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A two parameter study near the locking region of a semiconductor laser with phase-conjugate feedback

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ABSTRACT

A semiconductor laser subject to phase-conjugate optical feedback can be described by rate equations, which are mathematically delay differential equations (DDEs) with an infinite-dimensional phase space. We employ new numerical continuation techniques for DDEs to study the exact nature of the locking region in the parameter plane given by the feedback strength and the pump current. This reveals interesting dynamics, including heteroclinic bifurcations, near the locking region, leading to different scenarios of possible transitions into and out of locking. We show how several special points act as organizing centers for the dynamics.

Keywords: Semiconductor lasers, phase-conjugate optical feedback, locking region, bifurcation analysis

1. INTRODUCTION

In this paper we consider the dynamics of a semiconductor laser that receives phase-conjugate feedback (PCF) from a phase-conjugating mirror (PCM) placed at distance L_{ext} from the laser; see Fig. 1. Unlike in the case of conventional optical feedback (COF) from a regular mirror, the PCM reverses the phase of the light and the reflected wave travels back along the same path as the incident wave. This is physically interesting because the system is self-aligning. Moreover, perturbations of the light in the external cavity cancel each other out in the two passes to and from the PCM. This produces a highly focused beam¹ of considerable stability that can be used for applications such as mode locking and phase locking.² Furthermore, PCF is known to reduce the laser noise considerably.^{3–5} Many interesting types of dynamics have been identified in the PCF laser, including periodic, quasiperiodic and chaotic output.^{1, 3, 4, 6}

The main difficulty with any laser system featuring (optical) feedback is that it leads to a mathematical description by a system of delay differential equations (DDEs). Since DDEs have an infinite-dimensional phase space, namely the space of continuous functions over the delay time (in our case the fixed delay time $\tau = 2L_{\text{ext}}/c$), they are inherently difficult to study. We will not dwell here on the theory of DDEs and instead refer the interested reader to, for example, Refs. [7–9] for details.

For a long time numerical integration (simulation) of the relevant rate equations was the only tool to study complicated dynamics in DDEs. This is now changing with the implementation of advanced continuation techniques in the package DDE-BIFTOOL,¹⁰ which allows one to find and follow steady states, periodic orbits and their bifurcations, and also homoclinic and heteroclinic orbits.¹¹

Continuation has not yet been widely used to study the dynamics and bifurcations in feedback lasers (or other DDEs arising from applications). First examples include the study of connecting bridges of periodic solutions in the COF laser in Refs. [12–14], similar work on a VCSEL in Ref. [15], and our previous work on the PCF laser in Refs. [16–18]. All these continuation studies followed steady states and periodic orbits as a *single parameter* is changed (usually the strength of the feedback).

In this paper we use numerical continuation to perform a *two-parameter bifurcation study* in the plane of feedback strength and pump current near the locking region of the PCF laser; more details, in particular on the details of numerical continuation in DDEs, can be found in Ref. [19]. We obtain a consistent bifurcation diagram explaining the different routes into and out of locking. This work follows on from our one-parameter study near the locking region of the PCF laser in Ref. [16]. We identify several codimension-two bifurcations, most importantly, a double-Hopf point and a heteroclinic bifurcation called a *T-point*, and show how they organize the dynamics of the PCF laser.

Figure 1: Sketch of a semiconductor laser with phase-conjugate feedback.

2. RATE EQUATIONS OF PCF LASER

A single-mode semiconductor laser receiving weak (instantaneous) PCF can be described by the three-dimensional rate equations

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2} \left[-i\alpha G_N(N(t) - N_{\text{sol}}) + \left(G(t) - \frac{1}{\tau_p} \right) \right] E(t) + \kappa E^*(t - \tau) \\ \frac{dN}{dt} &= \frac{I}{q} - \frac{N(t)}{\tau_e} - G(t) |E(t)|^2 \end{aligned} \quad (1)$$

for the evolution of the complex electric field $E(t) = E_x(t) + iE_y(t)$ and the population inversion $N(t)$.^{4,6} In system (1) nonlinear gain is included as $G(t) = G_N(N(t) - N_0)(1 - \epsilon P(t))$, where $\epsilon = 3.57 \times 10^{-8}$ is the nonlinear gain coefficient and $P(t) = |E(t)|^2$ the intensity. Parameter values are set to realistic values corresponding to a Ga-Al-As semiconductor laser,^{4,6} namely, the line-width enhancement factor $\alpha = 3$, the optical gain $G_N = 1190 \text{ s}^{-1}$, the photon lifetime $\tau_p = 1.4 \text{ ps}$, the magnitude of the electron charge $q = 1.6 \times 10^{-19} \text{ C}$, the electron lifetime $\tau_e = 2 \text{ ns}$, and the transparency electron number $N_0 = 1.64 \times 10^8$. Further, $N_{\text{sol}} = N_0 + 1 / (G_N \tau_p)$. The feedback term in system (1) is simply $\kappa E^*(t - \tau)$ under the common assumption that the relevant phase factors are zero.^{4,6} The feedback rate is κ and the external cavity round-trip time is τ , which we fix at the realistic value $\tau = 2/3 \text{ ns}$, corresponding to an external-cavity length of $L_{\text{ext}} \approx 10 \text{ cm}$.

An important property of system (1) is its $\mathbb{Z}_2$ -symmetry under the transformation $(E, N) \rightarrow (-E, N)$, that is, under a rotation over π of the complex E -plane. As a consequence, any attractor (or other invariant set) is either symmetric, or has a symmetric counterpart. The $\mathbb{Z}_2$ -symmetry allows the possibility of symmetry-breaking bifurcations and implies restrictions on the types of bifurcations of periodic orbits.^{6,20}

3. THE LOCKING REGION

The *locking region* of the PCF laser is the region where system (1) has a stable steady state solution. It represents a frequency match between the solitary laser and the pump lasers used in the four-wave mixing to produce the PCM.²¹ In the locking region the laser is frequency and phase locked, resulting in an extremely narrow line-width (even in the presence of spontaneous emission noise).³⁻⁵

With DDE-BIFTOOL we detected and followed the bifurcations involved in the locking mechanism of the PCF laser. This resulted in the bifurcation diagram in Fig. 2 showing curves of Hopf bifurcations $H_{1,2}$, curves of saddle-node bifurcations SN and pitchfork bifurcations PF , and the curve Het of heteroclinic connections between two saddle steady states that are each others symmetric counterparts. The curves $H_{1,2}$ are drawn

Figure 2: Bifurcations bounding the locking region of the PCF laser in the $(\kappa\tau, I)$ -plane.

dark when they are supercritical (the bifurcating periodic orbit is stable), and lighter when they are subcritical. Also plotted are individual points of saddle-node bifurcations of limit cycle (SL), period-doubling bifurcations (PD) and torus bifurcations (T). (These points can be detected by DDE-BIFTOOL in suitable one-parameter sections, but the corresponding bifurcation curves cannot be continued at present.)

The locking region is bounded on the left by the curve of saddle-node bifurcations SN and from below by the curve of pitchfork bifurcations PF . Its boundary on the right is formed by the (supercritical) leftmost parts of the curves H_1 and H_2 , respectively. Inside the locking region there is a curve Het of heteroclinic orbits. The area above Het , and between the curves SN and H_2 is a region of bistability: a periodic orbit and a pair of non-symmetric steady states coexist. There is a second region of bistability, to the right of H_2 and above Het , where two stable periodic solutions coexist.

The saddle-node bifurcation SN is responsible for the birth of a non-symmetric attracting and a non-symmetric saddle steady state (and their symmetric counterparts). The attracting steady state corresponds to locked behavior of the PCF laser. It destabilizes along the curves of Hopf bifurcations $H_{1,2}$ where the laser develops periodic oscillations of the power. Along the region where H_1 is subcritical the bifurcating periodic orbit is unstable and this Hopf bifurcation is preceded by the birth of a pair of periodic orbits at SL . This leads to a bistability at the boundary of the locking region; see also Ref. [16].

The pitchfork curve PF marks the loss of stability of the trivial steady state $(E, N) = (0, \frac{I\tau_e}{q})$. Physically, the PCF laser is in its off-state below PF and locked above PF . In other words, the pitchfork bifurcation constitutes the laser threshold in this system with $\mathbb{Z}_2$ -symmetry.

Figure 3. The bifurcation diagram in $(\kappa\tau, I)$ -space near the T-point bifurcation TP ; panel (a) also shows bifurcations of periodic orbits originating from the Hopf curve H_1 , and panel (b) also shows bifurcations of periodic orbits originating from the Hopf curve H_2 .

There is a special point, called a double-Hopf bifurcation (DH), where the curves $H_{1,2}$ intersect at $(\kappa\tau, I) \approx (0.893, 0.06589)$. At this codimension-two point there are two pairs of complex eigenvalues on the imaginary axis²² and the center manifold is four-dimensional. Near a double-Hopf point the system can bifurcate to a number of invariant objects, including two-dimensional tori. It appears that near DH there is a switch from a period-doubling route to chaos to a route to chaos via the break-up of a torus, as is evidenced by the points PD and T that we managed to find. However, at present we are not able to resolve the exact structure near the double-Hopf point DH , because curves of codimension-one bifurcations of periodic orbits cannot be computed.

A second special point is at $(\kappa\tau, I) \approx (0.225, 0.06433)$ where the heteroclinic curve Het ends at the saddle-node curve SN . This codimension-two bifurcation is called a saddle-node heteroclinic point (SNH); it is a saddle-node homoclinic bifurcation²² when dividing out the $\mathbb{Z}_2$ -symmetry of system (1). Below the point SNH the saddle-node bifurcation along SN takes place on a periodic orbit.¹⁶

4. HETEROCLINIC ORBITS

Along the curve Het there is a heteroclinic orbit between the two saddle steady states that are each other's counterparts under the $\mathbb{Z}_2$ -symmetry. This orbit was found and followed in the $(\kappa\tau, I)$ -plane with a recent addition to DDE-BIFTOOL.¹¹ Followed further away from the locking region, the curve Het starts to curl up and converge to a single point, as is shown in Fig. 3. The center point of this spiral is a codimension-two point known as a T-point²³ and it is marked TP . At TP the heteroclinic connection between the two non-symmetric steady states is destroyed. This is due to the creation of two new heteroclinic orbits. To show that this is indeed the case we must look at the heteroclinic orbits themselves.

Figure 4 shows heteroclinic orbits along the branch Het , projected onto (E, N) -space. Near the saddle-node bifurcation SN [Fig. 4(a)] the heteroclinic orbit is seen to leave one saddle steady state and spiral into its symmetric counterpart; this was also found in Ref. [16]. As one moves along the curve Het , the heteroclinic orbits starts to increase in size in (E, N) -space [Fig. 4(b) and (c)]. As the T-point TP is approached, the heteroclinic orbit continues to grow in (E, N) -space [Fig. 4(d)] until the heteroclinic orbit is seen to pass very near the origin of the E -plane [Fig. 4(e)]. At the same time, the value of the inversion N grows with a final rapid oscillation before ending up at the other non-symmetric saddle steady state. Indeed the heteroclinic orbit passes extremely close to the trivial steady state, as is shown in the enlargement in Fig. 4(f). This is a clear indication that we are very close to the T-point bifurcation.

To check this further, we computed the new heteroclinic orbits (involving the trivial steady state) at the T-point TP . Figure 5(a) shows the original heteroclinic orbit of codimension one between the two non-symmetric saddle steady states. Being so close to TP , this orbit starts at one of the non-symmetric steady states and then spends much time at the trivial steady state before a sudden oscillation back to the end non-symmetric steady state. This allows us to split up this orbit at TP into its two parts. The first of them is shown in Fig. 5(b), where the non-symmetric steady state connects to the trivial steady state. This is a connection of codimension-two, that is, it only exists exactly at the special point TP . The second heteroclinic orbit, from the trivial steady state back to the non-symmetric steady state, is shown in Fig. 5(c). This connection is in fact of codimension zero (owing to the dimensions of the respective stable and unstable manifolds), meaning that it exists in an entire neighborhood of the point TP .

It is known that near a T-point there are other codimension-one homoclinic and heteroclinic orbits.²³ However, it appears to be very difficult to find and follow these solutions, so that a study of the dynamics near TP remains a challenge for the future.

5. BIFURCATIONS OF PERIODIC ORBITS

We have already pointed out some bifurcations of periodic orbits near the locking region in Fig. 4. Now we discuss in more detail the local codimension-one bifurcations SL , PD and T shown in Fig. 3. In panel (a) we plotted bifurcations of the periodic orbit born along the Hopf curve H_1 and in panel (b) those of the periodic orbit born along the Hopf curve H_2 . The indicated points were found by detecting with DDE-BIFTOOL the respective changes of the Floquet multipliers in one-parameter continuation studies in the parameter $\kappa\tau$ for fixed values of the pump I .

Figure 4. Heteroclinic orbits along the curve Het ; from (a) to (e) $(\kappa\tau, I)$ takes the values $(0.314, 0.065264)$, $(1.303, 0.069026)$, $(2.085, 0.069635)$, $(2.201, 0.070357)$, and $(2.177, 0.070394)$. Panel (f) is an enlargement of (e) showing the part of the heteroclinic orbit close to the trivial steady state.

Figure 5. Codimension-one heteroclinic orbit very close to the T-point TP (a), and the corresponding codimension-two (b) and codimension-zero (c) heteroclinic orbits at the T-point TP.

Figure 3(a) shows a curve of period-doubling, closely followed in a certain region by a curve of torus bifurcation. It appears that the torus bifurcation curve starts and ends at the period-doubling curve, but this is difficult to verify at present.

The organization of bifurcations is much more complicated in Fig. 3(b). One clearly notices a region bounded by saddle-node bifurcations of limit cycles SL , with a cusp at its bottom. Along the curve SL a pair of periodic orbits bifurcates. The points of period-doubling and of torus bifurcations appear to form nice curves, but it is less clear where they start and end up, and exactly which points belong to which curve. It is known that the different curves can interact at special points, such as 1:1 and 1:2 resonance points. This is not uncommon in laser systems; see, for example, Ref. [24]. However, the exact structure of these bifurcations will only be revealed when the continuation of the respective curves becomes available.

6. CONCLUSIONS

We have provided a state-of-the-art two-parameter bifurcation analysis of the locking region of the PCF laser. The general picture is that the locking region is bounded by a saddle-node bifurcation and/or a heteroclinic bifurcation on one side and by Hopf bifurcations on the other, which is consistent with a previous one-parameter study.¹⁶ A curve of pitchfork bifurcations was shown to form the laser threshold. Of particular interest is a heteroclinic bifurcation that we followed in two parameters. We identified a double-Hopf point, a saddle-node heteroclinic bifurcation, and a T-point bifurcation as special points in the bifurcation diagram and showed how they organize the dynamics. Finally, we made a first attempt at mapping out bifurcations of periodic orbits in two parameters. In summary, we presented a consistent picture of the PCF laser near its locking region.

To our knowledge what was presented here is the first numerical continuation study of a DDE in two parameters. Our results showcase the usefulness of continuation tools for the study of complicated dynamics and bifurcations in lasers with (optical) feedback.

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REFERENCES

1. C. R. Giuliano, "Applications of optical phase conjugation," *Physics Today* **34**, pp. 27–35, April 1981.
2. G. R. Gray, D. H. DeTienne, and G. P. Agrawal, "Mode locking in semiconductor lasers by phase-conjugate optical feedback," *Opt. Lett.* **20**, pp. 1295–1297, 1995.
3. G. P. Agrawal and G. R. Gray, "Effect of phase-conjugate feedback on the noise characteristics of semiconductor lasers," *Phys. Rev. A* **46**, pp. 5890–5898, 1992.
4. G. R. Gray, D. Huang, and G. P. Agrawal, "Chaotic dynamics of semiconductor lasers with phase-conjugate feedback," *Phys. Rev. A* **49**, pp. 2096–2105, 1994.
5. G. H. M. Van Tartwijk, H. J. C. Van der Linden, and D. Lenstra, "Theory of a diode laser with phase-conjugate feedback," *Opt. Lett.* **17**, pp. 1590–1592, 1995.
6. B. Krauskopf, G. R. Gray, and D. Lenstra, "Semiconductor laser with phase-conjugate feedback: Dynamics and bifurcations," *Phys. Rev. E* **58**, pp. 7190–7196, 1998.
7. O. Diekmann, S. A. Van Gils, S. M. V. Lunel, and H. O. Walther, *Delay Equations: Functional-, Complex-, and Nonlinear Analysis*, Springer, 1995.
8. J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer, 1993.
9. S. M. Verduyn Lunel and B. Krauskopf, "The mathematics of delay equations with an application to the Lang-Kobayashi equations," in *Fundamental Issues of Nonlinear Laser Dynamics*, B. Krauskopf and D. Lenstra (Eds.), AIP Conference Proceedings **548**, pp. 66–86, 2000.
10. K. Engelborghs, T. Luzyanina, and G. Samaey, "DDE-BIFTOOL v2.00: a Matlab package for bifurcation analysis of delay differential equations," Tech. Rep. TW-330, Department of Computer Science, K. U. Leuven, Belgium, 2000. (<http://www.cs.kuleuven.ac.be/~koen/delay/ddebiftool.shtml>)
11. G. Samaey, K. Engelborghs, and D. Roose, "Numerical computation of connecting orbits in delay differential equations," *Numer. Algorithms* **30**(3–4), pp. 335–352, 2002.
12. B. Haegeman, K. Engelborghs, D. Roose, D. Pieroux, and T. Erneux, "Stability and rupture of bifurcation bridges in semiconductor lasers subject to optical feedback," *Phys. Rev. E* **66** (046216), 2002.
13. D. Pieroux, T. Erneux, B. Haegeman, K. Engelborghs, and D. Roose, "Bridges of periodic solutions and tori in semiconductor lasers subject to delay," *Phys. Rev. Lett.* **87** (193901), 2001.
14. D. Pieroux, T. Erneux, T. Luzyanina, and K. Engelborghs, "Interacting pairs of periodic solutions lead to tori in lasers subject to delayed feedback," *Phys. Rev. E* **63** (036211), 2001.
15. M. Sciamanna, T. Erneux, F. Rogister, O. Deparis, P. Megret, and M. Blondel, "Bifurcation bridges between external-cavity modes lead to polarization self-modulation in vertical-cavity surface-emitting lasers," *Phys. Rev. A* **65** (041801(R)), 2002.
16. K. Green and B. Krauskopf, "Global bifurcations and bistability at the locking boundaries of a semiconductor laser with phase-conjugate feedback," *Phys. Rev. E* **66** (016220), 2002.
17. K. Green, B. Krauskopf, and K. Engelborghs, "Bistability and torus break-up in a semiconductor laser with phase-conjugate feedback," *Physica D* **173**, pp. 114–129, 2002.
18. K. Green and B. Krauskopf, "Bifurcation analysis of frequency locking in a semiconductor laser with phase-conjugate feedback," *Int. J. Bif. Chaos*, to appear.
19. K. Green, B. Krauskopf, and G. Samaey, "A two-parameter study of the locking region of a semiconductor laser subject to phase-conjugate feedback," Tech. Rep. 2002.11, Applied Nonlinear Mathematics, 2002. (<http://www.enm.bris.ac.uk/anm/preprints/2002r11.html>)
20. B. Krauskopf, G. H. M. Van Tartwijk, and G. R. Gray, "Symmetry properties of lasers subject to optical feedback," *Opt. Commun.* **177**, pp. 347–353, 2000.
21. G. H. M. Van Tartwijk and G. P. Agrawal, "Laser instabilities: a modern perspective," *Prog. Quantum Electron.* **22**, pp. 43–122, 1998.
22. Y. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer, 1995.
23. P. Glendinning and C. Sparrow, "T-points: A codimension two heteroclinic bifurcation," *J. Statist. Phys.* **43**, pp. 479–488, 1986.
24. S. Wicczorek, B. Krauskopf, and D. Lenstra, "Unnested islands of period-doubling in an injected semiconductor laser," *Phys. Rev. E* **64** (056204), 2001.