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Null Geodesics in Perturbed Spacetimes

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ABSTRACT

We present a generalization and refinement of the Sachs-Wolfe technique which unifies many of the approaches taken to date and clarifies both the physical and the mathematical character of the method. We illustrate the formalism with a calculation of the behavior of light passing a moving lens on a Minkowski background.

Subject headings: gravitation—cosmology: cosmic microwave background—cosmology: gravitational lensing

I. Introduction

A perfectly homogeneous and isotropic cosmic microwave background radiation (CMBR) is possible only in a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime (Ehlers, Geren and Sachs, 1968). For this reason, the extreme isotropy of the CMBR has often been taken to indicate that our Universe is, in some sense, well approximated by one of the FLRW models. Nevertheless, there is structure in the Universe that must induce distortions into the CMBR.

In the usual scenarios, the structure that we observe today evolves via gravitational instability from small inhomogeneities of the matter fields at the epoch of matter-radiation equality, which occurs at temperature $T_{EQ} \sim 5.5\Omega_0 h^2$ eV (where $h = H_0/100 \text{ kms}^{-1} \text{ Mpc}^{-1}$), when the formation of galaxies and clusters by gravitational collapse begins. (Peebles and Yu, 1970; Bond and Efstathiou, 1984; Bond and Efstathiou, 1987; Suto, Gouda, and Sugiyama, 1990). An important prediction of gravitational instability models was first noticed by Sachs and Wolfe (1967) who showed that even small amplitude perturbations of FLRW spacetimes at recombination (temperature $T_{REC} \sim .31$ eV, notably close to the epoch of matter-radiation equality) cause appreciable temperature fluctuations in the CMBR.

There are two main problems associated with a Sachs-Wolfe calculation; relating the metric perturbations to interesting physical perturbations of the matter fields, and determining the behavior of null geodesics in the perturbed spacetime. Solving the first problem amounts to constructing the physical model to be investigated. The behavior of electromagnetic radiation in the model can then be understood in the geometric optics limit by examining the behavior of null geodesics (Jordan, Ehlers, and Sachs, 1961). The original work of Sachs and Wolfe (1967) concerned itself with a particular perturbed Einstein de-Sitter spacetime and with a single observable property of its null rays, the redshift. This is an important quantity

because it is directly, and simply, related to the temperature profile of a thermal radiation field such as the CMBR. Much subsequent research has been devoted to constructing perturbations about Einstein de-Sitter spacetime which embody specific physical characteristics (e.g. spheroidal perturbations) and using the equivalent of the Sachs-Wolfe formula appropriate for the specific perturbations in order to understand the temperature pattern of the background radiation in these models (Grishchuk and Zel'dovich, 1978; Linder, 1988a; Argüeso and Martinez-Gonzalez, 1989; Argüeso, Martinez-Gonzalez, and Sanz, 1989). A formula for the redshift of null geodesics valid for perturbations about the curved FLRW models as well as the flat was produced by Anile and Motta in 1976.

Since gravitational lensing may be thought of as an aspect of the theory of perturbed null geodesics, some researchers have used Sachs-Wolfe like calculations to investigate the bending of light rays. Linder has considered the spatial components of the perturbed geodesic equation on an Einstein de-Sitter background, in addition to the timelike component, and has analyzed questions usually found in the realm of gravitational lens theory with a Sachs-Wolfe type calculation (Linder, 1988b, 1990). Martinez-Gonzalez, Sanz, and Silk (1990) have exploited the fact that, under certain assumptions, metric perturbations of the flat FLRW spacetime may be expressed by a small, scalar potential which obeys an expanding-space analog of the usual Poisson equation. They use this result to write formulae for both redshift and deflection angle in physically transparent forms.

A complete generalization of the Sachs-Wolfe formalism should solve for the timelike and spacelike components of null geodesics in a metric perturbed spacetime of arbitrary background. Such a generalization is presented here for the first time. When the background is taken to be Einstein-de Sitter our technique agrees with that of Sachs and Wolfe for the expression of the photon redshift and with that of Linder (1988b) for the perturbed photon path. When the background is taken to be a curved FLRW spacetime, our results agree with those of Anile and Motta for the photon redshift (the quantity with which they were concerned), but our method, unlike theirs, can also be used to describe the spatial components of the perturbed photon wavevector or lensing. Their formalism is inadequate for these issues because the spatial components of their primary equation (equation (7) of Anile and Motta, 1976), do not take into account the distinct nature of the background and perturbed paths to the correct order.

Since our formalism applies to general backgrounds it is able to handle interesting cases which were not previously amenable to a Sachs-Wolfe type calculation. For example, our method may be used to investigate perturbations of a Schwarzschild background by long-wavelength gravitational waves. We also expect that current developments in mathematical cosmology will benefit from our more general formulation of the Sachs-Wolfe technique. For instance, to date work has focused on models where the density perturbations are small, in the sense that the fractional density fluctuations $\delta\rho/\rho \ll 1$. However, recent work by Futamase (1989) and Jacobs, Linder, and Wagoner (1992) allows a description of a spacetime with *large* density contrasts by a *small* metric perturbation. While the background spacetime

is not necessarily FLRW, “clumpy” cosmologies described in this way could be excellent models of our Universe. In addition, the notion of an inhomogeneous spacetime behaving, “on average”, like an FLRW spacetime is not rigorously understood and there is no reason to believe that future models will behave like linear perturbations of Einstein-de Sitter space. We also hope that the technique presented here will be able to clarify a difficult point in lensing theory, the nature of the distance factors in the lens equation. Because of the difficulty in working with the rigorous optical scalar equations (Sachs, 1961), the relationship between the formal mathematics of lensing and the, very successful, use of the lens equation is not yet clear (Futamase and Sasaki, 1989; Watanabe and Tomita, 1990). The relationship of our technique to the optical scalar equations is the subject of a forthcoming paper (Pyne and Birkinshaw, work in progress).

This paper is organized as follows. In section II we formulate an equation describing the local separation of geodesics of the perturbed metric from geodesics of the background. We then show that this equation is formally integrable as a power series in the curvature of the background spacetime. In section III we use our solution to find the behavior of light passing near a moving lens on a Minkowski background, obtaining results in agreement with those obtained to date using other approaches. In section IV we summarize our results.

II. Null Geodesics in Perturbed Spacetimes

(a) Conventions. In the following, Greek indices run over 0,1,2,3 while Latin indices run over 1,2,3, the zeroth component is time-like, and the summation convention is assumed. The metric signature is +2 and we take $G = c = 1$. The Riemann tensor convention is defined by equation (A4).

(b) A Perturbed Jacobi Equation. Our starting point is the usual formalism of metric perturbation theory. We consider the true metric to be the sum of a background metric $g_{\mu\nu}^{(0)}$, and a small perturbation, $h_{\mu\nu}$,

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}. \quad (1)$$

We define $g^{\mu\nu}$ and $g^{(0)\mu\nu}$ by

$$\begin{aligned} g^{\mu\alpha} g_{\alpha\nu} &= \delta_{\nu}^{\mu} \quad , \\ g^{(0)\mu\alpha} g_{\alpha\nu}^{(0)} &= \delta_{\nu}^{\mu} \quad . \end{aligned} \quad (2)$$

It is then a simple matter to verify that to order h

$$g^{\mu\nu} = g^{(0)\mu\nu} - h^{\mu\nu} \quad (3)$$

where $h^{\mu\nu}$ is defined by

$$h^{\mu\nu} = g^{(0)\mu\alpha} g^{(0)\nu\beta} h_{\alpha\beta} \quad . \quad (4)$$

As usual (in a metric perturbation theory), we raise and lower all tensor indices with the background metric.

The Levi-Civita connection of $g_{\mu\nu}$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma} (g_{\sigma\beta,\alpha} + g_{\alpha\sigma,\beta} - g_{\alpha\beta,\sigma}) \quad (5)$$

can be split into zeroth and first order components in h

$$\Gamma^\mu_{\alpha\beta} = \Gamma^{(0)\mu}_{\alpha\beta} + \Gamma^{(1)\mu}_{\alpha\beta} \quad (6)$$

where

$$\Gamma^{(0)\mu}_{\alpha\beta} = \frac{1}{2} g^{(0)\mu\sigma} (g^{(0)}_{\sigma\beta,\alpha} + g^{(0)}_{\alpha\sigma,\beta} - g^{(0)}_{\alpha\beta,\sigma}) \quad (7)$$

and

$$\begin{aligned}
\Gamma^{(1)\mu}_{\alpha\beta} &= \frac{1}{2}g^{(0)\mu\sigma} (h_{\sigma\beta,\alpha} + h_{\alpha\sigma,\beta} - h_{\alpha\beta,\sigma}) \\
&\quad - \frac{1}{2}h^{\mu\sigma} (g^{(0)}_{\sigma\beta,\alpha} + g^{(0)}_{\alpha\sigma,\beta} - g^{(0)}_{\alpha\beta,\sigma}) \\
&= \frac{1}{2}g^{(0)\mu\sigma} (h_{\sigma\beta;\alpha} + h_{\alpha\sigma;\beta} - h_{\alpha\beta;\sigma}).
\end{aligned} \tag{8}$$

The semicolon above and in what follows denotes covariant differentiation with respect to the Levi-Civita connection of $g^{(0)}_{\mu\nu}$. We note that $\Gamma^{(1)\mu}_{\alpha\beta}$ is tensorial, at least in the usual restricted sense of metric perturbation theory (that is, under infinitesimal co-ordinate transformations).

Let $x^{(0)\mu}(\lambda)$ be a geodesic of the background spacetime with λ affine. We will sometimes refer to $x^{(0)\mu}(\lambda)$ as the “unperturbed path”. It satisfies the geodesic equation in the unperturbed spacetime, ($\cdot = d/d\lambda$)

$$\ddot{x}^{(0)\mu} + \Gamma^{(0)\mu}_{\alpha\beta} \dot{x}^{(0)\alpha} \dot{x}^{(0)\beta} = 0. \tag{9}$$

The notation $\Gamma^{(0)\mu}_{\alpha\beta} (x^{(0)})$ is meant to convey explicitly that the connection terms are evaluated on the unperturbed path.

Now consider the expression

$$x^\mu(\lambda) = x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda) \tag{10}$$

where $x^{(0)\mu}(\lambda)$ is the unperturbed geodesic. At this point, both $x^\mu(\lambda)$ and $x^{(1)\mu}(\lambda)$ are unspecified and so equation (10) can be considered to define either x^μ or $x^{(1)\mu}$ once the other is given. We will derive conditions on $x^{(1)\mu}(\lambda)$ which will be necessary and sufficient if $x^\mu(\lambda)$ is to be geodesic in the perturbed spacetime. In the following derivation, we will truncate three expressions by discarding terms of quadratic order in the products of $x^{(1)\mu}$ and $\dot{x}^{(1)\mu}$. We will often refer to this as “working to first order.” The consistency conditions for the truncations made will be discussed at the end of the derivation.

Differentiating equation (10) twice and using equation (9) gives

$$\ddot{x}^\mu = -\Gamma^{(0)\mu}_{\alpha\beta}(x^{(0)})\dot{x}^{(0)\alpha}\dot{x}^{(0)\beta} + \dot{x}^{(1)\mu}. \quad (11)$$

On the other hand, if $x^\mu(\lambda)$ is to be an affinely parameterized geodesic of the perturbed spacetime we must have

$$\begin{aligned} \ddot{x}^\mu &= -\Gamma^\mu_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta \\ &= -\Gamma^{(0)\mu}_{\alpha\beta}(x)\left(\dot{x}^{(0)\alpha}\dot{x}^{(0)\beta} + 2\dot{x}^{(0)\alpha}\dot{x}^{(1)\beta}\right) \\ &\quad - \Gamma^{(1)\mu}_{\alpha\beta}(x)\dot{x}^{(0)\alpha}\dot{x}^{(0)\beta}, \end{aligned} \quad (12)$$

where we have used equations (6) and (10), keeping terms only to first order. Provided no singularities intervene (which we can ensure by keeping to small enough neighborhoods and regular points), the connection terms near the perturbed path, x , may be expanded about their values on the unperturbed path, $x^{(0)}$, as

$$\begin{aligned} \Gamma^{(0)\mu}_{\alpha\beta}(x) &= \Gamma^{(0)\mu}_{\alpha\beta}(x^{(0)}) + \Gamma^{(0)\mu}_{\alpha\beta,\tau}(x^{(0)})x^{(1)\tau} + \dots \\ \Gamma^{(1)\mu}_{\alpha\beta}(x) &= \Gamma^{(1)\mu}_{\alpha\beta}(x^{(0)}) + \dots \end{aligned} \quad (13)$$

Substituting these expansions into equation (12) gives

$$\begin{aligned} \ddot{x}^\mu &= -\Gamma^{(0)\mu}_{\alpha\beta}(x^{(0)})\dot{x}^{(0)\alpha}\dot{x}^{(0)\beta} - \Gamma^{(1)\mu}_{\alpha\beta}(x^{(0)})\dot{x}^{(0)\alpha}\dot{x}^{(0)\beta} \\ &\quad - 2\Gamma^{(0)\mu}_{\alpha\beta}(x^{(0)})\dot{x}^{(0)\alpha}\dot{x}^{(1)\beta} - \Gamma^{(0)\mu}_{\alpha\beta,\tau}(x^{(0)})\dot{x}^{(0)\alpha}\dot{x}^{(0)\beta}x^{(1)\tau}. \end{aligned} \quad (14)$$

Comparing equations (14) and (11) we conclude that $x^\mu(\lambda)$, defined by equation (10), will be an affinely parametrized geodesic of the perturbed spacetime provided $x^{(1)\mu}(\lambda)$ satisfies the system of four coupled, second-order differential equations

$$\left(\frac{d^2}{d\lambda^2} + A\frac{d}{d\lambda} + B\right)x^{(1)} = f \quad , \quad (15)$$

where the 4×4 matrices A and B and the four-vector f are defined by

$$\begin{aligned} A^\mu{}_\alpha &= 2\Gamma^{(0)\mu}{}_{\tau\alpha}k^{(0)\tau} \\ B^\mu{}_\alpha &= \Gamma^{(0)\mu}{}_{\tau\sigma,\alpha}k^{(0)\tau}k^{(0)\sigma} \\ f^\mu &= -\Gamma^{(1)\mu}{}_{\tau\sigma}k^{(0)\tau}k^{(0)\sigma} \end{aligned} \quad . \quad (16)$$

Anticipating the interpretation of null geodesics as photon paths we have written $\dot{x}^{(0)}$ as $k^{(0)}$. A 4×4 identity matrix is the implied coefficient of the second-order derivative in equation (15). We are using the matrix notation as a convenient shorthand for unambiguous summations, e.g. $(Bx^{(1)})^\mu = B^\mu{}_\alpha x^{(1)\alpha}$. We emphasize that equation (15) is to hold along some segment of the unperturbed path and generates solutions for the separation, $x^{(1)}$, of the perturbed path relative to the background path.

We still need to discuss where the above equation is actually valid, that is, the appropriate consistency criteria for the approximations made. We will use Futamase's (1988) goodness-of-fit parameters for this purpose. Let the typical magnitude of a component of h relative to that of a component of $g^{(0)}$ be written ϵ^2 . Write the typical scales of h and $g^{(0)}$ as l and L , respectively. Put $\kappa = l/L$. Also, for simplicity (though without loss of generality), assume that the initial conditions appropriate for the sought-after solution are given at $\lambda = 0$. Let $k^{(0)}$ denote the magnitude of a typical component of the unperturbed wavevector at $\lambda = 0$. The consistency criteria for the approximations made above may then be found with the help of equations (9), above, and (20) below to be $\epsilon^2 \ll 1$, $\epsilon^2 \ll \kappa$, and

$$\epsilon^2 L \ln \left(\frac{\lambda k^{(0)}}{L} + 1 \right) \ll \min(l, L). \quad (17)$$

This last condition may be replaced with the stricter $\epsilon^2 \lambda k^{(0)} \ll \min(l, L)$.

In appendix A we show that equation (15) is simply a perturbation of the Jacobi equation of the background spacetime. In its most elegant form, then, our equation reads

$$\nabla_{k^{(0)}}^2 x^{(1)} - R(k^{(0)}, x^{(1)})k^{(0)} = f \quad (18)$$

where ∇ is the Christoffel connection of the background spacetime and R is its curvature tensor (with implicit superscript (0)). Remembering the tensorial nature of f , the tensorial character of our equation is now immediate. The physical meaning of the equation is also clear. The geodesic of the perturbed metric differs from that of the background because of

- (1) a force-like term associated with the perturbation (encoded in f) and
- (2) the usual geodesic deviation induced by the background spacetime.

(c) Comment on Initial Data. We now address the issue of what are appropriate initial data for the perturbed Jacobi equation. We know that the geodesic which we seek to construct is uniquely specified by a point of that geodesic, p , and the tangent vector to the geodesic at that point, k_p . As equation (10) indicates, we can construct a segment of the desired geodesic locally about p from any geodesic of the background which (in the naive sense) passes close to p with tangent vector close to k_p . An important point, however, is that the initial data specifying the desired geodesic does not, in general, generate a null geodesic of the background because k_p is generally not null in both the perturbed and background metrics. As we wish to construct null geodesics of the full metric using a null geodesic of the background, we conclude that the initial condition $k^{(1)}(\lambda_p) = 0$ is not appropriate for our equation. However, in the regions far from a localized perturbation (where, loosely, $h \rightarrow 0$) this subtlety is entirely avoided.

It is instructive to examine this point in another way. Suppose that we have chosen a null geodesic of the background and that a solution, $x^{(1)}(\lambda)$, of the perturbed Jacobi equation has been found. The condition that the constructed geodesic be null, to order h , in the perturbed spacetime is

$$\begin{aligned} g_{\mu\nu}(x)k^\mu k^\nu &= h_{\mu\nu}(x^{(0)})k^{(0)\mu}k^{(0)\nu} + 2g_{\mu\nu}^{(0)}(x^{(0)})k^{(0)\mu}k^{(1)\nu} \\ &\quad + 2g_{\mu\nu,\rho}^{(0)}(x^{(0)})x^{(1)\rho}k^{(0)\mu}k^{(0)\nu} \\ &= 0. \end{aligned} \tag{19}$$

Making use of the co-ordinate invariance of scalar quantities, we evaluate the last term on the RHS of the first equality by choosing co-ordinates adapted to $g^{(0)}$ at any given point so that we can replace the ordinary derivative with the covariant derivative of $g^{(0)}$. Metric compatibility then tells us that this term vanishes. We are left with the condition that everywhere along the background geodesic we must have

$$g_{\mu\nu}^{(0)}k^{(0)\mu}k^{(1)\nu} = -\frac{1}{2}h_{\mu\nu}k^{(0)\mu}k^{(0)\nu}. \tag{20}$$

We now show that this condition can always be enforced. We choose to work with co-ordinates for which the background connection coefficients vanish along the unperturbed path. In these co-ordinates, equation (15) implies

$$\begin{aligned}\frac{d}{d\lambda} \left(g_{\mu\nu}^{(0)} k^{(0)\mu} k^{(1)\nu} \right) &= g_{\mu\nu}^{(0)} k^{(0)\mu} \left(-\Gamma^{(0)\nu}_{\alpha\gamma,\beta} k^{(0)\alpha} k^{(0)\gamma} x^{(1)\beta} - \Gamma^{(1)\nu}_{\alpha\beta} k^{(0)\alpha} k^{(0)\beta} \right) \\ &= -g_{\mu\nu}^{(0)} \Gamma^{(1)\nu}_{\alpha\beta} k^{(0)\mu} k^{(0)\alpha} k^{(0)\beta}\end{aligned}\tag{21}$$

where the first term on the RHS vanishes as a consequence of the identity

$$R^{(0)}_{\mu\alpha\beta\gamma} k^{(0)\mu} k^{(0)\alpha} k^{(0)\gamma} x^{(1)\beta} = 0$$

written out in the choosen co-ordinates. The RHS of equation (21) may be shown, with the aid of equation (8), to be

$$= -\frac{1}{2} k^{(0)\mu} k^{(0)\alpha} k^{(0)\beta} \left(h_{\alpha\beta,\mu} - \Gamma^{(0)\rho}_{\alpha\mu} h_{\rho\beta} - \Gamma^{(0)\rho}_{\mu\beta} h_{\alpha\rho} \right).\tag{22}$$

We have thus shown that

$$\frac{d}{d\lambda} \left(g_{\mu\nu}^{(0)} k^{(0)\mu} k^{(1)\nu} \right) = \frac{D}{d\lambda} \left(-\frac{1}{2} h_{\mu\nu} k^{(0)\mu} k^{(0)\nu} \right)\tag{23}$$

where $D/d\lambda$ is the covariant derivative of $g^{(0)}$ along $x^{(0)}$.

On scalars we have $D/d\lambda = d/d\lambda$ so we can conclude that if equation (20) is satisfied at any point along the background geodesic it will be satisfied at every point. Notice that it is very simple to impose equation (20) at the initial point but that, in general, $k^{(1)} = 0$ is not an acceptable solution because at a given point a null vector in the perturbed metric is not necessarily null in the background metric. The calculation above thus amounts to a direct check that $\langle k^{(0)} + k^{(1)}, k^{(0)} + k^{(1)} \rangle_{g^{(0)+h}}$ is a constant of geodesic motion to order h , as it must be if this procedure is to be meaningful (the angle brackets denote the inner product with respect to the metric of the subscript).

(d) A Solution to the Perturbed Jacobi Equation. Having found the domain for which our equation, (15), is valid and having shown that the equation generates null paths from null paths, we construct a formal solution. The techniques we will use are simply the matrix analogues of familiar methods for dealing with ordinary differential equations. First we perform a change of variables in order to eliminate the first derivative term in our equation. To accomplish this, let $P(\lambda, a)$ be a 4×4 matrix function of two real arguments, and $v(\lambda)$ a vector such that $x^{(1)} = Pv$. Then in terms of v , equation (15) becomes

$$\ddot{v} + P^{-1} (2\dot{P} + AP) \dot{v} + P^{-1} (\ddot{P} + A\dot{P} + BP) v = P^{-1} f \quad (24)$$

where we have assumed P non-singular (justified below). Now choose P to satisfy

$$\dot{P} = -\frac{A}{2}P, \quad (25)$$

The solution is the path-ordered exponential,

$$P(\lambda, a) = \mathcal{P} \exp \left(-\frac{1}{2} \int_a^\lambda A(\tau) d\tau \right) \quad (26)$$

(a short introduction to the path-ordering symbol, $\mathcal{P}$, is provided in appendix B). We remark that P is exactly Synge's parallel propagator (Synge, 1960). We will call P the connector, following De Felice and Clarke (1990). Anticipating our final result, we see that the effect of the change of variables above is basically to untangle the co-ordinate basis, just as if we were working with a parallel propagated tetrad frame.

In writing equation (26) we have set $P(a, a) = 1_d$, the four-dimensional identity matrix. This initial condition and equation (25) can be used to show the important identity

$$P(\lambda, \lambda_1)P(\lambda_1, a) = P(\lambda, a) \quad (27)$$

and so

$$P(\lambda, a)^{-1} = P(a, \lambda) \quad (28)$$

proving that P is invertible.

With P chosen in this manner, equation (24) now becomes

$$\ddot{v} + P^{-1} \left(- \left(\frac{A}{2} \right)^2 - \frac{\dot{A}}{2} + B \right) P v = P^{-1} f \quad (29)$$

Written out explicitly using equation (16) the quantity in parentheses reveals itself to be $-R^{(0)\mu}{}_{\nu\rho\sigma} k^{(0)\nu} k^{(0)\rho}$, which we will write as the matrix $-R$, using notation consistent with (18).

We write (29) as a first-order system

$$\frac{d}{d\lambda} \begin{pmatrix} v \\ \dot{v} \end{pmatrix} - \begin{pmatrix} 0 & 1_d \\ P^{-1} R P & 0 \end{pmatrix} \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ P^{-1} f \end{pmatrix} \quad (30)$$

which can be solved by constructing the transition matrix of the system, $U(\lambda, a)$, which is the solution of its homogeneous part (Humi and Miller, 1988). This is the familiar method of Greens function solution. We will use the terms transition matrix and Greens function interchangeably in referring to $U(\lambda, a)$. Another path-ordering is needed to actually construct the transition matrix, with the result

$$U(\lambda, a) = \mathcal{P} \exp \left(\int_a^\lambda \begin{pmatrix} 0 & 1_d \\ P(\tau, a)^{-1} R(\tau) P(\tau, a) & 0 \end{pmatrix} d\tau \right) \quad (31)$$

where the boundary condition $U(a, a) = 1_d$ has been imposed. As a result, equations (27) and (28) remain valid with U written in place of P . Defining the column vectors $y = (v, \dot{v})$ and $s = (0, P^{-1} f)$ the formal solution to equation (29) with initial value boundary data is

$$y(\lambda) = U(\lambda, a)y(a) + \int_a^\lambda U(\lambda, \tau)s(\tau) d\tau \quad (32)$$

Confirmation that this expression solves equation (29) is provided by differentiation.

While (32) is a formal solution to the perturbed Jacobi equation (29; and hence 15), at this point its utility is far from evident. In particular, the presence of two path-ordered exponentials makes the calculations difficult. Had we not made the initial change of variables from $x^{(1)}$ to v we could have solved the system without recourse to the connector. We would then need only a single path-ordering, that responsible for constructing the appropriate Greens function. Appendix C outlines this approach. We feel, however, that the strength of the particular transcription of the solution to the perturbed Jacobi equation given above, equation (32), lies in its explicit geometric character. This is most evident by choosing co-ordinates for which the Christoffel coefficients of the background vanish along the background geodesic. In this case equation (32) applies with $P = 1_d$, $s = (0, f)$, $y = (x^{(1)}, k^{(1)})$ and

$$U(\lambda, a) = \mathcal{P} \exp \left(\int_a^\lambda \begin{pmatrix} 0 & 1_d \\ R & 0 \end{pmatrix} d\tau \right) \quad (33)$$

clearly expressing the relevance of the background curvature to the solution.

III. The Moving Lens

As an illustration of our method, we calculate the asymptotic behavior of a light ray passing by a point mass moving on a Minkowski background. Elements of this scenario have been analyzed by Birkinshaw and Gull (1983) without a fully general-relativistic treatment. Our formalism provides an easy and unified way of handling the problem in the context of general relativity. This particular case is especially simple because Minkowski space is flat, allowing us to choose co-ordinates such that $A = B = 0$, and hence allowing equation (15) to be solved by direct integration. In appendix D we show how the propagator techniques of section II can be used to gain an equivalent solution.

(a) The Metric. Let primed co-ordinates denote the frame in which the lens is at rest at the origin of co-ordinates. In this frame the metric is of the usual

Schwarzschild form,

$$ds'^2 = - \left(1 - \frac{2m}{r'}\right) dt'^2 + \left(1 - \frac{2m}{r'}\right)^{-1} dr'^2 + r'^2 d\theta'^2 + r'^2 \sin^2 \theta' d\phi'^2 \quad , \quad (34)$$

where m is the mass of the lens. Expanding to first order in m/r' we break the metric up into a sum of the Minkowski metric in spherical co-ordinates and a perturbation of the form

$$h_{\alpha'\beta'} = \frac{2m}{r'} \left(\delta_{\alpha'}^0 \delta_{\beta'}^0 + \delta_{\alpha'}^1 \delta_{\beta'}^1 \right). \quad (35)$$

Transforming to Cartesian co-ordinates puts the background Minkowski metric into its usual form, $\text{diag}(-1, 1, 1, 1)$, and puts the perturbation into the form

$$\begin{aligned} h_{0'0'}(x') &= \frac{2m}{r'} \\ h_{0'i'}(x') &= h_{i'0'}(x') = 0 \\ h_{i'j'}(x') &= \frac{2m}{r'^3} x_{i'} x_{j'} \end{aligned} \quad (36)$$

where

$$r' = [x'^2 + y'^2 + z'^2]^{1/2}. \quad (37)$$

Note that the $x_{i'}$ appearing in equation (36) are not vectors but the primed co-ordinate functions. In particular no index raising or lowering via a metric is taking place, $x_{i'} = x^{i'}$. This remark will apply equally well to the unprimed co-ordinate functions.

We are interested in the background metric and metric perturbation in the frame in which the lens appears to be moving with velocity $\underline{v}$, with the underbar denoting a three-vector quantity. We will use unprimed variables for this frame. If we agree to work only to linear order in v the connection between our two sets of co-ordinates is simply a linearized Lorentz transformation

$$\begin{aligned} t' &= t - \underline{v} \cdot \underline{x} \\ \underline{x}' &= \underline{x} - \underline{v}t. \end{aligned} \quad (38)$$

where the dot product is a convenient shorthand for summation of spatial indices, e.g. $\underline{v} \cdot \underline{x} = v^i x_i$. Under this transformation the background Minkowski metric is unchanged while the metric perturbation becomes

$$\begin{aligned} h_{00} &= \frac{2m}{r} \\ h_{0i} &= h_{i0} = -\frac{2m}{r} v_i - \frac{2m}{r^3} (\underline{v} \cdot \underline{x}) x_i \\ h_{ij} &= \frac{2m}{r^3} (x_i x_j - x_j v_i t - x_i v_j t) \end{aligned} \quad (39)$$

with

$$r = \sqrt{(\underline{x} - \underline{v}t) \cdot (\underline{x} - \underline{v}t)} . \quad (40)$$

The global Galilean transformation, (38), may be used to define the three-vector $\underline{v}$ because the background is Minkowski and because products $\mathcal{O}(vh)$ are ignored, so that $v^i = v_i$. Although this is not globally valid in general relativity, the approximations adopted are consistent to the order claimed, as may be discovered by repeating the calculation using the weak-field equations of general relativity, where the perturbation can be written $\text{diag}(-2\phi, -2\phi, -2\phi, -2\phi)$ with ϕ the Newtonian potential of the perturbation, taken to be a moving point mass (see e.g. Weinberg, 1972).

(b) Connection Terms. Simple calculation using equations (8) and (39) produces (to $\mathcal{O}(v, h)$),

$$\begin{aligned} \Gamma_{00}^{(1)0} &= -\frac{m}{r^3} \underline{v} \cdot \underline{x} \\ \Gamma_{0i}^{(1)0} &= \frac{m}{r^3} (x_i - v_i t) \\ \Gamma_{jk}^{(1)0} &= \frac{2m}{r^3} (\underline{v} \cdot \underline{x}) \delta_{jk} - \frac{m}{r^3} (v_j x_k + v_k x_j) - \frac{3m}{r^5} (\underline{v} \cdot \underline{x}) x_j x_k \\ \Gamma_{00}^{(1)i} &= \frac{m}{r^3} (x^i - v^i t) \\ \Gamma_{0j}^{(1)i} &= \frac{3m}{r^5} (\underline{v} \cdot \underline{x}) x_i x_j + \frac{m}{r^3} (x_j v_i - 3x_i v_j) \\ \Gamma_{jk}^{(1)i} &= \frac{2m}{r^3} (x_i - v_i t) \delta_{jk} - \frac{3m}{r^5} (x_i x_j x_k - x_i x_j v_k t - x_j x_k v_i t - x_i x_k v_j t) . \end{aligned} \quad (41)$$

(c) The Unperturbed Path. Let $t = 0$ be the time of closest approach of the photon and lens, and use co-ordinates centered on the lens with co-ordinate axes

chosen so that at $t = 0$ the photon wavevector, $\underline{n}_0$, points in the y direction and the photon position vector, $\underline{r}_0$, lies along the z axis. With this choice of co-ordinates the unperturbed path of the photon is simply $x^{(0)} = (\lambda, 0, \lambda, r_0)$ where $r_0 = |\underline{r}_0|$ is a constant of the motion.

Analysis of the consistency criteria for our method quickly reveals that we may work over the entire unperturbed path provided $\epsilon^2 \ll 1$ and $\epsilon^2 \ll \kappa$. The first of these inequalities limits us to regions of spacetime where the Newtonian potential of the perturbing mass is small, and the second restricts the mass to move at non-relativistic velocities. The second of these conditions we have already imposed. The first amounts to restricting our attention to weak lensing scenarios, that is, lensing for which the impact parameter is much larger than twice the Schwarzschild radius of the lens.

(d) The Bend Angle. We will call the deflection angle in the yz -plane the bend angle. In the small angle approximation it is given by

$$\begin{aligned}\theta_{\text{bend}} &= \frac{dz}{dy} \\ &= \frac{\dot{z}}{\dot{y}} \\ &= \frac{\dot{z}^{(1)}}{1 + \dot{y}^{(1)}} \\ &\approx \dot{z}^{(1)} = - \int_{-\infty}^{+\infty} \left(\Gamma_{00}^{(1)3} + 2\Gamma_{02}^{(1)3} + \Gamma_{22}^{(1)3} \right) d\lambda\end{aligned}\tag{42}$$

(using equation (15) in its first integral form, with $A = B = 0$). Substituting the co-ordinate values on the unperturbed path into the expressions for the connection coefficients, switching variables to $s = \lambda/r_0$, and working to first order in v gives

$$\theta_{\text{bend}} = -\frac{m}{r_0} \int_{-\infty}^{+\infty} \frac{ds}{(1+s^2)^{5/2}} \left[3(1-2v_y) - v_z s + 2v_z s^3 + \frac{15s(v_z + sv_y)}{1+s^2} \right]. \tag{43}$$

which may be integrated trivially to yield the result

$$\theta_{\text{bend}} = -\frac{4m}{r_0} (1 - v_y) \tag{44}$$

or, in three-vector form,

$$\theta_{\text{bend}} = -\frac{4m}{r_0} (1 - \underline{v} \cdot \underline{n}_0). \tag{45}$$

which is valid to first order in m/r_0 and v . This is the first result of our method and a short calculation shows that it agrees with a Lorentz transformation of the usual (static) lens deflection angle to a frame in which the lens moves with velocity $\underline{v}$.

(e) The Frequency Shift. The frequency shift of the photon between emission, e , and observation, o , is defined by

$$\begin{aligned}\frac{\Delta\nu}{\nu} &= \frac{\nu_o - \nu_e}{\nu_e} \\ &= (1 + z)^{-1} - 1.\end{aligned}\tag{46}$$

where the redshift, z , is given by

$$(1 + z)^{-1} = \frac{(k \cdot u)_o}{(k \cdot u)_e}.$$

with u_o and u_e the four-velocities of the observer and emitter, respectively. Keeping terms only to linear order,

$$(1 + z)^{-1} = \frac{k_\mu^{(0)}(o)u^{(0)\mu}(o) + k_\mu^{(0)}(o)u^{(1)\mu}(o) + k_\mu^{(1)}(o)u^{(0)\mu}(o)}{k_\mu^{(0)}(e)u^{(0)\mu}(e) + k_\mu^{(0)}(e)u^{(1)\mu}(e) + k_\mu^{(1)}(e)u^{(0)\mu}(e)}.\tag{47}$$

We take the observer and emitter to be far from the lens and at rest relative to each other (letting the observer and the emitter have some relative motion would merely give rise to the usual Doppler terms). To be precise, the observer and emitter four-velocities are parallel translates of each other along a path passing far from the lens, this relationship being, at least asymptotically, path-independent. This allows us to write $u^{(0)}(o) = u^{(0)}(e) = (1, 0, 0, 0)$. Furthermore, in the asymptotic limit we are considering, $u^{(1)}(o) = u^{(1)}(e) = (0, 0, 0, 0)$. Making these substitutions in equation (47) and combining the resulting expression with equation (46) yields

$$\frac{\Delta\nu}{\nu} = \frac{k_0^{(0)}(o) + k_0^{(1)}(o)}{k_0^{(0)}(e) + k_0^{(1)}(e)} - 1.\tag{48}$$

We showed in section II that for a perturbation of restricted scale we could self-consistently assume that the photon is unperturbed at emission, so that $k^{(1)}(e) = 0$.

In addition, $k_0^{(0)} = g_{0\mu}^{(0)} k^{(0)\mu} = -\dot{t}^{(0)} = -1$ at both emission and reception, from which we deduce

$$\begin{aligned}\frac{\Delta\nu}{\nu} &= -k_0^{(1)}(o) \\ &= k^{(1)0}(o) \equiv \dot{t}^{(1)}(o).\end{aligned}\tag{49}$$

Taking our emission point to be $\lambda = -\infty$ and our reception point to be $\lambda = +\infty$ equations (49) and (15) lead to

$$\frac{\Delta\nu}{\nu} = - \int_{-\infty}^{+\infty} \left(\Gamma_{00}^{(1)0} + 2\Gamma_{02}^{(1)0} + \Gamma_{22}^{(1)0} \right) d\lambda.\tag{50}$$

Substituting in the co-ordinate values for the unperturbed path, transforming variables to $s = \lambda/r_0$, and performing a few algebraic manipulations leads to

$$\frac{\Delta\nu}{\nu} = -\frac{m}{r_0} \int_{-\infty}^{+\infty} \frac{ds}{[1 - 2sv_z + (1 - 2v_y)s^2]^{5/2}} \left[v_z + s(2 - 3v_y) - 6s^2v_z + 2s^3(1 - 5v_y) \right],\tag{51}$$

from which an expansion to $\mathcal{O}(v)$ yields

$$\frac{\Delta\nu}{\nu} = -\frac{4m}{r_0} v_z\tag{52}$$

or, in three-vector notation

$$\frac{\Delta\nu}{\nu} = -\frac{4m}{r_0^2} \underline{v} \cdot \underline{r}_0.\tag{53}$$

Like the formula for the bend angle, this result agrees with previous calculations of the frequency shift (Birkinshaw and Gull, 1983): to $\mathcal{O}(v)$ the frequency shift arises from transverse motion of the lens.

(f) The Skew Angle. We will call the angular deflection out of the yz -plane the skew angle. The special relativistic treatment of the scattering of a photon by a moving point mass represents an instantaneous interaction between the photon and the lens and so, necessarily, gives a vanishing skew angle. Our method accounts for

the interaction of the photon and the lens over the entire photon path and confirms that the skew angle vanishes by direct calculation to order mv/r_0 .

In the small angle approximation the skew angle is given by

$$\begin{aligned}
\theta_{\text{skew}} &= \frac{dx}{dy} \\
&= \frac{\dot{x}}{\dot{y}} \\
&= \frac{\dot{x}^{(1)}}{1 + \dot{y}^{(1)}} \\
&\approx \dot{x}^{(1)} = - \int_{-\infty}^{+\infty} \left(\Gamma_{00}^{(1)1} + 2\Gamma_{02}^{(1)1} + \Gamma_{22}^{(1)1} \right) d\lambda.
\end{aligned} \tag{54}$$

which becomes

$$\begin{aligned}
\theta_{\text{skew}} &= \frac{mv_x}{r_0} \int_{-\infty}^{+\infty} \frac{s ds}{[1 - 2v_z s + (1 - 2v_y) s^2]^{3/2}} \\
&\quad - \frac{3mv_x}{r_0} \int_{-\infty}^{+\infty} \frac{s^3 ds}{[1 - 2v_z s + (1 - 2v_y) s^2]^{5/2}}.
\end{aligned} \tag{55}$$

which vanishes to first order in v .

IV. Summary

We have described a method for constructing null geodesics in arbitrary metric perturbed spacetimes using null geodesics of the background spacetime. The method fully generalizes the usual Sachs-Wolfe technique for calculating temperature fluctuations of the CMBR in metric perturbed spacetimes. Because our method constructs both the spatial and timelike components of the perturbed geodesic it is able to address questions of interest in gravitational lens theory, such as the bend angles of the true path relative to the unperturbed path. We have provided an explicit illustration with the calculation of the behavior of a photon passing a moving lens. A forthcoming paper will show how our method can be used to calculate other quantities of gravitational lens theory, such as the amplification undergone by a bundle of light rays passing a given perturbation.

Appendix A: Rewriting the Jacobi Operator

We prove the equivalence of equations (15) and (18). This short calculation has an interesting history. It is implicit in Weinberg (1972), section 6.10. We produced

the following proof in the course of trying to gain a physical understanding of equation (15). After we had done so, a comment in Burke (1985) drew our attention to a paper by Faulkner and Flannery (1978) where the same calculation appears in a different context. It appears that this calculation is either obvious or not depending on who is presenting it. It was not obvious to us, and we feel its importance to the physical understanding of equation (15) warrants its inclusion here.

Let $D/d\lambda$ denote covariant differentiation along the curve $x^{(0)}$ with the background connection. Then for an arbitrary vector v

$$\frac{D}{d\lambda}v^\mu = \frac{dv^\mu}{d\lambda} + \Gamma_{\alpha\beta}^{(0)\mu}k^{(0)\alpha}v^\beta \quad (A1)$$

and

$$\begin{aligned} \frac{D^2}{d\lambda^2}v^\mu &= \frac{d^2v^\mu}{d\lambda^2} + \Gamma_{\alpha\beta,\gamma}^{(0)\mu}k^{(0)\alpha}k^{(0)\gamma}v^\beta \\ &\quad + \Gamma_{\alpha\beta}^{(0)\mu}\frac{dk^{(0)\alpha}}{d\lambda}v^\beta + 2\Gamma_{\alpha\beta}^{(0)\mu}k^{(0)\alpha}\frac{dv^\beta}{d\lambda} \\ &\quad + \Gamma_{\alpha\beta}^{(0)\mu}k^{(0)\alpha}\Gamma_{\sigma\rho}^{(0)\beta}k^{(0)\sigma}v^\rho. \end{aligned} \quad (A2)$$

Using the geodesic equation for $k^{(0)}$ this becomes

$$\begin{aligned} \frac{D^2}{d\lambda^2}v^\mu &= \frac{d^2v^\mu}{d\lambda^2} + 2\Gamma_{\alpha\beta}^{(0)\mu}k^{(0)\alpha}\frac{dv^\beta}{d\lambda} \\ &\quad + \Gamma_{\alpha\beta,\sigma}^{(0)\mu}k^{(0)\alpha}k^{(0)\sigma}v^\beta + \Gamma_{\alpha\beta}^{(0)\mu}\Gamma_{\sigma\rho}^{(0)\alpha}k^{(0)\beta}k^{(0)\sigma}v^\rho \\ &\quad - \Gamma_{\alpha\beta}^{(0)\mu}\Gamma_{\sigma\rho}^{(0)\alpha}k^{(0)\sigma}k^{(0)\rho}v^\beta \end{aligned} \quad (A3)$$

The Riemann tensor of the background is given by

$$R^{(0)\mu}{}_{\alpha\beta\sigma} = \left(\Gamma_{\alpha\sigma,\beta}^{(0)\mu} - \Gamma_{\alpha\beta,\sigma}^{(0)\mu} + \Gamma_{\beta\rho}^{(0)\mu}\Gamma_{\alpha\sigma}^{(0)\rho} - \Gamma_{\sigma\rho}^{(0)\mu}\Gamma_{\alpha\beta}^{(0)\rho} \right) \quad (A4)$$

so that (A3) can be written

$$\frac{D^2}{d\lambda^2}v^\mu - R^{(0)\mu}{}_{\alpha\beta\sigma}k^{(0)\alpha}k^{(0)\beta}v^\sigma = \frac{d^2v^\mu}{d\lambda^2} + 2\Gamma_{\alpha\beta}^{(0)\mu}k^{(0)\alpha}\frac{dv^\beta}{d\lambda} + \Gamma_{\alpha\beta,\sigma}^{(0)\mu}k^{(0)\alpha}k^{(0)\beta}v^\sigma \quad (A5)$$

Comparing this equation to equation (15) completes the proof.

Appendix B: The Path-Ordering

This appendix provides a brief introduction to the use of the path-ordering symbol. Recall that in section II we needed to solve a matrix system of the form

$$\dot{P} = MP \quad (B1)$$

subject to $P(a, a) = 1_d$. If the quantities P and M were functions, the solution of this system would be the usual exponential. An exponential with matrix argument will *not* work, however, for the reason that a matrix function and its derivative matrix do not, in general, commute.

Rewriting equation (B1) as the integral system

$$P(\lambda, a) = 1_d + \int_a^\lambda M(\tau)P(\tau, a) d\tau \quad (B2)$$

allows a possible solution, order by order in M , to be written down by iteration,

$$P(\lambda, a) = 1_d + \int_a^\lambda M(\tau) d\tau + \int_a^\lambda M(\tau) d\tau \int_a^\tau M(\tau') d\tau' + \dots \quad (B3)$$

A simple proof of convergence then shows that this expression is, in fact, a valid solution. The region of integration for the n -th order term (in M) is known as the n -simplex (the n -dimensional analog of the triangle). We can extend the region of integration to the n -cube in two steps. At each order we must symmetrize the n -fold product $M(\tau)M(\tau')\dots M(\tau'')$ completely in its arguments. This extends

the argument of the n -simplex integral to an argument defined over the entire n -cube and ensures that each of the $n!$ simplices in the n -cube contributes the same amount to the total integral over the n -cube, an amount equal to the value of the original integral over the single simplex. Our second step, then, is to divide each term by its overcounting factor, $n!$ at n -th order. These combinatorial factors yield the exponential. Thus the path-ordering symbol amounts to a notice to perform an integration written over a cube only over its “lowest” (in the sense of the discussion above) simplex.

Appendix C: Another Transcription of the Solution

In this appendix we present another form for the solution of the perturbed Jacobi equation, needing only a single path-ordering but containing terms whose geometrical meanings are more obscure than those in equation (32). We first write equation (15) as an eight-dimensional first order system,

$$\begin{pmatrix} \dot{x}^{(1)} \\ \dot{k}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 1_d \\ -B & -A \end{pmatrix} \begin{pmatrix} x^{(1)} \\ k^{(1)} \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} \quad (C1)$$

where 1_d is a 4×4 identity matrix and A , B and f are given by equation (16). From this point we will denote the 8×8 matrix which is the coefficient matrix of the associated homogeneous system by M .

We can now proceed to solve equation (C1) in exactly the same manner as we solved equation (30) in section II. We first obtain the associated transition matrix, U , which solves

$$\frac{d}{d\lambda} U(\lambda, a) = M(\lambda) U(\lambda, a) \quad (C2)$$

subject to $U(a, a) = 1_d$. We note that the equation for U is solved by a path-ordered exponential

$$U(\lambda, a) = \mathcal{P} \exp \left(\int_a^\lambda M(\tau) d\tau \right) \quad (C3)$$

Having obtained U it is easy to check by straightforward differentiation that

equation (C1) is solved by, with $y := (x^{(1)}, k^{(1)})$ and $s := (0, f)$,

$$y(\lambda) = U(\lambda, a)y(a) + \int_a^\lambda U(\lambda, \tau)s(\tau) d\tau \quad (C4)$$

This is the solution we desired to obtain.

The equivalence of this solution and that given in the main body of the paper is easily established by working in the co-ordinate system for which the Christoffel coefficients vanish along the unperturbed path. The different appearance of the two solutions is essentially equivalent to the two different ways of writing the Jacobi equation. The usual expression for the Jacobi operator, involving covariant derivatives and the Riemann tensor, has the advantage of being an obviously geometrical quantity. Writing the Jacobi operator as the LHS of equation (15) obscures its geometrical meaning but requires fewer computations of coefficients. The solutions mirror the strengths and disadvantages of the two starting formulations although they are completely equivalent.

Appendix D: The Greens Function of Minkowski Space

In this appendix we construct the Greens function appropriate to a perturbed Minkowski space and show its equivalence to direct integration of equation (15). In co-ordinates for which the connection terms of the Minkowski background vanish, the connector is the identity and the transition matrix can be calculated instantly from equation (33) with $R = 0$, yielding

$$U(\lambda, a) = \begin{pmatrix} 1_d & (\lambda - a)1_d \\ 0 & 1_d \end{pmatrix} \quad (D1)$$

Simple matrix multiplication verifies that this solution satisfies conditions (27) and (28) (rewritten with U in place of P). It is easy to see that the bottom four rows yield a solution for the photon wavevector equivalent to simple integration of the RHS of equation (15). To see that the photon paths calculated by the two techniques agree, start from equation (32). Assuming vanishing initial data for convenience and using the form of the propagator given in equation (D1) yields

$$x^{(1)}(\lambda) = \lambda \int_a^\lambda s(\tau) d\tau - \int_a^\lambda \tau s(\tau) d\tau \quad (D2)$$

On the other hand, a direct second integration of equation (15) gives

$$x^{(1)}(\lambda) = \int_a^\lambda k^{(1)}(\tau) d\tau = \int_a^\lambda d\tau \int_a^\tau s(\tau') d\tau' \quad (D3)$$

which can be seen to be equivalent to (D2) after integration by parts.

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References

- Anile, A. M., & Motta, S. 1976, *ApJ*, 207, 685
- Argüeso, F., & Martinez-Gonzalez, E. 1989, *MNRAS*, 238, 1431
- Argüeso, F., Martinez-Gonzalez, E., & Sanz, J. L. 1989, *ApJ*, 336, 69
- Birkinshaw, M., & Gull, S. F. 1983, *Nature*, 302, 315
- Bond, J. R., & Efstathiou, G. 1984, *ApJ*, 285, L45
- Bond, J. R., & Efstathiou, G. 1987, *MNRAS*, 226, 655
- Burke, W. L. 1985, *Applied Differential Geometry*, Cambridge University Press, Cambridge
- De Felice, F., & Clarke, C. J. S. 1990, *Relativity on Curved Manifolds*, Cambridge University Press, Cambridge
- Ehlers, J., Geren, P., & Sachs, R. K. 1968, *J. Math. Phys.*, 9, 1344
- Faulkner, J., & Flannery, B. P. 1978, *ApJ*, 220, 1125
- Futamase, T. 1988, *Phys. Rev. Lett.*, 61, 2175
- Futamase, T. 1989, *MNRAS*, 237, 187
- Futamase, T., & Sasaki, M. 1989, *Phys. Rev. D*, 40, 2502
- Grishchuk, L. P., & Zeldovich, Ya. B. 1978, *Soviet Astronomy-AJ*, 22, 125
- Humi, M., & Miller, W. 1988, *Second Course in Ordinary Differential Equations for Scientists and Engineers*, Springer-Verlag, New York
- Jacobs, M. W., Linder, E. V., & Wagoner, R. V. 1992, *Phys. Rev. D*, 45, R3292
- Jordan, P., Ehlers, J., & Sachs, R. K. 1961, *Akad. Wiss. Lit. Mainz Abh. Math.-Nat. Kl.*, 1, 3
- Linder, E. V. 1988a, *ApJ*, 326, 517
- Linder, E. V. 1988b, *ApJ*, 328, 77
- Linder, E. V. 1990, *MNRAS*, 243, 362
- Martinez-Gonzalez, E., Sanz, J. L., & Silk, J. 1990, *ApJ*, 355, L5
- Peebles, P. J. E., & Yu, J. T. 1970, *ApJ*, 162, 815
- Sachs, R. K. 1961, *Proc. Roy. Soc. A*, 264, 309
- Sachs, R. K., & Wolfe, A. M. 1967, *ApJ*, 147, 73
- Suto, Y., Gouda, N., & Sugiyama, N. 1990, *ApJS*, 74, 665
- Synge, J. L. 1960, *Relativity: The General Theory*, North-Holland, Amsterdam
- Watanabe, K., & Tomita, K. 1990, *ApJ*, 355, 1
- Weinberg, S. 1972, *Gravitation and Cosmology*, John Wiley and Sons, New York