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Mode structure of delay-coupled semiconductor lasers: influence of the pump current

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Abstract. We consider two identical, mutually delay-coupled semiconductor lasers and show that their compound laser modes (CLMs) — the basic continuous wave solutions — depend rather sensitively on the pump current of the lasers. Specifically, we show with figures and accompanying animations how the underlying CLM structure and the associated locking region, where both lasers operate stably with the same frequency, change as a function of the pump current. In particular, our results provide a natural transition between rather different CLM structures that have been reported in the literature.

1. Introduction

Mutually delay-coupled semiconductor lasers (SLs) are receiving increasing attention because of possible applications, *e.g.*, for optical memories [13, 14] or ultra-fast optical clocks [15]. Moreover, as is argued *e.g.* in [7], delay-coupled SLs can be seen as a prototype example of delay-coupled oscillators in general.

Figure 1. Sketch of the mutually delay-coupled laser system.

The setup that we study here consists of two semiconductor lasers that are mutually delay-coupled in a face-to-face configuration via their electromagnetic fields; see Fig. 1. Apart from a possible difference between their free-running laser frequencies (describing a detuning Δ), the two lasers are assumed to be identical. An important element is the delay in the coupling, which arises from the finite propagation time of the light between the two spatially separated SLs. Due to the fast intrinsic time scales and the large bandwidth of SLs delay cannot be neglected and has a vital impact on the dynamics of the coupled system. This system has received quite some attention. Recently reported were synchronization for nonidentical SLs [1, 2], synchronization, symmetry breaking and high frequency oscillations for identical SLs [3, 5, 4], and different dynamics in the long and short coupling time regime [9, 8, 6, 7]. The impact of coupling on the mode structure and dynamics of the coupled system is studied in Refs. [11, 12].

We focus here on the structure of the *compound laser modes* (CLMs), which are continuous wave solutions of the two coupled lasers. Just as it is the case for the external cavity modes (ECMs) of a laser with conventional optical feedback (COF), the CLMs form the underlying skeleton of the system in the sense that their stability and overall structure are vital ingredients of the dynamics. A prominent example that stresses the utter importance of the underlying ECM structure is the Sisyphus cycle associated with low frequency fluctuations [16, 17]. In a similar way, complicated dynamics may consist of a succession of irregular and temporary visits of unstable CLMs. In other words, understanding the structure of the CLMs is an important step towards understanding the dynamics of the system.

It turns out that, owing to different possible types of CLMs, the CLM structure of delay-coupled SLs is much more complicated than that of the ECM structure of the COF laser. In Refs. [22, 23, 24] we found a characteristic structure of the CLMs for pump currents well above threshold. In Ref. [21], on the other hand, the case of a low pump current (near the solitary laser threshold) was considered and a very different CLM structure was found. This suggests that the CLM structure depends crucially on the level of pumping.

In this paper we address this sensitivity of the CLM structure on the pump current.

As in Refs. [22, 23, 24, 21], we consider a delay time on the order of the relaxation oscillations, which has the advantage that the number of CLMs is relatively small. Furthermore, in this regime the coupling phase is an important parameter [29]. This allows us to present a comprehensive geometrical picture of the CLM structure from small pump current, when the coupled laser system just starts to lase, to pump currents well above threshold. We consider both the case of zero detuning and of non-zero detuning between the two lasers. Furthermore, we show the influence of P on the locking region, where both lasers emit stably with the same frequency. We show how its shape in the plane of detuning Δ versus the coupling phase C_p changes in dependence of the pump current P . We also discuss what dynamics must be expected when the boundary of the locking region is crossed.

The dependence of the CLM structure and the locking region on the pump current is presented with images for representative choices of P . However, it is best observed in the accompanying animations. In particular, the images and animations clearly show how the two cases from Refs. [22, 23, 24] and Ref. [21] transform naturally into one another as a function of the pump current P .

Our results were obtained by performing a bifurcation analysis that describes how the CLMs appear, disappear and change their stability as parameters are changed; for an introduction to bifurcation analysis in the context of laser systems see [18]. While the paper is necessarily somewhat technical in places, our main results, and especially the animations, can be appreciated without a detailed knowledge of bifurcation theory. The key tool we used is *numerical continuation* with the software package DDE-BIFTOOL [19]. It allows one to find and follow (or continue) steady states and periodic solutions as parameters are changed and to detect bifurcations, such as changes of their stability. For a survey of how numerical continuation can be used for the bifurcation analysis of laser systems with delay, see [20].

2. Rate equation model and possible bifurcations of CLMs

We model the coupled laser system with rate equations for the normalized complex slowly-varying envelope of the optical fields $E_{1,2}$ and the normalized inversions $N_{1,2}$ (see [25] for details of the model):

$$\dot{E}_1 = (1 + i\alpha)N_1E_1 + \kappa e^{-iC_p}E_2(t - \tau) - i\Delta E_1, \quad (1)$$

$$\dot{E}_2 = (1 + i\alpha)N_2E_2 + \kappa e^{-iC_p}E_1(t - \tau) + i\Delta E_2, \quad (2)$$

$$T\dot{N}_1 = P - N_1 - (1 + 2N_1)|E_1|^2, \quad (3)$$

$$T\dot{N}_2 = P - N_2 - (1 + 2N_2)|E_2|^2. \quad (4)$$

In this dimensionless form time t is measured in units of the photon lifetime, which for SLs is typically on the order of picoseconds. The equations are written with respect to the mean optical frequency Ω of the free running SLs. Thus the optical fields are given by $E_{1,2}e^{i\Omega t}$. Apart from the difference in their solitary optical frequencies, which gives rise to the detuning Δ , the two lasers are identical. In particular, they have the

same pump current P (given with respect to the solitary laser threshold), self-phase modulation parameter α and carrier lifetime T . The mutual coupling is given by the second term of Eqs. (1) and (2). It contains the coupling rate κ , the delay time τ and the coupling phase $C_p = \Omega\tau$. The coupling phase C_p accounts for changes of the distance between the SLs on sub-wavelength scale, which have negligible influence on the delay time τ . This justifies to consider C_p as an independent parameter. As in [24] we adapt the physical meaningful values $\tau = 20.0$; $\kappa = 0.1$; $\alpha = 2.5$; $T = 392.0$.

We now consider the CLMs — the basic CW states of Eqs. (1)–(4). They are of the form

$$E_1(t) = R_1^s e^{i\omega^s t}, \quad E_2(t) = R_2^s e^{i\omega^s t + i\sigma}, \quad N_1(t) = N_1^s, \quad N_2(t) = N_2^s, \quad (5)$$

where ω^s is the common optical frequency (measured with respect to Ω) and σ is a phase difference between the electromagnetic fields of the two lasers. Furthermore, for a CLM the lasers have constant intensities $I_i^s = (R_i^s)^2$ and constant inversions N_i^s , which may be different for both lasers. Conceptually, the CLMs are the counterparts of the ECMs of a COF laser.

In the following we will discuss some general properties and possible bifurcations of the CLMs, in order to make our work more accessible to the reader not familiar with the approach of bifurcation theory and to connect to previous studies. This should portray a phenomenological picture of the possible bifurcations of CLMs.

Due to the fact that the CLMs have constant inversions and intensities, they are often referred to as *fixed points*, especially when representing the equations in terms of the intensities $I_{1,2}(t)$ and phases $\phi_{1,2}$; see *e.g.* Ref. [5]. Moreover, experimentally it is not possible yet to measure optical fields directly. In any practical case one is dealing with spectra (optical spectra and Rf-spectra) and time series of intensities. A CLM appears as a fixed point in the space of intensities and inversions. However, if the full dynamics of the optical field is taken into account, the CLMs (as well as the ECMs) are actually periodic orbits, because the optical field is oscillating with the optical frequency ω^s . Similar to the ECMs, this frequency is not the same for all CLMs, and thus cannot be ‘divided out globally’ [26].

We now briefly introduce the ways in which CLMs may appear, disappear or change their stability.

There are two possibilities how a CLM can emerge in system (1)–(4). The first is a *Hopf bifurcation of the off-state*, $(E_1, E_2, N_1, N_2) = (0, 0, P, P)$, of the coupled laser system. After this Hopf bifurcation a CLM appears, which means that the optical field starts to oscillate but the intensity is constant. In other words, a Hopf bifurcation of the off-state is physically the lasing threshold of the bifurcating CLM. (Note that in the intensity representation where $I_i^s = (R_i^s)^2$ this lasing threshold takes the form of a transcritical bifurcation [27].) The second possibility is a *saddle-node bifurcation of periodic orbits* in which a pair of CLMs with $R_{1,2}^s \neq 0$ is created, one of which is stable and the other unstable. In the following we simply refer to this bifurcation as a saddle-node bifurcation of CLMs. Note that for this bifurcation the intensities I_i^s of the lasers

do not go to zero.

The CLMs (as the ECMs of the COF laser) can undergo a third type of bifurcation, namely the intensity may start to oscillate with some, well-defined frequency that depends on the system parameters. When one monitors the Rf-spectrum, this bifurcation has the characteristics of a Hopf bifurcation: a new frequency appears and its amplitude shows a square root behavior as a function of parameters. Since a CLM is a periodic orbit this bifurcation is mathematically a torus bifurcation [26] but, as is common in the field, we refer to it as a *Hopf bifurcation of a CLM*.

Specific for the coupled laser system when the detuning Δ is zero, there is the possibility of a *pitchfork bifurcation of periodic orbits*. At this bifurcation (also called a symmetry breaking bifurcation) two asymmetrical CLMs are created from a symmetric CLM. Similar to the saddle-node bifurcation of a CLM, the intensities do not go to zero at the bifurcation point.

3. CLMs for zero detuning

We first consider the case of zero detuning, $\Delta = 0$, when there is the additional symmetry (in phase space) of exchanging the two lasers. In other words, any CLM either remains unchanged when exchanging the lasers or it comes with a symmetric counterpart. We distinguish between two types of CLMs; see [24] for details. First there are *constant-phase CLMs* (or symmetrical CLMs), which have $\sigma = 0$ and $\sigma = \pi$ and for which $N_1^s = N_2^s$ and, thus, $I_1^s = I_2^s$. Furthermore, there are *intermediate-phase CLMs* (or asymmetrical CLMs) for which the phase σ changes as a function of C_p and for which $N_1^s \neq N_2^s$ and $I_1^s \neq I_2^s$. From symmetry considerations (see *e.g.* Ref. [24]), it can be deduced that constant-phase CLMs are invariant under exchanging the lasers while intermediate-phase CLMs come in symmetric pairs.

Figure 2 shows curves of CLMs (also called branches) plotted in the $(\omega^s, N_{1,2}^s)$ -projection for different values of the pump current P ; the accompanying animation shows the continuous evolution of the branches as a function of P from the threshold of the coupled laser system up to the limit of high pump current where no further qualitative changes are observed. The gray areas in the panels corresponds to CLMs that have negative intensity and, hence, are non-physical. The horizontal line $N_{1,2} = P$ bounds this region; it corresponds to the lasing threshold of the respective CLM.

The curves shown in each panel of Fig. 2 are traced out by the CLMs when the coupling phase C_p is changed. For fixed C_p there are a finite number of CLMs. Generally, when C_p is decreased CLMs are born in pairs on the low frequency side and disappear in pairs on the high frequency side. In particular, all constant-phase CLMs lie on an ellipse. This ellipse is independent of the value of P , but a part of it may lie in the non-physical, grey region. (This ellipse is the exact counterpart of the well-known ellipse of ECMs of the COF laser.) Constant-phase CLMs are born in pairs in a saddle-node bifurcation marked by (+) and then move, one over the lower and one over the upper branch, towards the higher-frequency region. The constant-phase CLMs on the lower branch

Figure 2. Dependence on the pump current P of CLMs for $\Delta = 0$, shown in the $(\omega^s, N_{1,2}^s)$ -projection; Animation (1358k). The branches of CLMs are parametrized by C_p . Bold curves are stable regions; saddle-node bifurcations are marked by pluses (+), pitchfork bifurcations by diamonds ($\diamond$), and Hopf bifurcations by stars (*). The area $N_{1,2}^s > P$ is shaded gray. P takes the values indicated in the panels.

Figure 3. Enlarged view of the low-frequency region in the $(\omega^s, N_{1,2}^s)$ -projection for $\Delta = 0$. Same conventions as in Fig. 2. P takes the values indicated in the panels. Note the different scales of the axis. The insets show enlarged view around bifurcations.

may be stable: the boldfaced part of the constant-phase CLM ellipse is the locking region where both lasers operate with the same frequency and the same intensities. Figure 3 shows an enlarged view of the low-frequency region around the saddle-node bifurcation for different, representative values of P . It can be seen that for very low P [Fig. 3(a)] the saddle-node bifurcation is on the left-hand side of a Hopf bifurcation. This means that both constant-phase CLMs which are born in this saddle-node bifurcation are initially unstable. Quickly, one of them stabilizes in the Hopf bifurcation. For higher P [Fig. 3(b)] the order of saddle-node and Hopf bifurcation has exchanged. One of the constant-phase CLMs born in the saddle-node bifurcation is stable, the other one is unstable. The Hopf bifurcation bounding the locking region to the right ensures stable oscillations of the intensities. There are further Hopf bifurcations (all marked by $*$) [Fig. 2(a)–(h)]. In the high-frequency region the intensity of the lasers goes down to zero when they reach the line $N_{1,2} = P$ for low P [Fig. 2(a)–(d)]. The points where the ellipses intersect this line are Hopf bifurcations of the off-state. For sufficiently high P the entire ellipse consists of physical relevant CLMs. They disappear in a second saddle-node bifurcation in the high frequency region [Fig. 2(e)–(h)].

The intermediate-phase CLMs form all the other branches of CLMs in the panels of Fig. 2. For low values of P they do not exist [Fig. 2(a)], but around the solitary laser threshold of $P = 0$ an isola of intermediate-phase CLMs emerges [Fig. 2(b)], which is connected to the constant-phase CLMs at two pitchfork bifurcations, marked by diamonds ($\diamond$). See also enlargements around the pitchfork bifurcations in the

Figure 4. Dependence on the pump current P of CLMs for $\Delta = 0.02$, shown in the $(\omega^s, N_{1,2}^s)$ -projection; Animation (1468k). The branches of CLMs are parametrized by C_p , where red curves show N_1^s and blue curves show N_2^s . Bold curves are stable regions; saddle-node bifurcations are marked by pluses (+) and Hopf bifurcations by stars (*). The area $N_{1,2}^s > P$ is shaded gray. P takes the values indicated in the panels.

Figure 5. Enlarged view of the low-frequency region in the $(\omega^s, N_{1,2}^s)$ -projection, for $\Delta = 0.02$. Same conventions as in Fig. 4. P takes the values indicated in the panels. Note the different scales of the axis. The inset in panel (a) shows enlarged view around the bifurcations for the blue laser. The situation is similar for the red laser.

low frequency region of Fig. 3(b) and (c). When C_p is decreased the intermediate-phase CLMs are born in the left pitchfork bifurcation (in the low frequency region), move towards the right pitchfork bifurcation and disappear again. In the process the inversion of one of the lasers traces out the lower branch and the inversion of the other laser the upper branch. As P is increased further, the isola develops an increasingly complicated shape [Fig. 2(c)–(e)]. In particular, this involves the creation of extra saddle-node bifurcations (marked by (+)) where pairs of intermediate-phase CLMs appear or disappear as C_p is changed. We remark that Fig. 2(c) is very similar to the image of CLMs shown in Ref. [21]. Finally, the picture changes dramatically when two closed curves or isolas of intermediate-phase CLMs are shed off the main branch [Fig. 2(f)]. This happens when local branches of intermediate-phase CLMs re-connect in a different way in what is known as a saddle singularity transition; for more details on singularities and local bifurcations see, for example, [28]. The inversion of one laser is on the upper isola, while that of the other laser is on the lower isola. Intermediate-phase CLMs on the isolas appear and disappear in saddle-node bifurcations as C_p is changes. When P is increased even further [Fig. 2(g) and (h)] no further qualitative changes are observed. Instead a certain limiting situation is reached which is typical for pump currents quite far above threshold [22, 24].

Overall, the animation with Fig. 2 gives a very good impression of how the CLM structure for $\Delta = 0$ depends on and changes with the pump current P . In particular, it can be seen that the situation reported in Ref. [21] transforms naturally into the quite different situation for high pumping that is considered in Refs. [22, 24].

4. CLMs for non-zero detuning

As soon as $\Delta \neq 0$ the symmetry of exchanging the two lasers is broken (because it would involve changing Δ to $-\Delta$). As a consequence, pitchfork bifurcations cannot exist any longer; they are replaced by saddle-node bifurcations and continuing branches. Furthermore, there are no constant-phase CLMs any longer. All CLMs now have a

phase difference that changes with C_p and, as a consequence, different inversions and intensities; see also Refs. [22, 24]

The branches of CLMs for $\Delta = 0.02$ are plotted in the panels of Fig. 4 and in the associated animations in the $(\omega^s, N_{1,2}^s)$ -projection for the exact same values of P as in Fig. 2; the stability region, bifurcation points and the non-physical region are shown in the same way. Figure 4 shows that the situation for $\Delta \neq 0$ can be seen as ‘unfolding’ the special case of zero detuning. The two lasers can now be distinguished as a red-shifted laser 1 (with respect to the average solitary laser frequency Ω) and a blue shifted laser 2. The branches of CLMs are shown in Fig. 4 as red and blue curves, where the inversions $N_{1,2}^s$ of lasers 1 and 2 are shown in red and blue, respectively. Again, for fixed C_p there are finitely many CLMs. As a function of C_p , CLMs are born in saddle-node bifurcations. Figure 5 shows an enlarged view of the region around the saddle-node bifurcations in the low-frequency region for two different values of P . Also for non-zero detuning the order of the saddle-node bifurcation and the Hopf bifurcation changes. When C_p is decreased, the CLMs move along their respective branches, undergo several Hopf bifurcations and finally disappear at the line $N_{1,2}^s = P$ (Hopf bifurcation of the off state) or at another saddle-node bifurcation. This is shown in the animation with Fig. 6.

In contrast to the situation for zero detuning, the branches of CLMs in Fig. 4 deform substantially [Fig. 4(a) and (b)] and then unconnected isolas of extra CLMs appear [Fig. 4(c)]. These isolas grow and develop further saddle-node bifurcations [Fig. 4(d)] and finally split up in two [Fig. 4(e)], effectively creating two small isolas — the top one blue and the bottom one red. In another saddle transition another pair of isolas is created [Fig. 4(g)] — this time the top one is red and the bottom one is blue. Note that these four isolas taken together ‘unfold’ the two isolas that we found in Fig. 4(f). When P is increased further, we again reach a typical limiting situation for large pump currents. It is characterized by two horseshoe-shaped closed curves, one red and one blue, together with said four isolas. More details of the CLM structure as a function of the detuning Δ in this case can be found in Ref. [24].

Overall, Fig. 4 can be seen as an ‘unfolding’ of Fig. 2: as Δ is decreased to zero, the red and the blue curves move closer together and in the limit become the purple curves of Fig. 2. To allow for a better comparison between the two cases the animation with Fig. 7 shows the CLMs for zero and non-zero detuning side by side.

5. Changes to the locking region

The size of the locking region, the bold part on the branches of CLMs in Figs. 2 and 4, depends on the feedback phase C_p and on the detuning Δ . Consequently, it forms a region in the (Δ, C_p) -plane. The question arises of how the locking region depends on the pump current P . To answer this question we performed a three-parameter bifurcation study, the result of which is shown in Fig. 8 and the associated animation. The different panels show representative stills for the same values of P as in Figs. 2 and 4. Note that

Figure 6. The dependence of the CLMs on C_p for $\Delta = 0.02$ and $P = 0.141$; Animation (606k). The figure shows the CLMs for $C_p = 0$

Figure 7. Animation (1401k) that directly compares the case of zero detuning ($\Delta = 0$) and the case of non-zero detuning ($\Delta = 0.02$).

the image is π -periodic in C_p and reflectionally symmetric in Δ .

We first discuss the changes in the shape of the (green) locking region in Fig. 8 without considering the boundary curves involved; they are discussed below. For very low values of pumping there is no locking region at all. From a certain pump level onwards there appear disjoint locking regions that start to grow in size [Fig. 8(a)]. The fact that this occurs for negative values of P is due to the threshold reduction of the coupled laser system. For a given pump current these regions (which repeat with the period π of C_p) start to overlap [Fig. 8(b)]. This creates a small ‘channel’ where locking is not possible in a large region that stretches evenly from a maximal to a minimal value of the detuning Δ [Fig. 8(c)]. This channel then starts to become more pronounced and it grows in size [Fig. 8(d)–(g)]. All of this happens in a range of the pump current P below the solitary laser threshold. As P is increased further a limiting shape of the

Figure 8. Dependence on the pump current P of the locking region (green) in the (Δ, C_p) -projection; Animation (1518k). The black curves are Hopf bifurcations of the off-solution, the blue curves saddle-node bifurcations of CLMs, and the red curves Hopf bifurcations of CLMs. P takes the values indicated in the panels.

Figure 9. Dependence on the pump current P of the maximum width Δ_l of the locking region. Graphically extracted for each value of P .

locking region is reached. The locking region does not change qualitatively for even higher pump currents; compare [24].

Apart from the general shape of the locking region it is important to know what happens at its boundary. In other words, how does the locked solution become unstable as the locking region is left? There are three types of bifurcation curves plotted in Fig. 8 and, as P is increased, they interact to form a complicated web of curves. First, there are black curves of Hopf bifurcation of the off-state $(E_1, E_2, N_1, N_2) = (0, 0, P, P)$. They correspond to the two endpoints where ellipses intersect the line $N_{1,2}^s = P$ in Figs. 2(a)–(d) and 4(a)–(d). Further, there are saddle-node bifurcation curves of CLMs in blue and Hopf bifurcation curves of CLMs in red.

Note that most of the ‘web’ of bifurcation curves in Fig. 8 for larger values of P corresponds to bifurcations of CLMs that are already unstable. They were drawn for completeness and to give an impression of the overall increase in complexity with P , but a detailed discussion of the interplay of all of the different bifurcation curves in Fig. 8 is a mathematical challenge beyond the scope of this paper.

Here we concentrate our discussion on those curves that bound the (green) locking region. This is directly motivated from a physical perspective by the question of how the laser destabilizes when it leaves the locking boundary. The boundary of the locking region is initially made up of the black Hopf bifurcation curves of the off-state [Fig. 8(a)]. In other words, the intensity of the lasers goes to zero and the laser is off outside the locking region. Below the solitary laser threshold the only possibility for the system to lase is to profit from the mutual coupling. The mutual optical injection reduces the threshold of the coupled system. The most profitable situation is for $(\Delta, C_p) = (0, (k+0.5)*\pi)$, because then the injected optical field has its maximum when entering the other laser. In this case the coupled laser system is in stable CW-operation when the coupling is largest. In contrast, for pump currents above the solitary laser threshold [Fig. 8(h)] the maximum width of the locking region is at $(\Delta, C_p) = (0, k\pi)$. Thus, stable CW-operation is achieved best if each laser receives the least amount of light from the other laser.

When the channel of instabilities opens [Fig. 8(b)–(f)] its lower boundary is a Hopf bifurcation curve of CLMs. This means that the crossing of this boundary results in the onset of intensity oscillations. The upper boundary, on the other hand, is divided into two parts. While for larger values of Δ it is also a Hopf bifurcation of CLMs, for values of Δ close to zero this boundary consists of a saddle-node bifurcation of CLMs. When the saddle-node bifurcation is crossed, the stable CLM is lost and the system jumps to the next available attractor, which could be regular or irregular intensity fluctuations. Note that from a certain P on (after a singularity transition involving different blue saddle-node curves) the entire upper boundary of the locking region is a saddle-node bifurcation of CLMs, so that sudden jumps to nonlocked dynamics must be expected.

Finally, we show in Fig. 9 the overall maximal width, $\max |\Delta_l|$, of the locking region, that is, the Δ -range where locking can be found in an experiment. Note that $\max |\Delta_l|$ depends on the value of the feedback phase C_p , as can be seen clearly in Fig. 8 and the associated animation. Figure 9 shows how $\max |\Delta_l|$ changes as a function of on the pump rate P . Depending on P , there are different bifurcations that determine the maximum width of the locking region; compare Fig. 8. Note that the fastest changes takes place below and around the solitary laser threshold $P = 0$, where $\max |\Delta_l|$ is initially determined by the Hopf bifurcation of the off-state (black curve in Fig. 8). For intermediate P , on the other hand, $\max |\Delta_l|$ is attained at the intersection point of the Hopf bifurcation of the off-state and a Hopf bifurcation of a CLM (red curve in Fig. 8). Moreover, since the Hopf bifurcation curve of the off-state exhibits a ‘pillow-like’ shape, $\max |\Delta_l|$ increases with different slopes, or may even decrease slightly. Eventually, $\max |\Delta_l|$ is determined by the intersection of a Hopf bifurcation of a CLM and a saddle-node bifurcation of a CLM (blue curve in Fig. 8), leading to an almost linear decrease of the of $\max |\Delta_l|$. It seems plausible that $\max |\Delta_l|$ eventually reaches a constant positive value. To verify this a even more extensive numerical or asymptotic analysis would be necessary, but is beyond the scope of this study.

6. Conclusions

We demonstrated a strong dependence of the compound laser mode structure of two mutually delay-coupled identical semiconductor lasers on the pump current. The mode structure changes substantially below and around the solitary laser threshold and then reaches a limit that appears to be typical for sufficient pumping above threshold. The figures and animations of this paper show the transitions involved in a comprehensive geometrical way. No intricate knowledge of bifurcation theory is required to appreciate these transitions on the level of the region of stable locking. On the other hand, mathematically unraveling the intriguing interplay of different bifurcations remains an interesting challenge.

The size, shape and transitions at the boundaries of the locking region are of immediate physical relevance for locking experiments. An estimation of the maximum width of the locking region reveals the general trends: a broad maximum around solitary

laser threshold with a fast decrease for decreasing pump rates, and a linear decrease for increasing pump rates. The limiting case for even higher pump rate is left for further investigations.

We remark that the occurrence of stable CW-emission on a compound laser mode is mainly due to the short time delay we chose, which is on the order of the relaxation oscillation frequency. For longer coupling times the number of (unstable) modes increases and, therefore, their structure may become much more complicated; see, *e.g.*, Refs. [7, 3]. This is quite similar to the case of conventional optical feedback for short and long delay times; see, *e.g.*, Refs.[29, 16].

Indeed the compound laser modes — and this includes the unstable ones — form the backbone of the dynamics of the overall system in much the same way as the external cavity modes do for the laser with conventional optical feedback. An interesting and challenging topic of study are the very rich dynamics outside the locking region, some of which are reported in Ref. [7]. In conjunction with with Refs. [23, 24] the results presented here reveal an essential part of the underlying CLM structure of mutually delay-coupled lasers.

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