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Solution structure and dynamics of a semiconductor laser subject to feedback from two external filters

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ABSTRACT

We present an analysis of a semiconductor laser subject to filtered optical feedback from two filtering elements (2FOF). The motivation for this study comes from applications where two filters are used to control and stabilise the laser output. Compared to a laser with a single filtered optical feedback loop, the introduction of the second filter significantly influences the structure of the basic continuous-wave solutions, which are also known as external filtered modes (EFMs). We compute and represent the EFMs of the underlying delay differential equation model as surfaces in the space of frequency ω_s and inversion level N_s of the laser, and feedback phase difference dC_p . The quantity dC_p is a key parameter since it is associated with interference between the two filter fields and, hence, controls the effective feedback strength. We further show how the EFM surface in (ω_s, dC_p, N_s) -space changes upon variation of other filter parameters, in particular, the two delay times. Overall, the investigation of the EFM-surface provides a geometric approach to the multi-parameter analysis of the 2FOF laser, which allows for comprehensive insight into the solution structure and dynamics of the system.

Keywords: semiconductor laser, filtered feedback, external filtered modes

1. INTRODUCTION

Semiconductor lasers play a very important role in modern telecommunication networks, and their reliable operation is crucial for the overall network performance. For this reason alone, it is of great interest to find ways to stabilize the laser output. One such stabilization scheme is to consider a semiconductor laser subject to filtered optical feedback (FOF) from a single external filter — a system that has been extensively studied since 30 years.¹ Previous studies show that the introduction of FOF may indeed either improve the laser performance,^{2,3} but it may actually also disturb stable laser operation due to the emergence of more complicated dynamics in the system.^{1,4-7} It is noteworthy that the FOF laser may show so-called frequency oscillations, where the laser frequency oscillates but its intensity remains almost constant.⁸ Frequency oscillations are very different from the characteristic relaxation oscillations, and their period is of the order of the external round-trip time τ ; see also Refs. [7, 9, 10].

In this paper we consider a semiconductor laser receiving filtered optical feedback from two filtered optical feedback loops — a system that we refer to as the 2FOF laser. Our work is motivated by the suggested application of a 2FOF laser as a source in optical fibre networks. The underlying idea is that the second feedback loop provides additional control over the laser output. One of the possible realizations of the 2FOF laser is shown in Fig. 1. Here the filters are realized as Fabry-Pérot resonators, and the optical isolators in Fig. 1 ensure that there is no conventional optical feedback back into the laser.

Any optical feedback loop coupled to a semiconductor laser creates an external cavity, which allows the laser to operate at continuous-wave solutions called compound-cavity modes. For lasers with filtered feedback one speaks of external filtered modes (EFMs). For the single FOF laser it is known that the EFMs lie on closed curves in the (ω_s, N_s) -plane, called the EFM-components, which are traced out when the feedback phase C_p (the phase of the electromagnetic field that is fed back to the laser after one round trip in the feedback loop) is varied.¹¹ In fact, the feedback phase C_p has been identified as one of the main parameters of the FOF system.⁷ For the single FOF laser the dependence of the number of the EFM-components on the filter width Λ and the filter detuning Δ was presented in Ref. [12]; in particular, there may be at most two coexisting EFM-components.

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Parameter	Meaning	Value
α	linewidth enhancement factor	5
T	carrier lifetime $\times$ photon decay rate	100
P	pump parameter	3.5
$\kappa_1 = \kappa_2$	feedback rates	0.05
$\Lambda_1 = \Lambda_2$	filter widths	0.015
$\Delta_1 = \Delta_2$	filter detunings	0.0
C_p^1, C_p^2	feedback phases	2π -periodic
τ_1, τ_2	external cavity round-trip times	250, 500

Table 1. System parameters and their values.

find and follow solutions and determine their stability and bifurcations.¹⁵ We employ these tools here to determine the EFM structure of the 2FOF laser.

The starting point is our previous work in Ref. [16]. There we showed that for the case where both feedback loops have the same length there are maximally three EFM-components, and we determined how the number of the EFM-components depends on the filter detunings Δ_i and filter widths Λ_i . Important parameters in this context are the difference $d\tau = \tau_1 - \tau_2$ between the two delay times and the difference $dC_p = C_p^1 - C_p^2$ between the two feedback phases of the two filter loops. More specifically, we find it most convenient to represent the EFMs as surfaces in the space of frequency ω_s and inversion level N_s of the laser and the feedback phase difference dC_p . It is then possible to analyse how the EFM-surface changes when other parameters are varied. From this representation of the EFMs, information about the EFM-components can be inferred. Here we concentrate on the influence of the delay difference $d\tau$.

2. EXTERNAL FILTER MODES

An EFM is the basic cw-solutions of Eqs. (1)–(4) where the system lases with constant intensity at a fixed frequency ω_s ; this means that also the inversion and the amplitudes of the filter fields are constant. Mathematically, an EFM is a group orbit of the S^1 -symmetry and given by

$$(E(t), N(t), F_1(t), F_2(t)) = (E_s e^{i\omega_s t}, N_s, F_s^1 e^{i(\omega_s t + \phi_1)}, F_s^2 e^{i(\omega_s t + \phi_2)}). \quad (5)$$

Here, E_s , F_s^1 and F_s^2 are real and fixed amplitudes of laser and filter fields, and N_s is a fixed population inversion. Further, ω_s is a time independent frequency and ϕ_1, ϕ_2 are constant phase shifts between the laser field and the respective filter field.

As with other Lang-Kobayashi-type rate equations, to determine the EFMs one substitutes (5) into Eqs. (1)–(4) to obtain the transcendental equation

$$\Omega(\omega_s) = -\omega_s - \sqrt{1 + \alpha^2} \left(\frac{\kappa_1 \Lambda_1 \sin(\phi_1 + \arctan(\alpha))}{\sqrt{\Lambda_1^2 + (\omega_s - \Delta_1)^2}} + \frac{\kappa_2 \Lambda_2 \sin(\phi_2 + \arctan(\alpha))}{\sqrt{\Lambda_2^2 + (\omega_s - \Delta_2)^2}} \right), \quad (6)$$

for the possible values of the frequency ω_s of the EFMs,^{12,16} where

$$\phi_i = \omega_s \tau_i + C_p^i + \arctan\left(\frac{\omega_s - \Delta_i}{\Lambda_i}\right). \quad (7)$$

Once ω_s has been determined from Eq. (6), the constants E_s , F_s^1 , F_s^2 can be computed independently. By using an addition theorem for the sum of the two sine function in Eq. (6), one determines that all EFMs lie on curves within the envelope

Figure 2. Branches of EFMs in the (ω_s, N_s) -plane obtained by continuation. Grey circles are the EFMs for $C_p^1 = C_p^2 = 0$. The grey curves are the branches of EFMs calculated by continuation in C_p^2 for fixed C_p^1 . The dark vertical lines are the results of continuation of the EFMs for constant frequency ω_s (both C_p^1 and C_p^2 are independent continuation parameters). The outer black closed curve connecting all EFMs is the result of continuation in C_p^1 , with the additional restriction that $C_p^2 = C_p^1$. Here $\tau_1 = \tau_2 = 500$ and all other parameters values are as in Table 1.

given by

$$\Omega_e(\omega_s) = -\omega_s \pm \sqrt{1 + \alpha^2} \left(\frac{\kappa_1^2 \Lambda_1^2}{\Lambda_1^2 + (\omega_s - \Delta_1)^2} + \frac{\kappa_2^2 \Lambda_2^2}{\Lambda_2^2 + (\omega_s - \Delta_2)^2} + \frac{2\kappa_1 \kappa_2 \Lambda_1 \Lambda_2 \cos(C_p^2 - C_p^1 + \omega_s(\tau_2 - \tau_1) + \arctan(\frac{\omega_s - \Delta_2}{\Lambda_2}) - \arctan(\frac{\omega_s - \Delta_1}{\Lambda_1}))}{\sqrt{\Lambda_1^2 + (\omega_s - \Delta_1)^2} \sqrt{\Lambda_2^2 + (\omega_s - \Delta_2)^2}} \right)^{1/2}. \quad (8)$$

The transcendental EFM equation (6) is complicated and depends on all parameters of Eqs. (1)–(4). Hence, its solutions can only be found numerically (except for certain very special choices of the parameters). We use here the numerical tool of continuation to find and then continue in parameters EFMs directly as solutions of Eqs. (1)–(4); specifically, we use the package DDE-BIFTOOL¹⁷ for this purpose, where we ensure that EFMs are isolated solutions as in Refs. [12, 16, 18]. In this way, we obtain solution curves of EFMs in dependence on chosen parameters.

2.1 EFM-components

The EFM-components for the single FOF laser are closed curves in the (ω_s, N_s) -plane that are parametrised by the feedback phase, and they form the basis for understanding the system dynamics.¹² For the 2FOF laser EFM-components also arise but they may now be parametrised by either of the two feedback phases C_p^1 and C_p^2 . Therefore, we now set all parameters for the both feedback loops to equal values ($\tau_1 = \tau_2 = 500$ and all other values as in Table 1) and calculate the branches of EFMs under variation of either C_p^1 or C_p^2 .

Figure 2 shows the EFMs for $C_p^1 = C_p^2 = 0$ as grey circles, together with three kinds of EFM branches, which we obtained by continuation of the EFMs in different parameters. The light grey curves are the branches of EFMs that are obtained from a continuation in C_p^2 for fixed $C_p^1 = 0$. The vertical lines are the results of continuation where both C_p^1 and C_p^2 are independent continuation parameters, but the frequency ω_s is kept fixed. Finally, the black closed outer curve connecting all the EFMs is obtained by continuation of the EFMs in C_p^1 while also ensuring that $C_p^2 = C_p^1$. This curve has

the shape of an EFM-component as one finds for the FOF laser in the form of an ‘ellipse’ that is deformed by the filter profile (note that $\Delta_1 = \Delta_2$ so that the filters are not detuned from the laser); compare with Ref. [12].

Note from Fig. 2 that by setting $C_p^2 = C_p^1$ we reduce system (1)–(4) to the single FOF laser with feedback rate 2κ , where $\kappa = \kappa_1 = \kappa_2$. Hence, the outer black curve is truly an EFM component of the FOF laser. What is more, Fig. 2 shows that the EFMs move from the outer closed curve towards the diagonal when the difference between the feedback phases C_p^1 and C_p^2 increases. To explore this behaviour and the reduction of the 2FOF laser to the single FOF laser further, we consider the feedback phase difference dC_p as a new independent parameter; note that, as C_p^i , the feedback phase parameter dC_p is 2π -periodic.

Continuations in which dC_p is kept constant result in closed branches of EFMs that are, in fact, EFM-components of the single FOF laser, where the feedback strength is less or equal to 2κ . In other words, the introduction of dC_p yields a non-trivial reduction of the 2FOF laser to the FOF laser with feedback strength and feedback phase given by

$$\kappa^{\text{eff}} = 2\kappa \cos\left(\frac{dC_p}{2}\right), \quad C_p^{\text{eff}} = \frac{C_p^1 + C_p^2}{2}. \quad (9)$$

Note that the effective feedback strength κ^{eff} is maximal and equal to 2κ for $dC_p = 0$ due to positive interference. The other extreme case occurs for $dC_p = \pi$ when negative interference leads to a cancellation of the two filter fields, so that $\kappa^{\text{eff}} = 0$ and the 2FOF laser reduces to a free-running laser (without feedback).

Because the corresponding branches of EFMs calculated for constant dC_p are actually the EFM-components of the FOF system, we refer to them as EFM-components of the 2FOF system from now on. The natural question is now how the number of EFM-components depends on the parameters of the system.

2.2 The EFM surface

To address this question we consider a new object: the surface of EFMs in (ω_s, dC_p, N_s) -space. In light of the discussion in the previous section, the EFM-surface can be thought of (and be computed as) a dC_p -dependent family of EFM-components; in other words, an intersection of the surface with a plane of fixed dC_p yields the corresponding EFM-component(s).

In Fig. 3 we present the EFM-surface for $\tau_1 = \tau_2 = 500$ and all other values as in Table 1. For this choice of parameters the surface is a compact object that repeats periodically in dC_p ; only one copy in the interval $[-\pi, \pi]$ is shown in Fig. 3. Also shown is the projection onto the (ω_s, dC_p) -plane, which takes the form of an ellipse, whose major axis lies on the line given by $dC_p = 0$. Each cross section through the EFM-surface for fixed dC_p corresponds to the EFM-component for the single FOF laser with feedback rate and feedback phase given by Eq. (9); two examples of sections, for $dC_p = 0$ and for $dC_p = -0.9\pi$ are shown in Fig. 3 (a). The corresponding EFM-components with the EFMs (dots) are shown in panels (b) and (c). Note that the EFM-component for $dC_p = 0$ in Fig. 3 (b) is maximal and forms the boundary of the projection of the EFM-surface onto the (ω_s, N_s) -plane; this EFM-component is the black closed outer curve from Fig. 2. Figure 3 (c) shows the EFM-component for $dC_p = -0.9\pi$. Now the laser is subject to a much reduced effective feedback strength and, hence, the EFM-component is now much smaller. When $dC_p = \pi$ the two filter fields interfere destructively, and as a result cancel each other. This means that the EFM-component shrinks down to a single point for $dC_p = \pi$, which is the solitary laser mode. Note that this point is the common point of two adjacent compact parts of the EFM-surface.

2.3 Dependence on the difference between the delay times

The EFM-surface shown in Fig. 3 changes as parameters are changed. Indeed, understanding the influence of the many different parameters of the system on the EFM-surface is a major challenge that will be addressed elsewhere. Here we show how the EFM-surface is influenced by changing the delay difference $d\tau = \tau_1 - \tau_2$ from the value $d\tau = 0$ in Fig. 3; all other parameters are kept fixed at values in Table 1.

In fact, varying the length of one of the two feedback loops, and hence $d\tau$, causes quite a dramatic change to what EFM-components one may find for fixed dC_p . We find that the EFM-surface provides a simple geometrical explanation of this change — for any value of $d\tau$ and any choice of dC_p . This is demonstrated in Fig. 4, where we show the EFM-surface in (ω_s, dC_p, N_s) -space for $\tau_1 = 500$, $\tau_2 = 250$, that is, for $d\tau = 250$. Panel (a) shows one compact piece of the EFM-surface. Also shown is the boundary curve, again an ellipse, of the projection of the EFM-surface onto the (ω_s, dC_p) -plane.

Figure 3. Panel (a) shows the EFM-surface for $\tau_1 = \tau_2 = 500$ in (ω_s, dC_p, N_s) -space; also shown are the two planes for $dC_p = -0.9\pi$ and $dC_p = 0$ and the boundary curve of the projection onto (ω_s, dC_p) -plane. Panels (b) and (c) show EFM-components in the (ω_s, N_s) -plane for $dC_p = 0$ and $dC_p = -0.9\pi$, respectively; the grey circles are EFMs.

Notice that the EFM-surface is now ‘titled’ in the dC_p -direction and extends over several 2π periods of dC_p , from about $dC_p = -8\pi$ to $dC_p = 8\pi$; compare with Fig. 3 (a). The boundary of the projection of the EFM-surface onto the (ω_s, N_s) -plane remains unchanged but, as a result of the tilting for nonzero $d\tau$, it is no longer given by the EFM-component for $dC_p = 0$.

It is important to realize that there are infinitely many copies (all 2π -translates in dC_p) of the single compact surface in Fig. 4 (a). These other copies are shown in panel (b) in projection onto the (ω_s, dC_p) -plane where $dC_p \in [-8\pi, 8\pi]$; the shadow of the surface in Fig. 4 (a) is shaded. It is clearly visible that the major axes of the repeated ellipses, and hence the EFM surface, are titled with respect to the ω_s -axis for $d\tau \neq 0$.

As before, the EFM-components are found by considering the intersection of a plane for $dC_p = \text{const}$ with the entire EFM-surface, that is, with all copies in Fig. 4(b). An alternative geometric viewpoint is shown in Fig. 4(c), where the shadow of the EFM surface is shown in the (ω_s, dC_p) -plane over one fundamental 2π -interval of $dC_p \in [-\pi, \pi]$. The shaded region in Fig. 4(b) can be obtained by connecting translated copies of the respective seven disjoint pieces in Fig. 4(c).

Figure 4. Panel (a) shows the EFM-surface for $\tau_1 = 500$ and $\tau_2 = 250$ in (ω_s, dC_p, N_s) -space; also shown is the plane for $dC_p = 0$ and the boundary curve of the projection onto (ω_s, dC_p) -plane. Panels (b) and (c) show projections of the EFM-surface onto the (ω_s, dC_p) -plane, for $dC_p \in [-8\pi, 8\pi]$ and $dC_p \in [-8\pi, 8\pi]$, respectively.

Figure 5 shows the three-dimensional counterpart of Fig. 4(c), that is, the entire EFM-surface (consisting of seven disjoint pieces) in (ω_s, dC_p, N_s) -space for $dC_p \in [-\pi, \pi]$. Indeed, translated copies of the seven disjoint pieces join to form the single compact piece of the EFM-surface from Fig. 4 (a). The fact that there are several disjoint pieces for $dC_p \in [-\pi, \pi]$ is a direct consequence of the tilting of the EFM-surface for $d\tau \neq 0$. The planes for $dC_p = -\pi$, $dC_p = 0$ and $dC_p = \pi$ in Fig. 5 (a) give further insight into the EFM surface, and what its intersection with a plane of constant dC_p looks like.

These intersections are exactly the EFM-components, and four examples for different values of dC_p are shown in Fig. 5 (b)–(e). The insets illustrate the fact that the central adjacent EFM-components are not connected with each other, except for the case of a single solitary laser mode for $dC_p = -\pi$ in panel (e). Figure 5 (b) shows a group of seven EFM-components for $dC_p = -3\pi/4$; for this value of dC_p there is a small EFM-component on the left (for negative ω_s), which is the intersection of the plane for $dC_p = -3\pi/4$ with the left-most piece of the EFM-surface in Fig. 5 (a). As dC_p is increased, the EFM-components move, growing on the left and shrinking on the right; see Fig. 5 (c) and (d). At the same time, the EFMs for $C_p^1 = 0$ (grey circles) move along the EFM-components; their number per EFM component may

Figure 5. The EFM-surface in the (ω_s, dC_p, N_s) -space for $\tau_1 = 500$ and $\tau_2 = 250$ and $dC_p \in [-\pi, \pi]$ (a), and the EFM-components for $dC_p = -3/4\pi$ (b), $dC_p = 0$ (c), $dC_p = 3/4\pi$ (d), $dC_p = \pi$ (e). Dots correspond to set of EFMs for $C_p^1 = 0$. Inserts illustrate the fact all loops are separated. The only exception is the central EFM-component for $dC_p = \pi$ which has a shape of tilted eight. Vertical line in the inserts indicate $\omega_s = 0$. Values of the other parameters are in Table 1.

change by two in saddle-node bifurcations. Note that the EFM-components in panel (c) are for $dC_p = 0$ and that they are

Figure 6. The discrete frequencies of the EFMs, given by roots of the transcendental equation (6), arise as intersection points (grey dots) of a curve inside the (grey) envelope with the diagonal. The two panels are for $dC_p = 0$ (a) and for $dC_p = \pi$ (b), while $\tau_1 = 500$ and $\tau_2 = 250$.

symmetrical under the operation $(\omega_s, N_s) \mapsto (-\omega_s, -N_s)$; compare with Fig. 5 (a). Finally, panel (e) is for the special case of $dC_p = \pi$, which has the same symmetry property. Note from the inset that the central EFM-component has the shape of a tilted figure eight, so that there are only five separate EFM-components. This structure of the EFM-components for $dC_p = \pi$ is as that presented in Ref. [19] for a laser operating exactly at the minimum of a periodic filter profile. This observation can be explained as follows. Effectively, for $dC_p = \pi$ the laser is operating at the filter minimum given by $\kappa_{\text{eff}} = 0$. In the presence of nonzero $d\tau$, the interference between the two filters results in a periodic filter profile, which explains the existence of additional EFM-components, of which there are four in Fig. 5 (a). Note that the number of additional EFM-components depends on the amount of tilting of the EFM-surface, which increases with $d\tau$. In other words, for nonzero $d\tau$ the 2FOF laser may possess a large number of EFM-components.

Figure 6 shows the EFM frequencies for $d\tau = 250$ for the two cases $dC_p = 0$ of positive interference, and $dC_p = \pi$ of negative interference between the two filters. More specifically, in this interpretation we plot $\Omega(\omega_s) + \omega_s$ as a function of ω_s . Hence, the roots of Eq. (6) arise as intersection points (grey dots) of a curve, inside the (grey) envelope given by Eq. (8) with the diagonal. The difference with the case of a single FOF laser, which was considered in the same way in Ref. [12], is that Eq. (8) for the envelope does not reduce to a polynomial. In other words, the dependence of the number of EFM-components on the different parameters of the 2FOF laser is much more complicated. In particular, this number is not bounded for the 2FOF laser (it is at most two for the single FOF laser¹²).

Figure 6 (a) for $dC_p = 0$ correspond to Fig. 5 (c), and Fig. 6 (b) for $dC_p = \pi$ corresponds to Fig. 5 (e). Note that for $dC_p = 0$ the maximum of the filter field generated by interference between the two filters is at $\omega_s = 0$, while for $dC_p = \pi$ there are two maxima surrounding a minimum at $\omega_s = 0$. There are other local maxima of the envelope, which decay

away from $\omega_s = 0$. Note that only those 'bubbles' of the envelope that intersect the diagonal may actually contain EFMs. Overall, Figure 6 illustrates further that the 2FOF laser allows us to make a smooth transition between the two extreme cases in Figure 6 by changing dC_p from 0 to π .

3. CONCLUSIONS

We presented a study of the EFM structure of the 2FOF laser, where we presented the EFMs as a surface that is periodic in the feedback phase difference dC_p . This geometric approach allows to draw conclusions on the dependence of the EFMs on system parameters. As a concrete example, and motivated by case of a single FOF laser, we considered here the EFM-components in the plane of EFM frequency and laser inversion. For the special case that the two filter loops have zero detuning and the same feedback strength and filter width, we showed how a difference between the two external round-trip times gives rise to any number of EFM-components. In particular, we found that dC_p is a crucial parameter that describes how the overall effective feedback field arises from the two filter loops.

Ongoing work concentrates on the dependence of the EFM-surface on other parameters, including the two detunings and the filter width of the filters. Initial investigations show that the EFM-surface may bifurcate in several different ways to give rise to different numbers of possible EFM-components, even for the case that the two external round-trip times are identical.¹⁶ The next and challenging step is to consider the stability properties of the EFMs. More specifically, there are regions of stable EFMs on the EFM surface that are bounded by saddle-node and Hopf bifurcations. It is a major challenge to understand how the stability regions change under variation of the many system parameters.

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