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Random matrix theory and critical phenomena in quantum spin chains

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We compute critical properties of a general class of quantum spin chains which are quadratic in the Fermi operators and can be solved exactly under certain symmetry constraints related to the classical compact groups $U(N)$, $O(N)$, and $Sp(2N)$. In particular we calculate critical exponents s , ν , and z , corresponding to the energy gap, correlation length, and dynamic exponent, respectively. We also compute the ground state correlators $\langle \sigma_i^x \sigma_{i+n}^x \rangle_g$, $\langle \sigma_i^y \sigma_{i+n}^y \rangle_g$, and $\langle \prod_{i=1}^n \sigma_i^z \rangle_g$, all of which display quasi-long-range order with a critical exponent dependent upon system parameters. Our approach establishes universality of the exponents for the class of systems in question.

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Statistical mechanical models, such as the Ising model [1], enable the theoretical study of critical points and phase transitions, for example, those in magnetic systems. It is believed, though not yet rigorously proved, that such transitions have universal features, with asymptotic singularities defined by critical exponents whose values are the same within wide classes of systems. These exponents are expected to depend only on global characteristics, such as symmetry and dimensionality. This is the case for both classical and quantum phase transitions [2,3]. It is the symmetry dependence which we seek to expose in this paper.

Previously, critical exponents have been calculated rigorously for some exactly solvable models [4], and renormalization group techniques [5] have been developed to compute them in general classes of systems and to explain their universality. However, understanding universality in a mathematically rigorous way remains a major open problem. We seek to address this problem here with the use of techniques from random matrix theory, which provide a natural environment for studying symmetry-related phenomena. The idea is to make use of the results from [6,7], extending the application of random matrix theory to compute critical properties of the same general class of quantum spin chains as was considered in [6,7], enabling us to expose an explicit dependence of critical behavior on system symmetries, and establishing universality of the critical exponents for this large class of spin chains.

Random matrix theory [8] has found applications in many areas of mathematical physics ranging from quantum gravity [9], quantum chaos [10], and optics [11], to entanglement problems in quantum spin chains [6,7,12–15]. In particular, for translationally invariant spin systems (e.g., the XX model), the entanglement entropy can be expressed as the determinant of a Toeplitz matrix [6,7]. Such matrices are important in many areas of physics, including Osanger's calculation of the diagonal spin-spin correlation of the two-dimensional classical Ising model in 1946; see, for example, the review by Deift, Its, and Krasovsky [16]. The connection with random matrices then comes about because averages over the unitary group $U(N)$ (often referred to in the physics literature as the circular

unitary ensemble or CUE) may also be expressed as Toeplitz determinants.

The study of entanglement in spin chains was extended in [6,7] to encompass Hamiltonians which possess symmetries characterizing that of the Haar measure of each of the classical compact groups $U(N)$, $O^+(2N)$, $Sp(2N)$, $O^+(2N+1)$, $O^-(2N+1)$, and $O^-(2N+2)$. This can be thought of as an example of the general classification of quantum systems developed by Altland and Zirnbauer [17–19]. In each case it was found that the entanglement could be expressed as the determinant of matrices with combinations of Toeplitz plus Hankel structures, the asymptotics of which can be computed using proven instances of the Fisher-Hartwig conjecture and its generalizations. In all cases, it was found that changes in the analytic properties of the Fisher-Hartwig symbol of these matrices are the signature of phase transitions. The entanglement of a system and its critical properties are closely related, since when discussing the ground state (also a pure state), any correlations must be due to entanglement. It is this relationship that we seek to exploit now.

In Sec. I we recapitulate the results from [6,7] that we need in this paper. In Sec. II we compute the critical exponents s , ν , and z , which are related to the energy gap, correlation length, and dynamic exponent, respectively, and in Sec. III we compute the ground state correlators $\langle \sigma_i^x \sigma_{i+n}^x \rangle_g$, $\langle \sigma_i^y \sigma_{i+n}^y \rangle_g$, and $\langle \prod_{i=1}^n \sigma_i^z \rangle_g$, which exhibit quasi-long-range order behavior when the quantum system is gapless, decreasing as a power law of the distance n , with an exponent dependent upon the symmetry class. In a companion paper to this [20], we then use the mapping between one-dimensional quantum spin chains and two-dimensional classical spin models to extend our results to classical systems.

I. A CLASS OF EXACTLY SOLVABLE ONE-DIMENSIONAL QUANTUM SPIN CHAINS

We consider the general class of one-dimensional quantum systems of M spin-1/2 particles in an external field h given by the following Hamiltonian quadratic in the Fermi operators b_j :

$$\mathcal{H} = \sum_{j,k} \left(A_{jk} b_j^\dagger b_k + \frac{\gamma}{2} B_{jk} (b_j^\dagger b_k^\dagger - b_j b_k) \right) - 2h \sum_{j=1}^M b_j^\dagger b_j, \quad (1)$$

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where the b_j s satisfy the usual Fermi commutation relations

$$\{b_j^\dagger, b_k\} = \delta_{j,k}, \quad \{b_j^\dagger, b_k^\dagger\} = \{b_j, b_k\} = (b_j^\dagger)^2 = (b_j)^2 = 0. \quad (2)$$

Periodic boundary conditions $b_{M+j} = b_j$ are assumed, and the measure of anisotropy γ is real, with $0 \leq \gamma \leq 1$. The matrix $A_{j,k}$ must be Hermitian and $B_{j,k}$ antisymmetric, and without loss of generality we consider them both to contain only real parameters.

It is known [21] that such a model can be exactly diagonalized such that (1) becomes

$$\mathcal{H} = \sum_q |\Lambda_q| \eta_q^\dagger \eta_q + C, \quad (3)$$

where $|\Lambda_q|$ is the dispersion relation determined by $A_{j,k}$ and $B_{j,k}$, η_q are Fermi operators, and C is a constant given by

$$C = \frac{1}{2} \sum_{q=1}^M (A_{qq} - 2h - |\Lambda_q|). \quad (4)$$

Restricting the Hamiltonian (1) to possess symmetries characterizing the Haar measure of each of the classical compact groups $U(N)$, $O^+(2N)$, $Sp(2N)$, $O^+(2N+1)$, $O^-(2N+1)$, and $O^-(2N+2)$,¹ Keating and Mezzadri obtained [6,7]

$$\begin{aligned} \Lambda_q &= 2^{p+1} \left(\Gamma + \sum_{k=1}^L (a(k) \cos kq + i b(k) \sin kq) \right) \\ &= 2^{p+1} (a_q + i b_q), \end{aligned} \quad (5)$$

with real and imaginary parts² given by

$$a_q = \Gamma + \sum_{k=1}^L a(k) \cos kq, \quad \text{and} \quad b_q = \sum_{k=1}^L b(k) \sin kq, \quad (6)$$

and

$$\Gamma = \frac{1}{2} \begin{cases} a(0), & L = \frac{M-1}{2}, & \text{if } M \text{ is odd,} \\ a(0) + (-1)^l a(\frac{M}{2}), & L = \frac{M}{2} - 1, & \text{if } M \text{ is even,} \end{cases} \quad (7)$$

with

$$p = \begin{cases} 0 & \text{for } U(N) \text{ symmetry,} \\ 1 & \text{for all other symmetry classes.} \end{cases} \quad (8)$$

Here q is the wave number

$$q = \frac{2\pi l}{M}, \quad (9)$$

with

$$l = 0, \dots, M-1, \quad (10)$$

for translationally invariant systems.³

These symmetry constraints were achieved using real functions $a(j)$ and $b(j)$, even and odd functions of $\mathbb{Z}/M\mathbb{Z}$,

¹For an introduction see [6].

²Note for all symmetry classes other than $U(N)$, $\gamma = 0$ and thus $b_q = b(k) = 0$, and Λ_q is real.

³For the other symmetry classes see [6].

respectively, to dictate the entries of matrices $A_{j,k}$ and $B_{j,k}$, as reported in Table I in Appendix A.

II. ENERGY SPECTRUM AND CRITICAL EXPONENTS

From (3), we see that the class of quantum systems (1) is gapped whenever

$$|\Lambda_{q_c}| > 0, \quad (11)$$

where q_c is the value of q for which $|\Lambda_q|$ has an absolute minimum, and the energy gap Δ is given by

$$\Delta = |\Lambda_{q_c}|. \quad (12)$$

A signature of a continuous quantum phase transition [2,3] for a gapped system is that the energy gap Δ vanishes as the critical point is approached according to the power law

$$\Delta \sim |g - g_c|^s, \quad (13)$$

where s is the mass gap exponent. This behavior holds as $g \rightarrow g_c$ from above or below, where g is the parameter driving the phase transition (such as an external field) with critical value g_c .

In addition, as the critical point is approached, we expect the divergence of the characteristic length scale ξ [2,3] to take the form

$$\xi \sim |g - g_c|^{-\nu}, \quad (14)$$

where ν is the *correlation length critical exponent*. This length scale is often the correlation length, determining the exponential decay of correlations.

The ratio between the two exponents in (13) and (14) is called the *dynamic critical exponent* z [2,3],

$$z = \frac{s}{\nu}. \quad (15)$$

In addition, the dynamic exponent governs the vanishing of the dispersion relation (the energy spectrum in momentum space) $|\Lambda_q|$ as a function of q [22–24];

$$|\Lambda_{q \rightarrow 0}| \sim q^z, \quad (16)$$

where q is a label in momentum space. If the energy gap Δ is also given by $|\Lambda_{q=0}|$, then the dynamical critical behavior at the point $q = 0$ has the scaling form given by [22–24]

$$|\Lambda_{q \rightarrow 0}| \sim q^z (1 + (q\xi)^{-z}), \quad \text{as } q \rightarrow 0. \quad (17)$$

To compute these exponents s , ν , and z for our class of quantum spin chains (1) we consider the following cases.

A. $\gamma = 0$

Assuming that $\gamma = 0$, thus considering all symmetry classes, we find that Λ_q is a real valued function of the external parameter Γ .

In this case, the energy gap is given by

$$\Delta = |\Lambda_{q_c}| = 2^{p+1} \left| \Gamma + \sum_{k=1}^L a(k) \cos kq_c \right|, \quad (18)$$

which vanishes (the system becomes gapless) at a critical value Γ_c satisfying

$$\Gamma_c = - \sum_{k=1}^L a(k) \cos kq_c. \quad (19)$$

Comparing with (13), we find that $s = 1$ for this critical point (19), and from (15) we have the following relationship between the dynamic critical exponent z and the correlation length exponent ν :

$$z = \frac{1}{\nu}. \quad (20)$$

From (16), we see that z is the exponent governing the vanishing of $|\Lambda_q|$ as a function of q ; thus for the critical points

given by (19), where the value of $q = q_c$ corresponds to an extremum, we will always have even z .

For example, any system with

$$\sum_{k=1}^L a(k)k^2 \cos kq_c \neq 0 \quad (21)$$

will belong to the same universality class with $z = 2$ and $\nu = \frac{1}{2}$. An example is the well known nearest neighbor ($L = 1$) quantum XX model (isotropic XY model).

B. $\gamma \neq 0$

For $\gamma \neq 0$ we see from Table I in the Appendix that we are restricted to systems with $U(N)$ symmetry only. In this situation the energy gap is given by

$$\Delta = |\Lambda_{q_c}| = 2^{p+1} \sqrt{\left(\Gamma + \sum_{k=1}^L a(k) \cos kq_c\right)^2 + \left(\sum_{k=1}^L b(k) \sin kq_c\right)^2}. \quad (22)$$

There are now several possibilities.

First, assuming $q_c = 0, \pi$ we recover the results from Sec. II A. Alternatively if $\gamma = 1$ and $a(k) = b(k)$ for all k , (22) becomes

$$\Delta = |\Lambda_{q_c}| = 2^{p+1} \sqrt{\Gamma^2 + \sum_{k=1}^L \left(a^2(k) + 2a(k) \left(\Gamma \cos kq_c + \sum_{l=1}^{L-k} a(k+l) \cos lq_c \right) \right)}. \quad (23)$$

In this case, any system with

$$\sum_{k=1}^L \left(a^2(k) + 2a(k) \left(\Gamma \cos kq_c + \sum_{l=1}^{L-k} a(k+l) \cos lq_c \right) \right) = 2\Gamma c + c^2, \quad (24)$$

where c is a constant dependent upon the interaction coefficients $a(k)$, will belong to the same universality class, and all such systems have $z = \nu = 1$. An example is the well known nearest neighbor ($L = 1$) quantum Ising model (anisotropic XY model) where $q_c = \pi$ and $c = a(1)$.

III. CORRELATOR

In this section we compute the following correlators for the general class of quantum spin chains (1) restricted to $U(N)$ symmetry only and with $\gamma = 0$, in the limit $n \rightarrow \infty$:

$$\begin{aligned} \left\langle \prod_{l=i}^{i+n} \sigma_l^z \right\rangle_g &= (-1)^n \det \mathcal{M}_n, \\ \langle \sigma_i^x \sigma_{i+n}^x \rangle_g &= (-1)^n \det \mathcal{M}_n^x, \\ \langle \sigma_i^y \sigma_{i+n}^y \rangle_g &= (-1)^n \det \mathcal{M}_n^y, \end{aligned} \quad (25)$$

where $\langle \cdot \rangle_g$ is the expectation value with respect to the ground state of the quantum system. In order to compute the correlators (25) for anisotropic ($\gamma \neq 0$) Hamiltonians, it would require evaluating the determinants of block-Toeplitz matrices of the type discussed in [12]. For simplicity, we do not treat this case as it would be technically considerably more complicated, but it would not add substantially different physical information.

As summarized in Table I in the Appendix, it was found in [6,7] that for all symmetries considered, the matrices $\mathcal{M}_n$ are combinations of Toeplitz plus Hankel matrices

$$\mathcal{M}_n = T_n \pm H_n, \quad (26)$$

where T_n represents a Toeplitz matrix and H_n represents a Hankel matrix:

$$\begin{aligned} T_n[g] &= \{g_{j-k}\}_{j,k=0,\dots,n-1}, \\ H_n[g] &= \{g_{j+k+c}\}_{j,k=0,\dots,n-1}, \end{aligned} \quad (27)$$

where c is a constant given by Table I, and g_l are the Fourier coefficients of the symbol $g^{\mathcal{M}}(\theta)$ given by

$$g^{\mathcal{M}}(\theta) = \frac{\Lambda(\theta)}{|\Lambda(\theta)|}, \quad (28)$$

with $\Lambda(\theta)$ obtained by taking the limit $M \rightarrow \infty$ in (5).

The matrices $\mathcal{M}_n^x$ and $\mathcal{M}_n^y$ can be written as the combination of Toeplitz plus Hankel matrices

$$\mathcal{M}_n^x = T_n^x \pm H_n^x, \quad \mathcal{M}_n^y = T_n^y \pm H_n^y, \quad (29)$$

where T_n^x and T_n^y represent Toeplitz matrices with the same structure as in (27), but now the g_l s are Fourier coefficients of the symbols $g^{T^x}(\theta) = e^{i\theta} g^{\mathcal{M}}(\theta)$ and $g^{T^y}(\theta) = e^{-i\theta} g^{\mathcal{M}}(\theta)$, respectively. Similarly H_n^x and H_n^y represent Hankel matrices with entries given by Fourier coefficients of the symbols $g^{H^x}(\theta) = g^{H^y}(\theta) = e^{-i\theta} g^{\mathcal{M}}(\theta)$.

When $\gamma = 0$, $\mathcal{M}_n$ is a symmetric matrix with symbol $g^{\mathcal{M}}(\theta)$ given by (28). In this case when the system is gapped and away from the critical point,⁴ $g^{\mathcal{M}}(\theta)$ is a constant taking values ± 1 , and $\mathcal{M}_n = \pm I$. Therefore, we find that

$$\left\langle \prod_{l=i}^{i+n} \sigma_l^z \right\rangle_g = \pm 1, \quad \text{and} \quad \langle \sigma_i^x \sigma_{i+n}^x \rangle_g = \langle \sigma_i^y \sigma_{i+n}^y \rangle_g = 0. \quad (30)$$

When the system is gapless, $g^{\mathcal{M}}(\theta)$ is a piecewise continuous even function taking values 1 and -1 and has discontinuities at all points θ_r satisfying

$$\Lambda(\theta_r) = 0, \quad (31)$$

with the additional condition that the first nonzero derivative of $\Lambda(\theta)$ at θ_r is odd.

Symbols of this type can be written in Fisher-Hartwig form:

$$g^{\mathcal{M}}(\theta) = \phi(\theta) \prod_{r=1}^L u_{\alpha_r}(\theta - \theta_r) t_{\beta_r}(\theta - \theta_r), \quad (32)$$

where ϕ is smooth, has winding number zero, and

$$t_{\beta}(\theta) = e^{-i\beta(\pi-\theta)}, \quad 0 \leq \theta < 2\pi, \quad \beta \notin \mathbb{Z}, \quad (33)$$

$$u_{\alpha}(\theta) = (2 - 2\cos\theta)^{\alpha}, \quad \text{Re } \alpha > -\frac{1}{2},$$

where L is the number of discontinuities in the interval $[0, 2\pi)$. Note here that the term $(2 - 2\cos\theta)^{\alpha}$ has a zero if $\text{Re } \alpha > 0$, and a pole if $\text{Re } \alpha < 0$, and an oscillating discontinuity if $\text{Re } \alpha = 0$ but $\text{Im } \alpha_r \neq 0$. The term $e^{i\beta(\theta-\pi)}$ is a function with a jump discontinuity with limit $e^{-i\beta\pi}$ ($e^{i\beta\pi}$) as $\theta \rightarrow +0$ ($\theta \rightarrow 2\pi - 0$).

In our case $g^{\mathcal{M}}(\theta)$ is even and has only discontinuities; hence, (32) can be simplified further as

$$g^{\mathcal{M}}(\theta) = \phi(\theta) \prod_{r=1}^R t_{\beta_r}(\theta - \theta_r) t_{-\beta_r}(\theta + \theta_r), \quad (34)$$

where now R is the number of discontinuities in the interval $[0, \pi)$ [thus $L = 2R$ when comparing to (32)].

We now make use of the generalized Fisher-Hartwig Theorem, conjectured by Basor and Tracy [25] and proved by Deift, Its, and Krasovsky [26] for $\beta_r \in \mathbb{C}$:

$$\det T_n[g] \sim \sum_{\text{Rep.}} e^{c_0 n} n^{-\sum_{r=1}^L \beta_r^2} E, \quad n \rightarrow \infty, \quad (35)$$

where c_0 is the zeroth Fourier coefficient of $\ln \phi(\theta)$, the sum is over the different representations of (34) corresponding to the minimum exponent $\sum_r \beta_r^2$, and E is a constant given by [6, 26]

$$E = e^{\sum_{k=1}^{\infty} k c_k c_{-k}} \prod_{r=1}^L (\phi_+(e^{i\theta_r}))^{-\beta_r} (\phi_-(e^{-i\theta_r}))^{\beta_r} \\ \times \prod_{0 \leq r \neq s \leq L} (1 - e^{i(\theta_s - \theta_r)})^{\beta_r \beta_s} \prod_{r=1}^L G(1 - \beta_j) G(1 + \beta_j), \quad (36)$$

where $G(x)$ is the Barnes' G function and

$$\ln \phi_+(t) = \sum_{j=1}^{\infty} c_j t^j, \quad \ln \phi_-(t) = \sum_{j=1}^{\infty} c_{-j} t^{-j}. \quad (37)$$

In our case we have 2^R representations for (34), all with the form⁵

$$\phi(\theta; \pm\theta_1, \dots, \pm\theta_R) = (-1)^R e^{i \sum_{r=1}^R \pm\theta_r} \quad \text{with} \quad \beta_r = \pm \frac{1}{2}, \quad (38)$$

where the sign in front of β_r matches the sign in front of θ_r and the 2^R representations correspond to the 2^R possible sign combinations for each θ_r .

Until now (35) has been proven for determinants of Toeplitz matrices only; thus, the following results only apply to our class of quantum systems (1) corresponding to $U(N)$ symmetry. As $n \rightarrow \infty$ we obtain

$$\left\langle \prod_{l=i}^{i+n} \sigma_l^z \right\rangle_g = n^{-\frac{R}{2}} (-1)^{n(R+1)} F \sum_{\text{Rep.}} \mathcal{Q},$$

$$\langle \sigma_i^x \sigma_{i+n}^x \rangle_g = n^{-\frac{R}{2}} (-1)^{Rn} F^x \sum_{\text{Rep.}} \mathcal{Q}, \quad (39)$$

$$\langle \sigma_i^y \sigma_{i+n}^y \rangle_g = n^{-\frac{R}{2}} (-1)^{Rn} F^y \sum_{\text{Rep.}} \mathcal{Q},$$

when the systems is gapless, with constants

$$F = \left(G\left(\frac{1}{2}\right) G\left(\frac{3}{2}\right) \right)^{2R} \prod_{r=1}^R |1 - e^{i2\theta_r}|^{-\frac{1}{2}}, \quad (40)$$

$$F^x = e^{-\sum_{k=1}^{\infty} \frac{1}{k}} F \quad \text{and} \quad F^y = e^{\sum_{k=1}^{\infty} \frac{1}{k}} F.$$

The term $\mathcal{Q}$ contains the factors which have a dependence on the representation of $\phi(\theta)$, given by

$$\mathcal{Q}(\pm\theta_1, \dots, \pm\theta_R) = e^{in \sum_{r=1}^R \pm\theta_r} \prod_{1 \leq r < s \leq R} \left| \frac{1 - e^{i(\theta_r - \theta_s)}}{1 - e^{i(\theta_r + \theta_s)}} \right|^{4\beta_s \beta_r}, \quad (41)$$

since in our case we have

$$\prod_{0 \leq r \neq s \leq L} (1 - e^{i(\theta_s - \theta_r)})^{\beta_r \beta_s} \\ = \prod_{r=1}^R |1 - e^{i2\theta_r}|^{-\frac{1}{2}} \prod_{1 \leq r < s \leq R} \left| \frac{1 - e^{i(\theta_r - \theta_s)}}{1 - e^{i(\theta_r + \theta_s)}} \right|^{4\beta_s \beta_r}. \quad (42)$$

Thus, we find that away from the critical point $|\Gamma| > \Gamma_c$, the asymptotics (30) are given by a constant, whereas below the critical external field $|\Gamma| < \Gamma_c$ (when the system is gapless), the correlator decays like a power law (39) as $n \rightarrow \infty$. Of particular interest is the fact that the exponent describing this power law depends upon the number of discontinuities of the symbol (28), which for a specific system with fixed interaction coefficients will vary with the external field Γ . In other

⁴That is, the external field $|\Gamma| > |\Gamma_c|$; thus, all the spins are aligned in the direction of the field.

⁵This representation assumes that (28) is $+1$ in the region surrounding $\theta = 0$.

words, the long-range order is not determined by a local order parameter but by the overall *topological* properties of the chain. This is reminiscent of the *Berezinskii-Kosterlitz-Thouless* (BKT) *topological transitions* in two-dimensional classical systems [27,28]. Recently, the link between BKT transitions and quantum phase transitions (QPTs) has been the object of intensive studies [29–32]. It would be interesting to understand the relation between BKT transitions and QPTs for the models we consider in this paper in more depth and we intend to pursue this direction of research in a future publication.

We see that of central importance to this work is the analytic behavior of the symbol of the matrix determinant obtained for the correlation functions and that it is the symmetries of the systems which shape the behavior of this

symbol. It is by using this symbol and its analytic properties that we are able to show how symmetries affect the critical properties.

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APPENDIX: SYMMETRY CLASSES

TABLE I. The structure of functions $a(j)$ and $b(j)$ dictating the entries of matrices $\tilde{\mathbf{A}} = \mathbf{A} - 2h\mathbf{I}$ and $\tilde{\mathbf{B}} = \gamma\mathbf{B}$, which reflect the respective symmetry groups. The g_l s are the Fourier coefficients of the symbol $g^M(\theta)$ of $\mathcal{M}_M$. Note that for all symmetry classes other than $U(N)$, $\gamma = 0$ and thus $\tilde{\mathbf{B}} = 0$.

Classical compact group	Structure of matrices $\tilde{A}_{j,k}$ ($\tilde{B}_{j,k}$)	Matrix entries $(\mathcal{M}_n)_{j,k}$
$U(N)$	$a(j-k)$ ($b(j-k)$)	$g_{j-k}, \quad j, k \geq 0$
$O^+(2N)$	$a(j-k) + a(j+k)$	$g_0 \quad \text{if } j = k = 0$ $\sqrt{2}g_l \quad \text{if}$ either $j = 0, k = l$ or $j = l, k = 0$
$Sp(2N)$	$a(j-k) - a(j+k+2)$	$g_{j-k} + g_{j+k}, \quad j, k > 0$
$O^\pm(2N+1)$	$a(j-k) \mp a(j+k+1)$	$g_{j-k} - g_{j+k+2}, \quad j, k \geq 0$
$O^-(2N+2)$	$a(j-k) - a(j+k+2)$	$g_{j-k} \mp g_{j+k+1}, \quad j, k \geq 0$ $g_{j-k} - g_{j+k+2}, \quad j, k \geq 0$

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