so-called “membrane stresses” $\mathbf{N}$ (a vector with units of force per length) with components N_s and N_θ in the arclength and azimuthal directions; N_θ is often called the hoop stress and we refer to N_s as the membrane tension. For Hookean elasticity, the membrane stresses are given by

$$N_s = \frac{Eb}{1 - \nu^2}(\varepsilon_s + \nu\varepsilon_\theta) \quad \text{and} \quad N_\theta = \frac{Eb}{1 - \nu^2}(\varepsilon_\theta + \nu\varepsilon_s), \quad (5.2)$$

where E and ν are, respectively, the Young’s modulus and Poisson’s ratio of the material.

Local equilibrium of the membrane requires

$$\frac{d}{ds}(N_s t_r) - \frac{N_\theta}{a} + f_r = 0 \quad (5.3a)$$

$$\frac{d}{ds}(N_s t_z) + f_z = 0, \quad (5.3b)$$

with boundary conditions

$$N_s \mathbf{t} + \boldsymbol{\gamma}^{(1)} = \mathbf{0} \quad \text{on} \quad s = s_1 \quad \text{and} \quad -N_s \mathbf{t} + \boldsymbol{\gamma}^{(2)} = \mathbf{0} \quad \text{on} \quad s = s_2, \quad (5.4)$$

where the $\boldsymbol{\gamma}^{(i)}$ are forces applied at the two ends of the tube $s = s_i$. Note that we can identify s with the axial coordinate z in the *undeformed* state. This will not hold in general in the deformed state, i.e., a material point with position s will be mapped from $z = s$ to $z = s + u_z(s)$.

5.2. Thin-film flow coupled to membrane stresses

The tube must be held at $Z \rightarrow \infty$ with an axial force F to keep it stationary in the lab frame. This results in axial elastic stress (tension T), which we identify with N_s , and which may make an additional contribution to the pressure through the curvature of the tube. For simplicity we will make the approximation that the axial strains are small, so that $z \approx s$. The thin film equation (2.1) remains valid, as long as the intruder is slender. The fluid-facing normal to the tube surface is $\mathbf{n} = -\mathbf{e}_r + u'_r(z)\mathbf{e}_z$, and to leading order, the lubrication forces on the membrane are

$$f_r = p(z), \quad f_z = \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{e}_z|_{\text{tube}} = -pu'_r(z) - \mu \frac{\partial v_z}{\partial r} \Big|_{\text{tube}} = -pu'_r(z) - p'(z) \frac{h}{2} - \frac{\mu V}{h}. \quad (5.5)$$

Using (5.1)–(5.2),

$$N_\theta = \frac{Eb}{1 - \nu^2} \left[\frac{u_r}{a} + \nu \left(u'_z + \frac{u_r'^2}{2} \right) \right] \quad \text{and} \quad T \equiv N_s = \frac{Eb}{1 - \nu^2} \left[\frac{\nu u_r}{a} + \left(u'_z + \frac{u_r'^2}{2} \right) \right]. \quad (5.6)$$

Eliminating u'_z between the two expressions, we obtain

$$N_\theta = Eb \frac{u_r}{a} + \nu T. \quad (5.7)$$

Substituting the above relation into (5.3) and using (5.5), we obtain ($T \equiv N_s$):

$$p = \frac{Eb}{a^2} u_r + \frac{\nu T}{a} - \frac{d}{dz} \left(T \frac{du_r}{dz} \right) \quad (5.8a)$$

$$\frac{dT}{dz} = p \frac{du_r}{dz} + \frac{dp}{dz} \frac{h}{2} + \frac{\mu V}{h}. \quad (5.8b)$$

Compared with (2.2) for the pressure, (5.8a) now contains two additional terms due to tension, where the tension is computed via (5.8b). Below we will see that in the limit of small displacements ($\delta_m \ll a$), the original formulation is recovered.

Substituting the z -derivative of (5.8a) into (2.1), and using $u_r = h + \delta$ for the radial deformation of the tube, eliminates the pressure to yield the first of a system of two

ordinary differential equations

$$\frac{du_r}{dz} + \frac{\nu a}{Eb} \frac{dT}{dz} - \frac{a^2}{Eb} \frac{d^2}{dz^2} \left(T \frac{du_r}{dz} \right) - \frac{6\mu V a^2}{Eb} \frac{(h - h^*)}{h^3} = 0 \quad (5.9a)$$

$$\frac{dT}{dz} = \frac{Eb}{2a^2} \frac{du_r^2}{dz} + \frac{\nu T}{a} \frac{du_r}{dz} - \frac{d}{dz} \left(T \frac{du_r}{dz} \right) \frac{du_r}{dz} + \mu V \frac{(4h - 3h^*)}{h^2}. \quad (5.9b)$$

The second equation (5.9b) is found from (5.8b), eliminating p using (5.8a), and dp/dz using (2.1). These equations are subject to the condition that T at $z \rightarrow -\infty$ is some imposed tension $T_{-\infty}$ and that the lubrication pressure decays, which then relates u_r and T as $Z \rightarrow \pm\infty$ through (5.9a). The particular case $T_{-\infty} = 0$ corresponds to a free end on the left, so that the tension T_{∞} at the right end exactly cancels the shear forces created by viscous friction, i.e., the force on the intruder. Thus, even with a free end, there is always tension to the right of the intruder that is needed to maintain steady-state translation.

We now consider again the limit of small velocities, for which h is small. In the “outer” (dry contact) region, balancing the singular term in (5.9a) with the du_r/dz term yields the same film thickness scale as before: $h_s = \left\{ (6\mu V a^2 \ell) / (Eb \delta_m) \right\}^{1/2}$. Balancing the term on the left of the equality in (5.9b) with the first term on the right yields the tension scale $T_s = Eb \delta_m^2 / a^2$. This contribution is produced by $p du_r/dz$ (see Eq. (5.8b)) and is due to the stretching of the tube over the intruder surface. Thus, the tension remains nonzero even in the dry (static) limit of vanishingly small speeds. However, its integral over z , which yields the force per circumference, is identically zero in the static limit, since in this case $dT/dz \propto d(u_r^2)/dz$, as $V = T = 0$. But this means that the right-hand side is an exact derivative, with u_r vanishing for $z \rightarrow \pm\infty$.

The parameter $\epsilon = \delta_m/a$, which is typically small, measures the deformation relative to the undeformed tube radius, while $k = a \delta_m / \ell^2$ is typically $O(1)$. Introducing the rescaled tension $\mathcal{T} = T/T_s$, the governing equations (5.9) rescale as

$$\frac{dU}{dZ} + \nu \epsilon \frac{d\mathcal{T}}{dZ} - k \epsilon \frac{d^2}{dZ^2} \left(\mathcal{T} \frac{dU}{dZ} \right) - \frac{(H - H^*)}{H^3} = 0 \quad (5.10a)$$

$$\frac{d\mathcal{T}}{dZ} = \frac{1}{2} \frac{d(U^2)}{dZ} + \nu \epsilon \mathcal{T} \frac{dU}{dZ} - k \epsilon \frac{d}{dZ} \left(\mathcal{T} \frac{dU}{dZ} \right) \frac{dU}{dZ} + \lambda^{1/2} \frac{(4H - 3H^*)}{6H^2}, \quad (5.10b)$$

where $U(Z) = u_r/\delta_m = \Delta + \lambda^{1/2} H$ is the dimensionless radial displacement of the tube as before. There are now four dimensionless parameters: the dimensionless speed λ , the relative deformation ϵ , the shape parameter k , and Poisson’s ratio ν . The force necessary to move the intruder relative to the lab frame is identically the difference in tension across the ends multiplied by the circumference of the tube. In dimensionless terms (recall that $\mathcal{F} = F/(2\pi Eb \delta_m^2/a)$):

$$\mathcal{F} = \mathcal{T}_{\infty} - \mathcal{T}_{-\infty}. \quad (5.11)$$

As discussed earlier, the tension $\mathcal{T}$ remains finite over the intruder surface due to the mechanics of the dry problem, even as $\lambda \rightarrow 0$. The limit of $\epsilon \ll 1$ corresponds to intruders that are only slightly larger than the tube ($\delta_m \ll a$). Setting $\epsilon = 0$ in (5.10a) corresponds to the problem considered in Section 3, in which the tension is decoupled from the thin film dynamics. Now we relax the assumption of small deformation, and treat the asymptotics of small speeds including the effects of tension.

5.3. Solution for $\lambda^{1/2} \ll 1$

In the limit $\lambda^{1/2} \ll 1$, we can again separate an inner from an outer problem. Outer problems occur over the dry contact region ($0 < Z < 1$), as well as over the non-contacting regions ($Z < 0$) and ($Z > 1$), which are now nontrivial owing to a finite tension. As before, the inner problems resolve boundary layers at the entry ($Z = 0$) and exit ($Z = 1$) regions and are necessary to match between outer problems. Hereafter we also assume that $T_{-\infty} = 0$ (the left end of the tube is force free), so that $\mathcal{T}_{\infty}$ is the force that must be exerted to move the object; this corresponds to no stretch of the tube in the static limit $\lambda = 0$. In this case, the static solution is given by $U(0 < Z < 1) = \Delta(Z)$, $U(Z \leq 0) = U(Z \geq 1) = 0$, with $\mathcal{T} = U^2/2$, from integrating (5.10b). There are two contact points, one at $Z = 0$ and another at $Z = 1$, where the non-contacting and contacting regions of the solution meet. We discuss how these features are modified and inform the solution for small nonzero λ .

We first consider the problem over the dry contact region ($0 < Z < 1$), where $\Delta = O(1)$ and $H = O(1)$, so $U \sim \Delta$. Collecting terms of order λ^0 , one obtains over the contact region

$$\Delta' + \nu\epsilon\mathcal{T}' - k\epsilon(\mathcal{T}\Delta')'' - \frac{H - H^*}{H^3} = 0 \quad (5.12a)$$

$$\mathcal{T}' = \Delta\Delta' + \nu\epsilon\mathcal{T}\Delta' - k\epsilon(\mathcal{T}\Delta')'\Delta', \quad (5.12b)$$

with the prime denoting derivative with respect to Z . The second equation is a linear equation of first order for $\mathcal{T}$, which can be solved for a given intruder shape $\Delta(Z)$. Substituting back into (5.12a) yields an outer equation which has the same form (3.1) as before, but with $S(Z)$ replaced by $\Delta' + \epsilon\nu\mathcal{T}\Delta' - \epsilon k(\mathcal{T}\Delta')'\Delta'$, which contains extra terms.

To obtain a more transparent solution, one can solve (5.12) to leading order in ϵ . To this end it is sufficient to consider (5.12b) to order ϵ^0 , since in (5.12a) $\mathcal{T}$ is multiplied by ϵ . Integrating (5.12b), this gives

$$\mathcal{T} = \Delta^2/2 + \mathcal{T}_0, \quad (5.13)$$

where $\mathcal{T}_0$ is a constant of integration. Now the equation replacing (3.1) is

$$\frac{H - H^*}{H^3} = \Delta' + \nu\epsilon\Delta\Delta' - \frac{k\epsilon}{2}(\Delta^2\Delta')'' - k\epsilon\mathcal{T}_0\Delta''' \equiv \bar{S}(Z), \quad (5.14)$$

where $\bar{S}(Z)$ is more complicated than $S(\Delta)$, but still given in terms of $\Delta(Z)$ alone.

However, as opposed to the case without tension, the outer problems in the regions outside contact ($Z > 1$ and $Z < 0$) no longer correspond to vanishing displacement, but represent a membrane under tension (in the case $\mathcal{T}_{-\infty} = 0$, only $Z > 1$ has tension). The lubrication pressure is small in these regions; taking (5.8a) without the lubrication pressure p , and constant tension $T \sim T_{\infty}$, one obtains (in dimensionless form)

$$U'' = \frac{U}{\epsilon k \mathcal{T}_{\infty}} + \frac{\nu}{k}, \quad (5.15)$$

to the right of the contact zone ($Z > 1$). This equation admits the solution (Rallabandi *et al.* 2019)

$$U = -\epsilon\nu\mathcal{T}_{\infty} + C \exp\left[-\frac{Z}{\sqrt{\epsilon k \mathcal{T}_{\infty}}}\right], \quad \text{for } Z > 1, \quad (5.16)$$

where the constant C is determined by matching to the inner solution at the exit. We

note that the static problem ($\lambda = 0$) with no applied stretch has $\mathcal{T}_\infty = 0$, and so by continuity we expect $\mathcal{T}_\infty \rightarrow 0$ as $\lambda \rightarrow 0$.

A similar analysis holds on the left on the intruder, where we now use the condition that $\mathcal{T}_\infty = 0$. The solution to the left of the intruder ($Z < 0$) is then $U = \mathcal{T} = 0$, and the situation collapses onto the case of $\epsilon = 0$ (flow is independent of tension). Near the left contact, $H = O(1)$ and $\Delta(0) = 0$, so we identify $U(0) = O(\lambda^{1/2})$. Substituting these conditions into (5.13) sets $\mathcal{T}_0 = 0$, as expected from the dry static problem. Analogous conditions at the right contact constrain C , although we do not detail these here.

The key findings of the preceding analysis are (i) that the tension $\mathcal{T} = 0$ on the left boundary and (ii) that the film thickness over the contact region is governed by (5.14), which has the same form as (3.1), but with the substitution $S(Z) \rightarrow \bar{S}(Z)$. For the same reasons as before the outer problem restricts

$$H^* \leq \frac{2}{3\sqrt{3\bar{S}_0}}, \quad (5.17)$$

where $\bar{S}_0 = \bar{S}(Z = 0)$, analogous to (3.3). As before, the inner solution around the entry region $Z = 0$ must set the value of H^* in conjunction with (5.17). This time we note that the tension near the entry region vanishes to leading order in $\lambda^{1/2}$, so that the similarity equation for the structure of the entry region remains the same as (3.5), but with the substitution $S_0 \mapsto \bar{S}_0$. Thus, the entry region of the lubrication layer accounting for tension has a structure that is closely related to the earlier analysis where tension was neglected at the outset. We therefore invoke the analysis from earlier sections and obtain $H^* = 2/(3\sqrt{3\bar{S}_0})$ [cf. (3.7)]. The right boundary layer retains its scaling with λ from the analysis of the earlier sections, although the shape is modified by the presence of a finite tension.

To obtain the force $\mathcal{F}$ on the intruder (which is identical to $\mathcal{T}_\infty$), we integrate (5.10b) for $\epsilon \ll 1$ [consistent with the approximation in (5.13)] and use (5.16) to find

$$\mathcal{T}_\infty = \frac{\epsilon^2 \nu^2 \mathcal{T}_\infty^2}{2} + \lambda^{1/2} \int_{-\infty}^{\infty} \frac{4H - 3H^*}{6H^2} dZ. \quad (5.18)$$

As before, we retain contributions of the integral from the outer solution $0 < Z < 1$ as the boundary layers contribute an $O(\lambda^{1/2})$ smaller force. Dropping terms of $O(\epsilon^2)$ and approximating $H \approx H^*$ for slender shapes, as before we find

$$\mathcal{F} = \mathcal{T}_\infty \approx \int_0^1 \frac{4H - 3H^*}{6H^2} dZ \simeq \frac{\sqrt{3}}{4} (\lambda \bar{S}_0)^{1/2}.$$

Evaluating $\bar{S}$ at $Z = 0$ according to (5.14), using that $\Delta(0) = 0$ and $\mathcal{T}_0 = 0$, we find $\bar{S}_0 = S_0 - k\epsilon S_0^3$. Thus our final result for the force becomes

$$\mathcal{F} \approx \frac{\sqrt{3}}{4} (\lambda (S_0 - k\epsilon S_0^3))^{1/2} \approx \frac{\sqrt{3}}{4} (\lambda S_0)^{1/2} \left(1 - \frac{\epsilon k}{2} S_0^2\right). \quad (5.19)$$

We observe that this result is consistent with the zero-tension result in the limit $\epsilon \rightarrow 0$ and with the static result for $\lambda \rightarrow 0$. Analysis of the boundary layer near the exit region shows that the earlier scaling $(Z - 1) = O(\lambda^{1/2})$ remains intact, but that the shape of the film thickness now depends on tension $\epsilon \mathcal{T}_\infty$. We do not analyze this region further since its features make only a subdominant contribution to $\mathcal{F}$.

FIGURE 7. Rescaled dimensionless force $\mathcal{F}/\sqrt{\lambda}$ versus ϵ at different λ , indicating numerical results (symbols) and the theoretical prediction (5.19) for small λ (thick line). The theory provides a good approximation to the numerical results for small λ up to $\epsilon \approx 0.4$. Observe that for small λ , the force decreases slightly with ϵ as predicted, whereas it increases slightly with ϵ for $\lambda \gtrsim 1$.

5.4. Comparison with numerical solutions

To verify the estimate (5.19) for $\mathcal{F}$, we numerically solve the system of governing equations (5.9) simultaneously for pressure P , displacement U and tension $\mathcal{T}$, this time focusing on generic symmetric shapes $\Delta(Z) = 4Z(1 - Z)$. The details of the numerical method are outlined in the Appendix. For simplicity, we also assume that the tube is incompressible ($\nu = 1/2$) and consider a spherical intruder ($R_z = a + \delta_m$), so that $\ell \approx 2\sqrt{2\delta_m a}$, yielding $k = a\delta_m/\ell^2 = 1/8$. In our numerical solutions we study the dependence of the drag force $\mathcal{F}$ on the dimensionless speed λ and the dimensionless radial displacement $\epsilon = \delta_m/a$.

In Fig. 7 we plot numerical results for the force $\mathcal{F}$ for different values of ϵ with a symmetric intruder. We find that the dependence of $\mathcal{F}$ on λ remains largely unchanged from the $\epsilon = 0$ (hoop stress only) limit. The force decreases with ϵ for small λ . The expression (5.19) is a good approximation of the numerical results for $\lambda \lesssim 0.1$ and remains quantitative at least up to $\epsilon = 0.3$. Increasing λ beyond unity, the effect is surprisingly reversed, with the force increasing with ϵ at a fixed λ . Thus, the main finding from the numerical solutions is that the inclusion of tension quantitatively modifies the force from its $\epsilon = 0$ value, while the dependence on λ remains quantitatively unchanged.

6. Discussion and Conclusions

We have analyzed the force required to drag a slender close-fitting particle or intruder through a lubricated elastic tube in the situation where the intruder is wider than the tube. After setting up a model in which the tube supports radial stresses only, we developed an asymptotic solution for small speeds. The force necessary to move the intruder through the tube increases with the velocity and exhibits power law behaviors in some limits. The force was found to have a simple dependence on the shape of the intruder. For asymmetric intruders, the force is smallest when the intruder translates with its narrow end first.

Of particular note is our conceptually simple result (3.14) and its refinement (5.19), which clarify the roles of asymmetry and tension. Since the film thickness is selected at

the entry, a smaller slope at the head produces a smaller resistance. It is tempting to apply this idea to problems like the motion of eggs through the oviduct, but the evidence for laying with either the pointed or the blunt end first is inconclusive (Bradfield 1951; Salamon & Kent 2014).

In our formulation, we have assumed that the tube is filled with fluid. Our numerical results show that for realistic conditions, the main contribution to the drag comes from the neighborhood of the intruder. Another possible setup would be a film of finite thickness pre-coating the tube, in which case the resistance necessary to maintain a Poiseuille flow far from the intruder would disappear. As long as the selected film thickness H^* is smaller than the precoated thickness, this will have a minor effect on the entry region, where fluid accumulates. However, the presence of a meniscus could change the exit region (without an effect on the drag), where the tube surface is starved of liquid. If the precoated thickness were thinner than the value set by H^* , the thickness of the lubricating layer would become independent of speed, so that the total drag would be proportional to the speed. The inclusion of the finite bending resistance of the tube may also modify the details of the entry and exit regions, though we again expect only a modest influence on the drag, which is dominated by the contact region.

Perhaps the most significant unresolved question concerns the validity of the lubrication equation, for example in the case of intruders that are not slender. It would be interesting to test our results using solvers for fluid-structure interactions which do not rely on that assumption (Heil 2004; Heil & Hazel 2006). Equally, one might consider non-linear elastic materials, and how the use of such a material might be used to reduce drag. However, our previous experimental and theoretical work on the “dry” version of the intruder problem (Rallabandi *et al.* 2019) indicated that a simple neo-Hookean assumption performs surprisingly well.

Acknowledgements

The authors thank E. Reyssat and M. C. Stoddard for fruitful discussions. J. E. acknowledges support by the Leverhulme Trust through International Academic Fellowship IAF2017-010.

Declaration of Interests. The authors report no conflict of interest.

Appendix A. Numerical solution with tension

For a numerical solution of the full steady-state problem combining membrane tension and lubrication we have to solve the system (5.10), and find the unknown parameter H^* as part of the solution. The strategy that we used to solve the hoop-stress-only ($\epsilon = 0$) version of the problem (guessing H^* and shooting to the correct value) fails in the general case since it is difficult to find a solution that connects large positive Z to large negative Z . Instead, we use an iterative solution strategy that allows relaxation to the correct steady-state solution. To this end we introduce a time-dependent form of (5.10) that involves a time-dependent thin-film equation coupled with the nonlinear equations for the deformation of the tube.

Using dimensionless variables and denoting dimensionless time by t , this system of

equations is

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P}{\partial Z} - H \right) \quad (\text{A } 1a)$$

$$P = U + \nu\epsilon\mathcal{T} - k\epsilon \frac{\partial}{\partial Z} \left(\mathcal{T} \frac{\partial U}{\partial Z} \right) \quad (\text{A } 1b)$$

$$\frac{\partial \mathcal{T}}{\partial Z} = P \frac{\partial U}{\partial Z} + \lambda^{1/2} \left(\frac{H}{2} \frac{\partial P}{\partial Z} + \frac{1}{6H} \right). \quad (\text{A } 1c)$$

These equations describe the evolution of the film thickness starting from arbitrary initial conditions; at steady state, (A 1a) reduces to (2.1). It is useful to eliminate P between (A 1b) and (A 1a) to obtain an evolution for U involving $\mathcal{T}$:

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial Z} \left\{ H^3 \left[\frac{\partial U}{\partial Z} + \nu\epsilon \frac{\partial \mathcal{T}}{\partial Z} - k\epsilon \frac{\partial^2}{\partial Z^2} \left(\mathcal{T} \frac{\partial U}{\partial Z} \right) \right] - H \right\}, \quad (\text{A } 2)$$

where we recall that $H = \lambda^{-1/2}(U - \Delta)$ by definition.

We solve the system (A 1a), (A 1c), (A 2), with a second-order central difference scheme in space and a first-order backward Euler integration in time. We start with initial guesses for the pressure $P^{(n)}(Z)$ and displacement $U^{(n)}(Z)$, where the superscript denotes time levels. We then integrate (A 1c) numerically, subject to the single condition $\mathcal{T}(Z \rightarrow -\infty) = 0$, to obtain the tension $\mathcal{T}^{(n)}(Z)$. Next, we integrate the fourth-order nonlinear equation (A 2) [after substituting $H = \lambda^{-1/2}(U - \Delta)$] through a time-interval δt using a linearized backward Euler scheme (see e.g. Moin 2010), subject to boundary conditions $U = -\epsilon\nu\mathcal{T}$ and $U' = -\epsilon\nu\mathcal{T}'$ at $Z \rightarrow \pm\infty$, which follow from (A 1b); see the discussion below. This yields the displacement at the next timestep $U^{(n+1)}(Z)$. We use this solution to estimate $\partial H/\partial t$ and finally solve (A 1a) for $P^{(n+1)}(Z)$ subject to vanishing pressure at both ends of the tube. This sequence of steps is repeated until steady-state solutions are obtained simultaneously for U , $\mathcal{T}$ and P . The axial force on the tube is then extracted from the solution as $\mathcal{F} = \mathcal{T}(Z \rightarrow \infty)$.

The numerical results including tension in the main text are for $\Delta(Z) = 4Z(1 - Z)$. Asymptotic solutions for T , U and P as $Z \rightarrow \pm\infty$ are useful as boundary conditions for the numerical solutions. Far away from the intruder ($Z \mapsto \pm\infty$), U remains bounded while $\Delta \sim -4Z^2$. At steady state, (A 1a) behaves as $P' \sim H^{-2}$, which (using $H = (U - \Delta)/\sqrt{\lambda}$) integrates to

$$P(Z \rightarrow \pm\infty) \sim -\frac{\lambda}{48Z^3}. \quad (\text{A } 3)$$

Substituting this relation into (A 1c) and integrating yields

$$\mathcal{T}(Z \rightarrow \pm\infty) \sim T_{\pm\infty} - \frac{\lambda}{6Z}. \quad (\text{A } 4)$$

Finally (A 1b) yields

$$U(Z \rightarrow \pm\infty) \sim -\nu\epsilon \left(T_{\pm\infty} - \frac{\lambda}{6Z} \right). \quad (\text{A } 5)$$

REFERENCES

- AUDOLY, B. & POMEAU, Y. 2010 *Elasticity and Geometry: from hair curls to the non-linear response of shells*. Oxford University Press.
- BARAKAT, J. M. & SHAQFEH, ERIC S. G. 2018a The steady motion of a closely fitting vesicle in a tube. *J. Fluid Mech.* **835**, 721.

- BARAKAT, J. M. & SHAQFEH, E. S. G. 2018*b* Stokes flow of vesicles in a circular tube. *J. Fluid Mech.* **851**, 606–635.
- BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press, Cambridge.
- BRADFIELD, J. R. G. 1951 Radiographic studies on the formation of the hen’s egg shell. *J. Exp. Biol.* **28**, 125–140.
- BRETHERTON, F. P. 1961 The motion of long bubbles in tubes. *J. Fluid Mech.* **10**, 166–188.
- CHATKOFF, M. L. 1975 A biophysicist’s view of ovum transport. *Gynecol. Obstet. Invest.* **6** (3-4), 105–122.
- DOWSON, D & HIGGINSON, GR 1959 A numerical solution to the elasto-hydrodynamic problem. *Journal of mechanical engineering science* **1** (1), 6–15.
- DOWSON, D. & HIGGINSON, G. R. 1966 *Elastohydrodynamic Lubrication, the Fundamentals of Roller and Gear Lubrication*. Pergamon Press, Oxford, UK.
- ESSINK, M., PANDEY, A., KARPITSCHKA, S., VENNER, K. & SNOEIJER, J. 2020 Regimes of soft lubrication. *arXiv preprint arXiv:2007.08020*.
- FITZ-GERALD, JM 1969 Mechanics of red-cell motion through very narrow capillaries. *Proc. Roy. Soc. Lond. B* **174** (1035), 193–227.
- FREUND, J. B. 2014 Numerical simulation of flowing blood cells. *Annu. Rev. Fluid Mech.* **46**, 67–95.
- HEIL, M. 2004 An efficient solver for the fully coupled solution of large-displacement fluid-structure interaction problems. *Comput. Methods Appl. Mech. Eng.* **193**, 1–23.
- HEIL, M. & HAZEL, A. L. 2006 *oomph-lib* – an object-oriented multi-physics finite-element library. In *Fluid-Structure Interaction* (ed. M. Schäfer & H.-J. Bungartz), pp. 19–49. Springer.
- LANDAU, L. D. & LIFSHITZ, E. M. 1984 *Elasticity*. Pergamon: Oxford.
- LIGHTHILL, M. J. 1968 Pressure-forcing of tightly fitting pellets along fluid-filled elastic tubes. *J. Fluid Mech.* **34**, 113–143.
- MOIN, P. 2010 *Fundamentals of Engineering Numerical Analysis*. Cambridge University Press, new York.
- RALLABANDI, B., MARTHELOT, J., JAMBON-PUILLET, E., BRUN, P.-T. & EGGERS, J. 2019 Curvature regularization near contacts with stretched elastic tubes. *Phys. Rev. Lett.* **123**, 168002.
- RALLABANDI, B., SAINTYVES, B., JULES, T., SALEZ, T., SCHÖNECKER, C., MAHADEVAN, L. & STONE, H. A. 2017 Rotation of an immersed cylinder sliding near a thin elastic coating. *Phys. Rev. Fluids* **2** (7), 074102.
- SAINTYVES, B., JULES, T., SALEZ, T. & MAHADEVAN, L. 2016 Self-sustained lift and low friction via soft lubrication. *Proc. Nat. Acad. Sci.* **113** (21), 5847–5849.
- SAINTYVES, B., RALLABANDI, B., JULES, T., AULT, J., SALEZ, T.S, SCHÖNECKER, C., STONE, H. A. & MAHADEVAN, L. 2020 Rotation of a submerged finite cylinder moving down a soft incline. *Soft Matter* **16** (16), 4000–4007.
- SALAMON, A. & KENT, J. P. 2014 Orientation of the egg at laying - is the pointed or the blunt end first? *J. Poultry Sci.* **13**, 316–318.
- SECOMB, T. W., SKALAK, R., ÖZKAYA, N. & GROSS, J. F. 1986 Flow of axisymmetric red blood cells in narrow capillaries. *J. Fluid Mech.* **163**, 405–423.
- SEKIMOTO, K. & LEIBLER, L. 1993 A mechanism for shear thickening of polymer-bearing surfaces: elasto-hydrodynamic coupling. *Europhys. Lett.* **23** (2), 113–117.
- SKOTHEIM, J. M. & MAHADEVAN, L. 2004 Soft lubrication. *Phys. Rev. Lett.* **92**, 245509.
- SNOEIJER, J. H., EGGERS, J. & VENNER, C. H. 2013 Similarity theory of lubricated Hertzian contacts. *Phys. Fluids* **25**, 101705.
- SZERI, A. Z. 2010 *Fluid Film Lubrication*. Cambridge University Press.
- TAKAGI, D. & BALMFORTH, N. J. 2011 Peristaltic pumping of rigid objects in an elastic tube. *J. Fluid Mech.* **672**, 219–244.
- TANI, M., CAMBAU, T., BICO, J. & REYSSAT, E. 2017 Motion of a rigid sphere through an elastic tube with a lubrication film. *APS* pp. L20–012.
- URZAY, J., LLEWELLYN SMITH, S. G. & GLOVER, B. J. 2007 The elastohydrodynamic force on a sphere near a soft wall. *Phys. Fluids* **19**, 103106.

- VANN, P. G. & FITZ-GERALD, J. M. 1982 Flow mechanics of red cell trains in very narrow capillaries. I. trains of uniform cells. *Microvasc. Res.* **24** (3), 296–313.
- VURGAFT, A., ELBAZ, S. B. & GAT, A. D. 2019 Forced motion of a cylinder within a liquid-filled elastic tube—a model of minimally invasive medical procedures. *J. Fluid Mech.* **881**, 1048–1072.