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On solitary waves of a piecewise-linear suspension bridge model

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Abstract

A further investigation of solitary wave solutions is undertaken for a piece-wise linear beam equation which has been proposed as a model for suspension bridges. Solutions are characterized as being of multi-trough or multi-wiggle type. It is proved that there at most five single-troughed waves, whereas there are infinitely many multi-troughed waves for a range of wave-speeds. Numerical shooting and continuation methods are used to shed light on the global structure of these solutions. Several open questions are highlighted.

1 Introductory Remarks

As remarked by McKenna & Walter [14], suspension bridges are known to exhibit travelling wave behaviour. To account for this, the equation [14]

$$v_{tt} + v_{xxxx} + v^+ = 1 \tag{1.1}$$

was investigated for travelling wave solutions. This led naturally to the study of solutions of the form $v(x, t) = 1 + u(x - ct)$ of the piecewise linear ordinary differential equation $u^{iv} + c^2 u'' + (u + 1)^+ - 1 = 0$ ¹. Solutions were obtained by solving explicitly the two halves of the equation, namely $u^{iv} + c^2 u'' - 1 = 0$ where $u \leq -1$ and $u^{iv} + c^2 u'' + u = 0$ where $u \geq -1$, and patching them together at $u = -1$. While solutions of the type described in the engineering literature [1] were found, the equation was not exhaustively studied and some interesting mathematical gaps remained.

For example, while solutions were shown to exist for some wave speeds c , it was not clear from [14] for what range of c the solutions existed. Although multiple solutions were observed, it was not clear how many solutions existed and how they could be characterized, or indeed if they persisted under a small perturbation in the nonlinearity in (1.1).

One of these questions was answered by McKenna & Chen [13], when it was shown using variational methods that a nontrivial solution existed for all c in the interval $(0, \sqrt{2})$. Problems of multiplicity and of characterization of the solutions were not addressed.

¹This equation also occurs in the engineering literature in [3, 12] where it is used to describe deflections in railroad tracks and undersea oil pipelines

The purpose of this paper is to fill some of these gaps and to highlight the remaining open problems of existence and multiplicity for this equation and to make some conjectures about those parts of the global picture which remain unproven.

We follow the idea of McKenna & Walter in that we first look for solutions which fall below the line $u = -1$ on a single interval centred about $x = 0$. The explicit solution is then matched at $u = -1$ with an explicit solution in the second region above that line which decays exponentially as x goes to infinity. (One has to check that this solution never crosses the line $u = -1$ again.) These solutions are the ones we will call “one-trough” solutions. Some of the ideas of this analysis were motivated by the results of [15] on periodic solutions of a related travelling-wave equation.

When we speak of “multiple solutions”, we obviously mean solutions which are not merely translations of each other but which are fundamentally different in some way. We can think of two different ways of distinguishing different types of solution. One is simply to count the number troughs the solution has, namely the number of times the solution drops below $u = -1$. As mentioned earlier, our first task, in section 2, is the construction of one-troughed solutions.

In section 3, using arguments of Devaney [8] and Buffoni, Champneys & Toland [5], we show that the one-troughed solutions can be used to construct infinitely many multi-troughed solutions which are “close to” superpositions of large translates of a single troughed solution. The key to such a result is in proving that the solutions constructed in section 2 are the transverse intersections of the stable and unstable manifolds of the zero equilibrium. This transversality also implies persistence under small perturbations to the nonlinearity in (1.1).

Section 4 presents analytic evidence for the non-existence of asymmetric one-troughed solutions, and some results on the limit $c \rightarrow 0$.

A second way to distinguish solutions is clear after a look at the different solutions shown in figure 5. Here, it is clear that for large negative u , the second derivative changes sign more often as we find larger and larger solutions. We could for example say that the solutions (a)-(e) are distinguished by having from none to four local maxima in the nonlinear range $u \leq -1$. The number of local maxima in an interval where the solution lies below $u \leq -1$ gives a different way of finding multiple solutions. We refer to such solutions as having “multi-wiggles”.

We have found some of these different types of solutions, and section 5 presents results on numerical computation and continuation with c of a sample of solutions. However, the global picture of how many types there are, and their dependence on the wave speed c remains unclear. We shall make further remarks on this in the concluding section of the paper.

2 Symmetric one-troughed waves

We are interested in homoclinic solutions to the origin of

$$u^{iv} + c^2 u'' + (1 + u)^+ - 1 = 0. \quad (2.1)$$

Note that (2.1) can be decomposed into the two linear differential equations

$$u^{iv} + c^2 u'' + u = 0 \quad \text{for } u > -1 \quad (\text{region 1}) \quad (2.2)$$

$$u^{iv} + c^2 u'' - 1 = 0 \quad \text{for } u < -1 \quad (\text{region 2}). \quad (2.3)$$

The linearization of (2.2) about the origin has characteristic equation

$$\mu^4 + c^2 \mu^2 + 1 = 0. \quad (2.4)$$

In what follows we shall be interested only in the case $c^2 < 2$, when the origin is a hyperbolic equilibrium (namely that there are linearly decaying and expanding eigendirections, which is a necessary condition for the existence of homoclinic orbits in this context). Specifically, for $0 \leq c^2 < 2$ (2.4) has solutions of the form

$$\mu = \pm\lambda \pm i\omega, \quad \text{where} \quad \lambda = \frac{\sqrt{2-c^2}}{2} \quad \text{and} \quad \omega = \frac{\sqrt{2+c^2}}{2}. \quad (2.5)$$

In this section we consider homoclinic solutions for which there are two x -values $x_1 < x_2$ such that $u(x_1) = u(x_2) = -1$, with $u(x)$ in region 1 for all $x < x_1$ and $x > x_1$, and $u(x)$ in region 2 for $x_1 < x < x_2$. Moreover, we shall consider only solutions for which at some $x^* \in (x_1, x_2)$, $u'(x^*) = u'''(x^*) = 0$. Since (2.1) is autonomous, we are free to choose $x^* = 0$ and then the reversing involution

$$R : (u, u', u'', u''') \rightarrow (u, -u', u'', -u'''), \quad t \rightarrow -t,$$

under which (2.1) is invariant, shows that such a solution will be symmetric under R . In particular, to find symmetric homoclinic solutions, it suffices to find a non-trivial solution $\{u(x) : x \in [-\infty, 0]\}$ such that $u(x) \rightarrow 0$ as $x \rightarrow -\infty$ and $u'(0) = u'''(0) = 0$. For future reference, we denote by $\mathcal{S}$ the so-called symmetric section of phase space

$$\mathcal{S} = \{(u, u', u'', u''') : u' = u''' = 0\}.$$

In region 1, the complete family of solutions in the unstable manifold of the origin can be written in the form

$$u_1(x) = Ae^{\lambda x} \cos(\omega x), \quad (2.6)$$

$$u'_1(x) = Ae^{\lambda x} [\lambda \cos(\omega x) - \omega \sin(\omega x)], \quad (2.7)$$

$$u''_1(x) = Ae^{\lambda x} [(\lambda - \omega^2) \cos(\omega x) - 2\lambda\omega \sin(\omega x)], \quad (2.8)$$

$$u'''_1(x) = Ae^{\lambda x} [(\lambda^3 - 3\lambda\omega^2) \cos(\omega x) + (\omega^3 - 3\lambda^2\omega) \sin(\omega x)], \quad (2.9)$$

parametrised by $A \in [a_0, a_0 e^{2\pi\lambda/\omega})$ for some constant $a_0 > 0$. We are interested in the smallest value x_1 of x such that solutions (2.6)–(2.9) satisfy $u_1(x_1) = -1$. We can parametrise such points in phase space by the value of

$$\theta = u'_1(x_1).$$

Then, we have

$$u_1(x_1) = -1 \quad u'_1(x_1) = \theta, \quad u''_1(x_1) = 1 + \sqrt{2-c^2} \theta, \quad u'''_1(x_1) = \sqrt{2-c^2} + (1-c^2) \theta. \quad (2.10)$$

It remains to specify the limits of θ such that solutions originate in the local unstable manifold, that is, such that $u_1(x)$ is in region 1 for all $x < x_1$. Appealing to the theory of paths as in section 3 of [5], it is not difficult to see that the maximum value of θ is zero, independent of $c^2 \in [0, 2)$. Again, simple geometry shows that a strict lower bound θ_{min} on θ is obtained by computing, forwards in x , the trajectory through the point parametrised by $\theta = 0$ until the next x -value $x^* > x_1$ such that $u_1(x^*) = -1$. Then $\theta_{min} := u'_1(x^*)$. Thus

$$\begin{aligned} -1 &= Ae^{\lambda(x^*+x_1)} \cos[\omega(x^*+x_1)] \\ \theta_{min} &= Ae^{\lambda(x^*+x_1)} [\lambda \cos[\omega(x^*+x_1)] - \omega \sin[\omega(x^*+x_1)]], \end{aligned}$$

Figure 1: The function $\theta_{min}(c^2)$

which, using (2.10) with $\theta = 0$ and (2.6) and (2.7), reduces to

$$e^{\lambda x^*} [\lambda \sin(\omega x^*) - \omega \cos(\omega x^*)] + \omega = 0 \quad (2.11)$$

$$e^{\lambda x^*} (\lambda^2 + \omega^2) \sin(\omega x^*) - \omega \theta_{min} = 0. \quad (2.12)$$

Upon using the expressions (2.5) for λ and ω in terms of c^2 , (2.11) defines an implicit equation for x^* . Again for simple geometric reasons, we seek a solution $x^* \in [\pi/\omega, 2\pi/\omega]$. It is not difficult to see that the derivative of the right-hand side of (2.11) is single signed in this interval and moreover that, for each $c^2 \in [0, 2)$, (2.11) has a unique solution in $[\pi/\omega, 2\pi/\omega]$. Substituting this value of x^* into (2.12) gives a unique value of θ_{min} . Figure 1 shows a graph of $\theta_{min}(c^2)$ obtained in this fashion using Maple [7], from which we note that θ_{min} is a monotone increasing function of c^2 with $\theta_{min}(0) = -52.158560\dots$

Next we consider solutions in region 2. Here the general solution can be written in the form

$$u_2(x) = \alpha \cos[c(x - x_1) + \phi] + \frac{(x - x_1)^2}{2c^2} + \beta(x - x_1) + \gamma, \quad (2.13)$$

with

$$u_2'(x) = -\alpha c \sin[c(x - x_1) + \phi] + (x - x_1)/c^2 + \beta \quad (2.14)$$

$$u_2''(x) = -\alpha c^2 \cos[c(x - x_1) + \phi] + 1/c^2 \quad (2.15)$$

$$u_2'''(x) = \alpha c^3 \sin[c(x - x_1) + \phi], \quad (2.16)$$

for arbitrary constants α , β , ϕ and γ . Matching this solution and its first three derivatives with the expressions (2.10) when $x = x_1$ leads to

$$\alpha \cos(\phi) + \gamma = -1 \quad (2.17)$$

$$-\alpha c \sin(\phi) + \beta = \theta \quad (2.18)$$

$$-\alpha c^2 \cos(\phi) + 1/c^2 = 1 + \sqrt{2 - c^2} \theta \quad (2.19)$$

$$\alpha c^3 \sin(\phi) = \sqrt{2 - c^2} + (1 - c^2) \theta. \quad (2.20)$$

Solving (2.19) and (2.20) for α and ϕ give

$$\alpha = \frac{\sqrt{1 + c^2 \theta^2}}{c^4} \quad (2.21)$$

$$\phi = \arctan \left(\frac{c(c^2 - 1)\theta - c\sqrt{2 - c^2}}{-1 + c^2 + c^2\sqrt{2 - c^2}\theta} \right). \quad (2.22)$$

Also using (2.19) and (2.20), $\cos(\phi)$ and $\sin(\phi)$ can be expressed in terms of θ and c . Substitution of the resulting expressions into (2.17) and (2.18) yields

$$\beta = \frac{(\sqrt{2 - c^2} + \theta)}{c^2} \quad (2.23)$$

$$\gamma = \frac{c^2\sqrt{2 - c^2}\theta - (c^4 - c^2 + 1)}{c^4}. \quad (2.24)$$

We can now apply the symmetry condition, namely that $u_2'(0) = u_2'''(0) = 0$. From (2.14) and (2.16) we have that at such a point

$$-x_1 + c^2\beta = 0, \quad -cx_1 + \phi = n\pi, \quad n \in \mathbb{Z}. \quad (2.25)$$

Elimination of x_1 and using the expressions (2.22) and (2.23) for ϕ and β yields the single condition

$$L(c, \theta, n) := n\pi - \arctan \left(\frac{c(c^2 - 1)\theta - c\sqrt{2 - c^2}}{-1 + c^2 + c^2\sqrt{2 - c^2}\theta} \right) + c\sqrt{2 - c^2} + c\theta = 0. \quad (2.26)$$

Thus the search for one-troughed symmetric homoclinic solutions of (2.1) has been reduced to the search for zeros of (2.26) for $0 < c^2 < 2$, $\theta \in (\theta_{\min}(c^2), 0)$ and $n \in \mathbb{Z}$.

Figure 2 shows graphs of $L(c, \theta, n)$ against θ computed by Maple for several different values of c^2 and for all n for which a zero occurs. The lowest curve in each case corresponds to $n = 0$, with arctangent measured between $-\pi/2$ and $3\pi/2$ using the sign of $\cos(\phi)$ to determine which branch to take. It is apparent from the graphs that, upon increasing c^2 from zero, the number of solutions initially increases (for example, at $c^2 = 0.5$ there are apparently five solutions) and then decreases. Whenever a solution is created or destroyed, it corresponds to a zero of L disappearing off the left of the graph. That is, θ passes through $\theta_{\min}$.

Figure 3 shows the zeros of L plotted as a function of c^2 . We note from the figure that there at most five solutions (corresponding to $n = 0, 1, 2, 3, 4$). Figures 4 and 5 show all the solutions found to exist by this method at $c^2 = 0.5$ and 1.5 respectively.

With reference to Figure 3(b), note that there is no solution for c sufficiently small. Moreover, there also exists a range of c^2 up to and including $c^2 = 2$ for which there are no solutions. Using Maple, we find these maximum and minimum values of c^2 to be

$$c_{\max}^2 = 1.997861, \quad c_{\min}^2 = 0.007612. \quad (2.27)$$

Thus we have proved the following.

Theorem 2.1 *There exists $0 < c_{\min}^2 < c_{\max}^2 < 2$, whose six-figure estimates are given by (2.27), such that for all $c^2 \in (c_{\min}^2, c_{\max}^2)$ there exist between one and five symmetric, one-troughed homoclinic orbits of (2.1). Moreover for $c^2 < c_{\min}^2$ and $c_{\max}^2 < c^2 < 2$ there are no symmetric one-troughed homoclinic orbits.*

Figure 2: The function $L(c^2, \theta, n)$ against θ for (a) $c^2 = 0.1$, (b) $c^2 = 0.5$, (c) $c^2 = 1.5$, (d) $c^2 = 1.95$. The left-hand end-point of θ is $\theta_{\min}(c^2)$ in each plot.

Figure 3: Zeros of $L(c, \theta, n)$ plotted as θ against c^2 . (a) $0 < c^2 < 2$ (b) more details near $c = 0$. The lowest curve of each plot is $\theta_{min}(c^2)$.

Figure 4: Two solutions at $c^2 = 1.5$ corresponding to $n = 0, 1$.

(a) **(b)** **(c)** **(d)** **(e)**

Figure 5: Five solutions at $c^2 = 0.5$ corresponding to $n = 0, 1, 2, 3, 4$.

Remarks

1. Note that θ passing through θ_{min} does not necessarily correspond to a homoclinic solution ceasing to exist, but rather that a *one-troughed* solution ceases to exist. It may be conjectured that emerging from such a point is a homoclinic solution with three troughs, that is, three x -intervals for which u is in region 2. Such solutions are not captured by the present analysis, and their investigation is left for future numerical continuation work.
2. Also, we have not yet considered the possibility of asymmetric one-troughed solutions.
3. Recall that McKenna & Chen [13] have proved using the Mountain Pass Lemma the existence of at least one homoclinic solution for all $0 < c^2 < 2$. We conclude that for $0 < c^2 < c_{min}^2$ and $c_{max}^2 < c^2 < 2$, the mountain pass solution is not single troughed.

3 Infinitely many multi-troughed solutions

The theory of Devaney [8] (see also [2]) shows that in a four-dimensional Hamiltonian system having an equilibrium with linearization of the form (2.2), the existence of one homoclinic orbit implies the existence of infinitely many others, provided the primary orbit represents the transverse intersection of the stable and unstable manifolds within the appropriate level set of the Hamiltonian. The extra solutions are multi-modal in that their graphs consist approximately of multiple copies of the primary homoclinic orbit, separated by an arbitrarily large number of half-oscillations of the linear problem, see section 3 of Buffoni, Champneys and Toland (1996) for more details.

In section 4.2 of Buffoni, Champneys and Toland (1996), necessary conditions were derived for the non-transversality of a symmetric homoclinic orbit in a reversible fourth-order Hamiltonian system. These conditions are expressed in terms of the tangent space to the unstable manifold $W^u(0)$ of the origin at a symmetric homoclinic point, that is at the point of intersection between a homoclinic orbit and the symmetric section. For this theory to apply we require that $W^u(0)$ has a well defined tangent space in a neighbourhood of the symmetric homoclinic point. Clearly this will be true for the homoclinic orbits so far constructed, provided that the θ -value for the homoclinic orbit is bounded away from 0 and θ_{min} .

Let $\mathbf{u} = (u, u', u'', u''')$ denote a general point in phase space, and let $\mathbf{u}_2(x)$ denote the portion of a homoclinic solution contained in region 2 with $\mathbf{u}_2(0) \in \mathcal{S}$. Let T denote the tangent space to the unstable manifold at the symmetric homoclinic point

$$T = T_{\mathbf{u}_2(0)}W^u(0).$$

Note that

$$\mathbf{u}'_2(0) = (0, u''_2(0), 0, 1 - c^2 u''_2(0)) \in T.$$

Then, by Lemma 4.1 of [5], there are only two possibilities for non-transversality, namely that there exists a vector $w \in \mathbb{R}^4$ such that $T = \text{span}\{\mathbf{u}'_2(0), w\}$ where either

1. $w \in \mathcal{S}$, i.e. $Rw = w$, or
2. $Rw = -w$.

It was argued in [5] (see Knobloch [10] for rigorous results), that the unfolding of case 1 in a reversible Hamiltonian system leads to the coalescence and subsequent disappearance of two symmetric homoclinic orbits, whereas case 2 leads to a pitchfork bifurcation of two asymmetric homoclinic orbits. To consider the possibility of either bifurcation occurring for equation (2.1) we consider the two cases separately.

In case 1, an extra restriction on w can be imposed owing to the Hamiltonian structure of (2.1), more specifically, that the function

$$H(u(x)) = u'''u' + \frac{c^2(u')^2}{2} - \frac{(u'')^2}{2} + \begin{cases} \frac{u^2}{2}, & u > -1 \\ -u - \frac{1}{2}, & u < -1 \end{cases} \quad (3.1)$$

is conserved along solutions. Moreover, For homoclinic orbits to the origin $H(u(x)) \equiv 0$. Therefore, w must lie in the tangent space to the zero energy manifold at $\mathbf{u}_2(0)$ which can be written as

$$\nabla H^\perp = (-1, 0, -u_2''(0), 0)^\perp$$

because $u_2(0) < -1$ for one-troughed symmetric homoclinic solutions. Hence

$$w = (-au_2''(0), 0, a, 0), \quad \text{for some } a \in \mathbb{R},$$

and

$$T = \{(-au_2''(0), bu_2''(0), a, b(1 - c^2u_2''(0))) : a, b \in \mathbb{R}\}.$$

Now, because solutions in the unstable manifold can be parametrised by θ , a complementary vector to $\mathbf{u}_2'(0)$ in T is

$$\frac{d}{d\theta}\mathbf{u}_2(0) = \left(\frac{du_2}{d\theta}(0), \frac{du_2'}{d\theta}(0), \frac{du_2''}{d\theta}(0), \frac{du_2'''}{d\theta}(0)\right) := (v_\theta, v_\theta', v_\theta'', v_\theta''').$$

Hence non-transversality according to case 1 occurs if $(v_\theta', v_\theta''') = b(u_2''(0), 1 - c^2u_2''(0))$, that is if

$$T_1(c^2, \theta) := v_\theta' (c^2u_2''(0) + 1) - v_\theta''' u_2''(0) = 0. \quad (3.2)$$

Case 2 is easier, because here

$$T = \{(0, a, 0, b) : a, b \in \mathbb{R}\},$$

and a suitable test function for non-transversality is

$$T_2(c^2, \theta) := \sqrt{(v_\theta)^2 + (v_\theta'')^2} = 0. \quad (3.3)$$

Note that T_1 and T_2^2 are smooth functions of θ and c^2 which can be expressed in closed form, see Appendix A. These expressions may be evaluated along the curves in the (c^2, θ) -plane depicted in Figure 3 corresponding to symmetric homoclinic orbits. Figure 6 shows the values, computed by Maple, of T_1 and T_2 as functions of c^2 along each homoclinic curve. Note from the figure that both test functions are everywhere non-zero along the homoclinic curves, although there is some doubt about the lowest curve in 6(b). In fact, it is found that this function, corresponding to $n = 0$, is positive and monotone decreasing with value 0.0129745 at $c^2 = c_{max}^2$. We therefore conclude that all homoclinic orbits so-far shown to exist are transverse. Thus we have the following result.

Theorem 3.1 *There exists $0 < c_{min}^2 < c_{max}^2 < 2$, whose six-figure estimates are given by (2.27), such that for all $c^2 \in (c_{min}^2, c_{max}^2)$ there exist infinitely many symmetric, multi-troughed homoclinic solutions of (2.1).*

Figure 6: (a) Test function T_2 given by (3.3) evaluated as a function of c^2 along each curve of symmetric homoclinic orbits. (b) Similarly for the test function T_1 given by (3.2).

Another consequence of transversality is that homoclinic solutions must persist under small perturbations which preserve the Hamiltonian and reversible structure. Moreover, these solutions continue to be transverse. Therefore we have

Theorem 3.2 *There exists an open dense set of within the topology of fourth-order, R -reversible Hamiltonian systems, containing the dynamical system equivalent to (2.1) for which there are infinitely many homoclinic orbits.*

4 Asymmetric one-troughed solutions

Here, as in section 2, we consider solutions for which there is a unique x -interval such that the solution is in region 2, but now we shall relax the requirement of symmetry. We shall again take the one parameter family of solutions in the unstable manifold parametrised by $\theta \in (\theta_{min}(c^2), 0)$ and seek solutions which, at $x = x_2 > 0$ such that $u_2(x_2) = -1$ satisfy the conditions to be in the local *stable* manifold. First we note that x_2 is defined implicitly by the smallest positive solution of $u_2(x_2) = -1$. That is, using expression (A.1) with x_1 replaced by $x_1 - x_2$ for $u_2(x_2)$,

$$(2 - 2c^2 - 2c^2\sqrt{2 - c^2}\theta) \cos(cx) + (+2c^3\theta - 2c\theta - 2c\sqrt{2 - c^2}) \sin(cx) - (c^4 + c^2\theta^2) = 2c^4, \quad (4.1)$$

where $x = x_2 - x_1$.

Next we need to characterize where solutions in the local stable manifold first intersect $u = -1$ as x decreases from $+\infty$. Solutions in the local stable manifold within region 1 have the form (up to translation in x)

$$u_3(x) = Ae^{-\lambda x} \cos(\omega x) \quad (4.2)$$

parametrised by $A \in [a_0, a_0 e^{2\pi\omega/\lambda})$ for some $a_0 > 0$. A similar analysis to that in section 2 shows that the intersection of such solutions with $u = -1$ can be parametrised by the value of

$$\psi = u'_3(x_2) \in (0, \theta_{\min}(c^2)),$$

where $\theta_{\min}$ is given by the solution of (2.11) and (2.12) and is plotted in Figure 1. Then we have

$$u_3(x_2) = -1, \quad u'_3(x_2) = \psi, \quad u''_3(x_2) = 1 - \sqrt{2 - c^2} \psi, \quad u'''_3(x_2) = (1 - c^2) \psi - \sqrt{2 - c^2}. \quad (4.3)$$

It remains to match at $x = x_2$ the first three derivatives of the solutions $u_2(x)$ and $u_3(x)$ given by (2.13) and (4.2). This leads to three conditions, namely

$$\begin{aligned} M_1 &:= u'_2(x_2) \in (-\theta_{\min}(c^2), 0), \\ M_2 &:= u''_2(x_2) - (1 - \sqrt{2 - c^2} u'_2(x_2)) = 0, \\ M_3 &:= u'''_2(x_2) - ((1 - c^2) u'_2(x_2) - \sqrt{2 - c^2}) = 0. \end{aligned}$$

The three expressions on the left-hand side can be written explicitly in terms of c^2 , θ and $(x_2 - x_1)$ as

$$\begin{aligned} M_1 &= (1/c^3) \left[(c^3\theta - c\theta - c\sqrt{2 - c^2}) A + (c^2 + c^2\sqrt{2 - c^2}\theta - 1) B \right. \\ &\quad \left. + (c(\theta + x_2 - x_1 + c\sqrt{2 - c^2})) \right] \end{aligned} \quad (4.4)$$

$$\begin{aligned} M_2 &= (1/c^3) \left[(2c^3 + 2c^3\sqrt{2 - c^2}\theta - c\sqrt{2 - c^2}\theta - 3c) A + (3c^2\theta + 2c^2\sqrt{2 - c^2} \right. \\ &\quad \left. - \sqrt{2 - c^2} - 2c^4\theta) B + (3c - 2c^3 + c\sqrt{2 - c^2}(\theta + x_2 - x_1)) \right], \end{aligned} \quad (4.5)$$

$$\begin{aligned} M_3 &= (1/c^3) \left[(c\theta + c\sqrt{2 - c^2} - c^3\theta) A + (1 - c^2 - c^2\sqrt{2 - c^2}\theta) B \right. \\ &\quad \left. + (c^3(\theta + x_2 - x_1) - c(\theta + x_2 - x_1) - c\sqrt{2 - c^2}) \right], \end{aligned} \quad (4.6)$$

in which

$$A = \cos(c(x_2 - x_1)) \quad \text{and} \quad B = \sin(c(x_2 - x_1)).$$

Figure 7 shows M_1 , M_2 and M_3 plotted as functions of θ for $c^2 = 0.5$ using Maple with $x_2 - x_1$ obtained numerically from (4.1). It can be observed from Figure 7(a) that there are five values of θ , $\theta_0 > \theta_1 > \dots > \theta_4$, for which $M_2 = M_3 = 0$. This information is a little hard to see directly from 7(a), but plots (c) and (d) show blow-ups for θ near θ_0 and θ_4 . In each blow-up, each of the functions M_2 and M_3 have two zeros locally (the extra zero of M_2 near θ_4 in (d) occurs for $\theta < \theta_{\min}$), but only one of these also corresponds to a zero of the other function. The structure near the other θ_i is similar. Figure 7(b) shows that when $\theta = \theta_i$, $u'_2(x_2) < -\theta_{\min}(0.5) = -24.300683$. Moreover, from that graph, notice that $u'_2(x_2) = -\theta_i$ (as seen by intersections with the 45° line). Note that this implies that $u'_1(x_1) = -u'_3(x_2)$, which is a necessary condition for a symmetric homoclinic orbit. The other five intersections with the 45° line correspond to where $u'_1(x_1) = -u'_3(x_2)$, but $u'''_1(x_1) \neq -u'''_3(x_2)$. More significantly, we have already proved that there are five symmetric one-troughed solutions for $c^2 = 0.5$, and the present analysis shows that there are a total of five one-troughed solutions independent of symmetry. Note too that from Figures 7 and 2(b) that the θ -values of the solutions found by the two different methods appear identical. These results, which are partly numerical, lead to a very strong indication, although not quite a proof of the following.

Proposition 4.1 *For $c^2 = 0.5$ there are no asymmetric one-troughed homoclinic solutions.*

Figure 7: Numerically calculated matching conditions for the existence of not necessarily symmetric one-troughed homoclinic orbits at $c^2 = 0.5$, $\theta \in (\theta_{\min}(0.5), 0)$. (a) M_2 and M_3 given by (4.5) and (4.6); (b) M_1 given by (4.4) and the line $y = -x$; (c) and (d), blow-ups of (a) near the first and last simultaneous zero of each function.

Since $c^2 = 0.5$ showed the maximum possible number of symmetric one-troughed solutions and that we have found, by transversality, that there is no asymmetric bifurcation from symmetric one-troughed homoclinic orbits, it would be natural to make the following conjecture.

Conjecture 4.2 *There are no asymmetric one-troughed homoclinic solutions to (2.1) for any $c^2 \in [0, 2)$.*

Of course, due to the theory of section 3, there are infinitely many asymmetric multi-troughed solutions of (2.1) for $c_{min}^2 < c^2 < c_{max}^2$.

5 Numerical results

We now turn to numerical methods for the computation and the continuation with c^2 of homoclinic solutions to (2.1). We shall restrict ourselves to the computation of symmetric solutions. The methods used are a Newton-based shooting algorithm and continuation using Auto [9]; the reader is referred to [5, 6] for the details. In order to avoid numerical problems associated with the non-smoothness of (2.1), we replaced $(1 + u)^+$ with

$$(1/2)(1 + u) + (1/2)\sqrt{(1 + u)^2 + \varepsilon}, \quad \text{where } \varepsilon = 1.0 \times 10^{-10}.$$

Figure 8 shows six solutions calculated by shooting at $c^2 = 1.5$, the symmetric point of each one of which occurs in a trough with one more wiggle than the previous one. Note that the first two of these are the single-troughed solutions depicted in Figure 4. Each of the other solutions has a total of three troughs. One might be tempted to conjecture that the solutions in plots (c), (d) and (e) — with three, four and five wiggles respectively — form a continuous branch with the corresponding solutions in Figure 5 which are all single-troughed. Numerical continuation results to follow shortly, however, show that this is not the case.

Figure 9 shows two examples of multi-modal homoclinic solutions which are indicative of the infinity of solutions which are known to exist according the theory of section 3. Both solutions may be described as bi-modal because they consist of two copies of one-troughed waves, separated by a large number of small oscillations. The solution in (a) is a bi-modal based on the one-wiggle solution, and that in (b) is based on the two-wiggle solution.

The great multiplicity of multi-troughed solutions is illustrated by the symmetric homoclinic orbit depicted in Figure 10, which consists of several troughs of increasing depth up to the point of symmetry. The middle trough has many wiggles. It is tempting to conjecture, from the evidence of this solution, that for each $c^2 \in (0, 2)$ solutions exist which have an arbitrarily deep trough. As, c^2 tends to 2 however, there should be more and more subsidiary troughs leading up to the deepest one.

Figure 11(a) shows bifurcation diagrams depicting the minimum of symmetric homoclinic orbits against c^2 . As well as the first five² solutions depicted in Figure 8, the branch of solutions which in the notation of [5] would be labeled **2**(2) is also depicted. This latter is a bi-modal solution based on the one-wiggle solution with a single local maximum (for $u > 0$) between its two troughs. Notice from the Figure that only two solutions appear to survive until $c^2 = 2$, the primary and **2**(2). Graphs of these solutions for c^2 close to this limit are depicted in Figure 11(b) and (c). Note that the primary solution has a negative minimum at the symmetry point whereas the **2**(2) solution has a positive maximum. Necessarily both solutions have minima below -1 , otherwise they would be solutions of a linear problem

²the sixth solution behaves similarly, but does not appear on this diagram because its minimum is always below -80

Figure 8: Six multi-wiggle solutions computed at $c^2 = 1.5$.

Figure 9: Two bi-modal solutions computed at $c^2 = 1.5$.

Figure 10: A deep, multi-troughed solution computed at $c^2 = 1.5$.

Figure 11: (a) Bifurcation diagram of (1--5) the first five solutions depicted in Figure 8 and (2(2)) the simplest bimodal solution associated with 1. (b) Solution on the branch 1 at $c^2 = 1.9984$. (c) Solution on the branch 2(2) at $c^2 = 1.9984$.

and hence not homoclinic solutions. Also, it can be just about made out from the Figure 11(b), that the primary solution has another trough below -1 before the trough containing its symmetric homoclinic point. Recall the result in section 2 that there are no single-troughed symmetric homoclinic solutions for $c^2 > 1.997$, and note that the solution plotted in for $c^2 = 1.9984$.

All other solutions depicted in Figure 11 lie on closed curves that have a limit point with respect to c^2 at a value of $c^2 < 2$. In the parlance of [5] these points are coalescences, caused by a zero of test function T_2 along the curve of homoclinic solutions. Figure 12 depicts the metamorphosis that occurs to three of the solutions as one traces a branch around the limit point from the starting solution depicted in Figure 8. In each case, two solutions are plotted at the same c^2 -value. The process of going around the limit point either adds an extra trough or metamorphoses the non-middle trough. In no case is the number of troughs decreased upon going around the limit point starting from those in Figure 8. Thus, apart from the 1 and 2 wiggle solutions, the five single-troughed solutions depicted at $c^2 = 0.5$ in Figure 5 do *not* form continuous branches with the corresponding solutions in Figure 8. We conjecture that these additional three solutions at $c^2 = 0.5$ disappear via coalescence soon after they cease to be single troughed.

All numerical evidence so far points to solutions not changing their qualitative shape (i.e. number of troughs, number of wiggles in each trough and number of oscillations between each trough) as c^2 is decreased. However, the amplitude of each solution increases. More precisely, the minimum of each trough appears to decrease uniformly as c^2 decreases, tending to $-\infty$ as $c^2 \rightarrow 0$. This observation accords with the result of [11] that all solutions tend to infinity in sup norm as $c^2 \rightarrow 0^+$.

A complete understanding of the global structure of the infinite number of homoclinic solutions is, of course, impossible to obtain using a finite number of numerical experiments.

6 Concluding remarks and open problems

At this time, we wish to summarize three separate ideas: what we have proved, what we have found via numerical experimentation, and what we suspect to be the mathematical truth, but which we cannot verify.

We have *proved* that there exist strictly nested intervals $(0, 2) \supset I_1 \supset I_2 \supset I_3 \supset I_4 \supset I_5$ such that for c^2 in the interval I_i , there are at least i solutions of the *one-troughed* type: namely, solutions which lie below the line $u = -1$ only on one interval of the x -axis. The largest of these intervals is $(0.007612, 1.997861)$. As shown in figure 5, these solutions are distinguished by the number of changes of sign in their second derivatives *below* $u = -1$ which we have been calling “wiggles”.

We have also proved, using the transversality arguments, that there exist infinitely many solutions which are approximately the superposition of large translates of the elementary solutions found explicitly. We do not know if one can get solutions which are approximately the superposition of two different solutions with the same wave speed c .

Numerically, we have used a combination of shooting, Newton’s method and continuation methods to “flesh out” the picture by continuing the bifurcation branches found analytically.

As shown in figure 11, they *appear to* continue to a turning point somewhat short of the point 2, and turn into a solution with either more wiggles or more troughs.

One branch, namely the one corresponding to the simplest wave (as shown in figure 5(a)), seems to persist all the way to $c^2 = 2$, while the rest turn at some point before this end point. We now come to our first mathematical problem:

Problem 6.1 *Show that there are always two solutions for all c such that $0 < c^2 < 2$*

Figure 12: (a) Two solutions on the branch 2 at $c^2 = 1.5$. (b) Two solutions on the branch 3 at $c^2 = 1.5$. (c) Two solutions on the branch 5 at $c^2 = 1.5$.

Where we were able to explicitly obtain a formula for the solutions, (in the intervals I_i), we were able to use transversality theory to patch these solutions to obtain infinitely many multi-trough solutions. It is natural to suspect that if the original primary solutions persist, so should these. This leads to the next problem, where we suggest the possibility of this result for all c .

Problem 6.2 *Show that these solutions can always be patched together to obtain infinitely many multi-troughed solutions which resemble superpositions of translates of these primary solutions.*

The next problem is motivated by figure 10, where a huge multi-wiggle solution is numerically constructed.

Problem 6.3 *Show that there are infinitely many solutions for any fixed c in $(0, 2^{1/2})$ with arbitrarily large numbers of sign-changes in the second derivative of the solution.*

Finally, we recall that a smoothed-out version of equation (2.1), namely

$$u^{iv} + c^2 u'' + e^u - 1 = 0 \quad (6.1)$$

has been shown [13] to give travelling wave solutions of the corresponding beam equation with remarkable interaction and stability properties. This leads to the question of how general a nonlinearity can be permitted for this global picture to survive.

Problem 6.4 *Show that these results continue to exist for a larger class of nonlinearities, preferably including the exponential nonlinearity mentioned above.*

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References

- [1] Amann, O.H., von Karman, T., & Woodruff, G.B., The Failure of the Tacoma Narrows Bridge, Federal Works Agency, 1941.
- [2] Belyakov, L.A. & Shil'nikov, L.P. Homoclinic curves and complex solitary waves. *Selecta Math. Sov.* **9** (1990) 219-228.
- [3] Blackmore, A. & Hunt, G.W. Principles of localised buckling for a strut on an elastoplastic foundation. *Transactions of the ASME* **63** (1996) 234-239.
- [4] Buffoni, B. Periodic and homoclinic orbits for Lorentz-Lagrangian systems via variational methods. *Nonlinear Analysis TMA*, **26** (1995) 443-462.
- [5] Buffoni, B., Champneys, A.R. & Toland J.F. Coalescence and bifurcation of a plethora of homoclinic orbits for an Hamiltonian system. *J. Dyn. Diff. Eqns.* **8** (1996) 221-281.

- [6] Champneys, A.R. & Spence, A. Hunting for homoclinic orbits in reversible systems: a shooting technique. *Adv. Comp. Math.* **1** (1993) 81-108.
- [7] Char, B.W. *et al. Maple V Language Reference Manual* Springer-Verlag: Berlin, 1995.
- [8] Devaney, R.L. Homoclinic orbits in Hamiltonian systems. *J. Diff. Eqns* **21** (1976) 431-438.
- [9] Doedel, E.J., Keller, H.B. & Kernévez, J. P. Numerical analysis and control of bifurcation problems. *Int. J. Bifurcation & Chaos*, **1** (1991) 493-520, 745-772.
- [10] Knobloch, J. Bifurcation of degenerate homoclinics in reversible and conservative systems. *J. Dyn. Diff. Eqns.*, submitted.
- [11] Lazer, A.C. & McKenna, P.J. Abstract, proceedings of AMS annual meeting, 1991.
- [12] Lin, Li & Adams, G.G., Beam on tensionless elastic foundation *J. Eng. Mech.*, 113 (1986), 542-553.
- [13] McKenna, P.J. & Chen, Yue Travelling waves in a nonlinearly suspended beam: theoretical results and numerical observations. *J. Diff. Eqns.*, to appear.
- [14] McKenna, P.J. & Walter, W. Traveling waves in a suspension bridge. *SIAM J. Appl. Math.* **50**, (1990) 703-15.
- [15] Peletier, L.A & Troy, W.C. Multi-bump periodic travelling waves in suspension bridges. Preprint, University of Leiden, 1996.

A Test functions for transversality

We derive here analytic expressions for $T_1(c^2, \theta)$ and $T_2(c^2, \theta)$ expressed implicitly in (3.2) and (3.3). First, we substitute

$$\cos(\phi) = \frac{c^2 - 1 + c^2 \sqrt{2 - c^2} \theta}{\sqrt{1 + c^2 \theta^2}}, \quad \sin(\phi) = \frac{c(c^2 - 1) \theta - c \sqrt{2 - c^2}}{\sqrt{1 + c^2 \theta^2}},$$

along with the expressions (2.21), (2.23) and (2.24) for α , β and γ , into (2.13)–(2.16) to obtain expressions for $u_2(x)$ and its derivatives. Evaluating at $x = 0$ one obtains

$$u_2(0) = \frac{(2 - 2c^2 - 2c^2 \sqrt{2 - c^2} \theta) \cos(cx_1) + (2c\theta + 2c\sqrt{2 - c^2} - 2c^3\theta) \sin(cx_1) - (c^4 + c^2\theta^2)}{2c^4} \quad (\text{A.1})$$

$$u_2'(0) = (1/c^3) \left[(c^3\theta - c\theta - c\sqrt{2 - c^2}) \cos(cx_1) + (1 - c^2 - c^2\sqrt{2 - c^2} \theta) \sin(cx_1) \right] \quad (\text{A.2})$$

$$u_2''(0) = (1/c^2) \left[(c^2 + c^2\sqrt{2 - c^2} \theta - 1) \cos(cx_1) + (c^3\theta - c\theta - c\sqrt{2 - c^2}) \sin(cx_1) + 1 \right] \quad (\text{A.3})$$

$$u_2'''(0) = (1/c) \left[(c\sqrt{2 - c^2} + c\theta - c^3\theta) \cos(cx_1) + (c^2 + c^2\sqrt{2 - c^2} \theta - 1) \sin(cx_1) \right] \quad (\text{A.4})$$

where, for a symmetric homoclinic point to occur at $x = 0$, the first equation of (2.25) gives that

$$x_1 = c^2\beta = \sqrt{2 - c^2} + \theta. \quad (\text{A.5})$$

Substituting for x_1 using (A.5) and taking derivatives of (A.1)–(A.4) with respect to θ one obtains

$$v_\theta = (\theta/c^2) \left[(1 - c^2) \cos(cx_1) + c\sqrt{2 - c^2} \sin(cx_1) - 1 \right] \quad (\text{A.6})$$

$$v'_\theta = (\theta/c) \left[-\sqrt{2 - c^2} \cos(cx_1) + (1 - c^2) \sin(cx_1) \right] \quad (\text{A.7})$$

$$v''_\theta = \theta \left[(c^2 - 1) \cos(cx_1) - c\sqrt{2 - c^2} \sin(cx_1) \right] \quad (\text{A.8})$$

$$v'''_\theta = c\theta \left[c\sqrt{2 - c^2} \cos(cx_1) + (c^2 - 1) \sin(cx_1) \right]. \quad (\text{A.9})$$

Taking the appropriate combinations of (A.6)–(A.9) and (A.3) according to (3.2) and (3.3) gives

$$\begin{aligned} T_1(c^2, \theta) = & (\theta^2/c^4) \left[(1 + 3c^4 - 4c^2 + 2c^8 - 4c^6) A^2 + (-2c^7\sqrt{2 - c^2} - 2c^3\sqrt{2 - c^2} + 2c\sqrt{2 - c^2} \right. \\ & \left. + 2c^5\sqrt{2 - c^2}) AB + (2c^2 - 2) A - 2c\sqrt{2 - c^2} B + (2c^2 + 2c^6 - c^8 - c^4 + 1) \right] \end{aligned}$$

and

$$\begin{aligned} T_2(c^2, \theta) = & (\theta/c^2) \left[(c^2 - 2\sqrt{2 - c^2}c^2 + 2c^4\sqrt{2 - c^2} + 4\theta c^4 - 2\theta c^4\sqrt{2 - c^2} - 2c^6\theta - \theta c^2 + 2c^6 - 4c^4 \right. \\ & \left. + 2\theta c^6\sqrt{2 - c^2}) A^2 + (2c^3\sqrt{2 - c^2} + c^3\theta + 2\theta c^7 - 4\theta c^5 - 2c^3\theta\sqrt{2 - c^2} - 2c^5\sqrt{2 - c^2} \right. \\ & \left. + 2c^5\theta\sqrt{2 - c^2} + c + 2c^5 - 4c^3) AB + (\sqrt{2 - c^2}c^2) A + (c^3 - c) B + (\theta c^2 - c^6 + c^6\theta \right. \\ & \left. - 2\theta c^4 - c^4\sqrt{2 - c^2} + \sqrt{2 - c^2}c^2 - \theta c^6\sqrt{2 - c^2} + 2c^4 + \theta c^4\sqrt{2 - c^2}) \right], \end{aligned}$$

in which

$$A = \cos(c\theta + c\sqrt{2 - c^2}) \quad \text{and} \quad B = \sin(c\theta + c\sqrt{2 - c^2}).$$