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Automatic balancing of a rigid rotor with misaligned shaft

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Abstract. We present a nonlinear analysis of the dynamics of an automatic ball balancer (ABB) for rotors
which are both eccentric and misaligned. The ABB consists of two or more ball bearings which are free
to travel around a circular race at a fixed distance from the shaft. The balls, after a transient response,
find a steady state which balances the rotor. Following the previous work of Green et al. at Bristol, we
have included the effect of shaft misalignment which causes the rotor to precess. This can be countered by
having two ABB races at different axial locations along the shaft. Mathematically, we use a Lagrangian
approach to derive the equations of motion for the system. It is found that, contrary to the case of flexible
rotors that are subject to eccentricity and shaft bending, there is no choice of co-ordinate system which
leads to autonomous governing equations. Simulations are then computed which illustrate the role of the
ball damping coefficient.

Introduction

One of the primary causes of vibration in rotating machinery is mass imbalance, this occurs when the
principal axis of the moment of inertia is not coincident with the axis of rotation. For a rigid rotor the
imbalance is usually eliminated by adding (or subtracting) correction masses in two distinct planes in
such a way as to realign and centre the principal axis. However when using this method, the rotor has
to be rebalanced every time its mass distribution changes. This limitation motivates the study of self-
compensating balancing devices, in which masses automatically redistribute themselves so as to eliminate
any imbalance.

One such device is the automatic ball balancer (ABB), which consists of a series of balls that are free
to travel, around a race filled with viscous fluid. This idea was initially proposed in 1904 [1], and since
then ABB’s have been incorporated into many products including optical disc drives, machine tools and
washing machines. The first study of an ABB was carried out by Thearle in 1932 [2], who demonstrated
the existence of a stable steady state at rotation speeds above the first critical frequency. More recently,
Chung et al. [3] have studied the stability of automatic balancing for an eccentric flexible rotor, Chao et al.
[4] have looked at the non-planar effects of an optical disk drive fitted with an ABB and Green et al. [5]
have focused on using numerical continuation to map the bifurcations between different attracting states
in a Jefcott rotor fitted with an ABB.

The rest of this paper is outlined as follows. In Section 2 we present a fully geometric model for
two-plane automatic balancing of a rigid rotor [6]. Our techniques are fully nonlinear and do not rely on
small angle or constant spin speed assumptions, and so may be applicable to many other fields, such as
celestial mechanics and the dynamics of spinning spacecraft [7]. In Section 3 we focus on the two-plane
autobalancing of a rotor with shaft misalignment and simulate responses for different values of the ball
damping parameter. Finally, in Section 4 we draw conclusions and discuss how we would make the system
amenable to bifurcation analysis.
Two-plane automatic balancing

Reference frames and generalized co-ordinates. In describing the dynamics of a rigid body it is useful to consider the relation between an $xyz$ frame fixed to the body and an inertial $XYZ$ space frame, see Figure 1. Firstly a translation is needed to place the $XYZ$ axes onto the $X'Y'Z'$ axes which is parallel to $XYZ$ but with the same origin as the $xyz$ body axes. Then the orientation of the $xyz$ body axes, relative to the $X'Y'Z'$ frame, can be specified through the use of Euler angles where we will use the convention set out below.

![Definition of co-ordinate system.](image)

Here, a rotation by an angle $\phi$ about the $X'$ axis results in the primed co-ordinate system $x'y'z'$. This intermediate sytem is then rotated by an angle $\theta$ about the $y'$ axis resulting in the double primed system $x''y''z''$. Finally, a rotation $\psi$ about the $z''$ axis gives the unprimed $xyz$ co-ordinate system. The full rotation $R$ can be concisely written in matrix form as

$$R = R_\psi R_\theta R_\phi,$$

where

$$R_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix}, \quad R_\theta = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}, \quad R_\psi = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ (2)

The use of three separate axes eliminates any polar singularity, furthermore in the small angle limit, $\phi$ and $\theta$ reduce to projection angles, which are commonly seen in the rotordynamics literature [6]. Thus the full co-ordinate transformation from $xyz$ to $XYZ$ is given by

$$r_{XYZ} = r_o + R^T r_{xyz},$$ (3)

where $r_{XYZ}$ and $r_{xyz}$ are the position vectors in the $XYZ$ and $xyz$ systems respectively, and $r_o = [X, Y, Z]$ is the position vector of the $xyz$ systems origin. Thus the generlized co-ordinates $q = [X, Y, Z, \phi, \theta, \psi]^T$ can be used to describe the motion of any rigid body.

Model. The model shown in Figure 2 consists of a balanced rotor of mass $M$ and moment of inertia $I = \text{diag}[I_x, I_y, I_z]$ with respect to the $xyz$ axes. The mass imbalance is introduced through two point masses $m_1, m_2$, fixed to the rotor in planes $z_1, z_2$, with eccentricities $\varepsilon_1, \varepsilon_2$, and angular positions $\alpha_1, \alpha_2$ so that
both static and couple imbalance are present. This setup is dynamically equivalent to the most general rigid
rotor, which has both an offset centre of mass and a misaligned principal axis. The autabalancing device
has two races \( R_1 \) and \( R_2 \), each fitted with \( n \) balls, which are represented as points of mass \( m_i \), in plane \( z_i \),
with eccentricity \( \varepsilon_i \) and angular position \( \alpha_i \), where \( i = 3 \ldots n + 2 \), and \( i = n + 3 \ldots 2n + 2 \) correspond
to the balls in \( R_1 \) and \( R_2 \) respectively. The supports \( S_1 \) and \( S_2 \), have rest positions \([0, 0, l_1]\) and \([0, 0, -l_2]\),
with stiffness \( K_j = \text{diag}[k_{xj}, k_{yj}, k_{zj}] \), and damping \( C_j = \text{diag}[c_{xj}, c_{yj}, c_{zj}] \), \( j = 1, 2 \) with respect to the
XYZ axes.

![Fig. 2: Two-plane autabalancing rotor model.](image)

**Lagrangian formulation of the equations of motion.** We begin by writing the position vectors of the
centre of mass (COM), imbalances and balancing balls in the XYZ co-ordinate system. Using (3) we have
\[
\mathbf{r}_M = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad \mathbf{r}_{m_i} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + R^T \begin{bmatrix} \varepsilon_i \cos \alpha_i \\ \varepsilon_i \sin \alpha_i \\ z_i \end{bmatrix}, \quad i = 1 \ldots 2n + 2. \tag{4}
\]
Without loss of generality we can take \( \alpha_1 = 0 \) and \( \alpha_2 = \gamma \) where \( \gamma \) is the constant phase between the two
imbalances.

Next we write the angular velocity in the body axis, since it is in these axes in which the inertia matrix
takes a diagonal form. As \( \omega_\phi \) is parallel to the space \( X \) axis, its components in the body axis are given by
applying the complete rotation \( R \). Now \( \omega_\phi \) lies along the intermediate \( y' \) axis and so we apply only the last
rotation \( R_\psi \), finally \( \omega_\psi \) lies along the body \( z \) axis and so needs no further work. Thus the components of
\( \omega \) in the body axis are given by
\[
\omega = R \begin{bmatrix} \phi \\ 0 \\ 0 \end{bmatrix} + R_\psi \begin{bmatrix} 0 \\ \theta \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \psi \end{bmatrix},
\]
\[
= \begin{bmatrix} \phi \cos \theta \cos \psi + \dot{\theta} \sin \psi \\ -\phi \cos \theta \sin \psi + \dot{\theta} \cos \psi \\ \phi \sin \theta + \dot{\psi} \end{bmatrix}. \tag{5}
\]
The kinetic energy \( T \) is now given by (see also the appendix)
\[
T = \frac{1}{2} \omega^T \omega + \frac{1}{2} M \dot{r}_M^2 + \frac{1}{2} \sum_{i=1}^{2n+2} m_i \dot{r}_{m_i}^2. \tag{6}
\]
The potential $V$ comes from the elastic energy due to the deflection $r_{s1}$ and $r_{s2}$ of the supports $S_1$ and $S_2$ thus

\[ V = \frac{1}{2} \sum_{j=1}^{2} r_{s_j}^T K r_{s_j}, \]

where

\[ r_{s_j} = \begin{bmatrix} x'_j \\ y'_j \\ z'_j \end{bmatrix} = \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \tilde{t}_1 = t_1, \quad \tilde{t}_2 = -t_2. \]

Similarly Rayleigh’s dissipation function $F$ can be written as

\[ F = \frac{1}{2} \sum_{j=1}^{2} \dot{r}_{s_j}^T C \dot{r}_{s_j} + \frac{1}{2} \sum_{i=3}^{2n+2} c_b \dot{\alpha}_i, \]

where the ball damping coefficient $c_b$, arises from the drag on the balls as they pass through a viscous fluid in the race.

The equations of motion can now be generated by applying Lagrange’s equations

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_k, \]

where $L = T - V$ is the Lagrangian, and $q = [X, Y, Z, \phi, \theta, \psi, \alpha_i]^T$. The COM is confined to the $XY$ plane, and the spin speed $\psi = \omega$, is assumed constant with $\omega = 40 \text{ rad s}^{-1}$. Both supports are identical, with stiffness $K_i = \text{diag}[k, k, k]$, $k = 50 \text{ N m}^{-1}$ and damping $C_i = \text{diag}[c, c, c]$, $c = 5 \text{ N s m}^{-1}$.

The imbalance masses have values $m_1 = m_2 = 0.04 \text{ kg}$, $e_1 = e_2 = 0.1 \text{ m}$, $z_1 = -z_2 = 0.1 \text{ m}$ and $\alpha_2 = \gamma = \pi$, which corresponds to a rotor whose imbalance occurs solely through misalignment. The ABB has parameters $m_i = 0.01 \text{ kg}$, $e_i = 0.1 \text{ m}$, $i = 3 \ldots 6$, $z_3 = z_4 = 0.25 \text{ m}$, $z_5 = z_6 = -0.25 \text{ m}$, corresponding to two races with two balls each. Finally the damping of the balls in the race $c_b$, is the parameter which is to be varied.

**Balancing of a misaligned rotor**

**Model parameters.** We have used the computer algebra system Maple to derive the equations of motion (omitted here) from (10). The equations are then transferred without resorting to linearization into Matlab where they are solved numerically. The model rotor used for simulation has parameters $M = 1 \text{ kg}$, $I_x = I_y = 1 \text{ kg m}^2$, $I_z = 0.5 \text{ kg m}^2$, $l_1 = l_2 = 0.5 \text{ m}$. The COM is confined to the $XY$ plane, and the spin speed $\psi = \omega$, is assumed constant with $\omega = 40 \text{ rad s}^{-1}$. Both supports are identical, with stiffness $K_i = \text{diag}[k, k, k]$, $k = 50 \text{ N m}^{-1}$ and damping $C_i = \text{diag}[c, c, c]$, $c = 5 \text{ N s m}^{-1}$.

The imbalance masses have values $m_1 = m_2 = 0.04 \text{ kg}$, $e_1 = e_2 = 0.1 \text{ m}$, $z_1 = -z_2 = 0.1 \text{ m}$ and $\alpha_2 = \gamma = \pi$, which corresponds to a rotor whose imbalance occurs solely through misalignment. The ABB has parameters $m_i = 0.01 \text{ kg}$, $e_i = 0.1 \text{ m}$, $i = 3 \ldots 6$, $z_3 = z_4 = 0.25 \text{ m}$, $z_5 = z_6 = -0.25 \text{ m}$, corresponding to two races with two balls each. Finally the damping of the balls in the race $c_b$, is the parameter which is to be varied.

**Simulations.** Numerical simulations which illustrate the influence of the ball damping coefficient $c_b$, are shown in Figure 3. The top graphs show the vibration amplitude of the first support $A_1 = |r_{s1}|$, and the ball angles are shown underneath.

In the underdamped regime shown on the left, the balls initially oscillate about the unstable balanced positions, before destabilising completely at around $t = 50 \text{ s}$. The resulting vibrations are an order of magnitude worse than the motion without the ABB. For high damping on the right, these oscillations are prevented, but the balls take a long time to reach the balanced state. Ideally we desire critical damping where the balls reach the equilibrium position in the minimum time without oscillation, as illustrated in the middle graphs.
Conclusion

We have presented a fully nonlinear geometric model for the two-plane balancing of a rigid rotor. In previous studies [3,5] of flexible and planar rotors, autonomous equations are derived by formulating the Lagrangian in the rotating frame. However, in the present case of a non-planar rigid rotor, the spin angle $\psi = \omega t$, is taken about a body $z$ axis which is not parallel to the space $Z$ axis. As a consequence, it is not possible to use the rotating frame to forge equations, which are both autonomous, and satisfy the condition that the rotor moves in the horizontal $Z = 0$ plane.

We have provided simulations for the two-plane automatic balancing of a misaligned rotor. It is shown that the two-race mechanism can eliminate not only vibrations due to radial motion, but also those due to angular displacement. However automatic balancing can only be achieved in regions of parameter space where the balanced state is stable, in particular an appropriate choice of the ball damping coefficient $c_b$, is essential for a successful working device.

It is known [6] that spin speed $\omega$, moment of inertia $I$, and support damping $C$, all influence the stability of balanced operation. This dependence will be investigated in future studies, where we envisage using numerical continuation, to compute bifurcation diagrams showing the boundaries of stability in various parameter planes [5].

References


APPENDIX

The explicit forms for the kinetic energy $T$, potential $V$ and Rayleigh’s dissipation function $F$ which are derived in Section 2 are given here

$$T = \frac{1}{2} I_x \left( \dot{\phi} \cos \theta \cos \psi + \dot{\theta} \sin \psi \right)^2 + \frac{1}{2} I_y \left( -\dot{\phi} \cos \theta \sin \psi + \dot{\theta} \cos \psi \right)^2 + \frac{1}{2} I_z \left( \dot{\phi} \sin \theta + \dot{\psi} \right)^2$$

$$+ \frac{1}{2} M \left( \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 \right)$$

$$+ \frac{1}{2} \sum_{i=1}^{2n+2} m_i \left[ \dot{X} + \varepsilon_i \left( -\sin \theta \dot{\theta} \cos (\psi + \alpha_i) - \cos \theta \sin (\psi + \alpha_i) (\dot{\psi} + \dot{\alpha}_i) \right) + z_i \cos \theta \dot{\theta} \right]^2$$

$$+ \left[ \dot{Y} + \varepsilon_i \left( \cos \theta \dot{\theta} \sin \phi \cos (\psi + \alpha_i) + \sin \theta \cos \phi \dot{\phi} \cos (\psi + \alpha_i) - \sin \theta \sin \phi \sin (\psi + \alpha_i) (\dot{\psi} + \dot{\alpha}_i) - \sin \phi \dot{\phi} \sin (\psi + \alpha_i) \right) \right]^2$$

$$+ \left[ \dot{Z} + \varepsilon_i \left( -\cos \theta \dot{\theta} \cos \phi \cos (\psi + \alpha_i) + \sin \theta \sin \phi \dot{\phi} \cos (\psi + \alpha_i) + \sin \theta \cos \phi \sin (\psi + \alpha_i) (\dot{\psi} + \dot{\alpha}_i) + \cos \phi \dot{\phi} \sin (\psi + \alpha_i) \right) \right]^2, \quad \text{(A-1)}$$

$$V = \frac{1}{2} \sum_{j=1}^{2} \left\{ k_{x_j} \left( X + \sin \theta \hat{l}_j \right)^2 + k_{y_j} \left( Y - \cos \theta \sin \phi \hat{l}_j \right)^2 + k_{z_j} \left( Z - (1 - \cos \theta \cos \phi) \hat{l}_j \right)^2 \right\}, \quad \text{(A-2)}$$

$$F = \frac{1}{2} \sum_{j=1}^{2} \left\{ c_{x_j} \left( \dot{X} + \cos \theta \dot{\theta} \hat{l}_j \right)^2 + c_{y_j} \left( \dot{Y} + \left( \sin \theta \dot{\theta} \sin \phi - \cos \theta \cos \phi \hat{l}_j \right)^2 \right. \right. \right.$$

$$\left. \left. \left. \left. + c_{z_j} \left( \dot{Z} - \left( \sin \dot{\theta} \cos \phi + \cos \theta \sin \phi \dot{l}_j \right) \right)^2 \right) \right\} + \frac{1}{2} \sum_{i=3}^{2n+2} c_i \dot{\alpha}_i. \quad \text{(A-3)}$$