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Phononic band gap design in honeycomb lattice with combinations of auxetic and conventional core

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Abstract

We present a novel design of a honeycomb lattice geometry that uses a seamless combination of conventional and auxetic cores, i.e. elements showing positive and negative Poisson's ratio. The design is aimed at tuning and improving the band structure of periodic cellular structures. The proposed cellular configurations show a significantly wide band gap at much lower frequencies compared to their pure counterparts, while still retaining their major dynamic features. Different topologies involving both auxetic inclusions in a conventional lattice and conversely hexagonal cellular inclusions in auxetic butterfly lattices are presented. For all these cases the impact of the varying degree of auxeticity on the band structure is evaluated. The proposed cellular designs may offer significant advantages in tuning high-frequency bandgap behaviour, which is relevant to phononics applications. The configurations shown in this paper may be made iso-volumetric and iso-weight to a given regular hexagonal topology, making possible to adapt the hybrid lattices to existing sandwich structures with fixed dimensions and weights. This work also features a comparative study of the wave speeds corresponding to different configurations vis-a vis those of a regular honeycomb to highlight the superior behavior of the combined hybrid lattice.

Keywords: Keyword 1, Periodic Structures 2, Auxetic Materials 3, Band Structure 4, Negative Poisson's ratio 5, Wave Propagation 6, Spectral Finite Element

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Introduction

Cellular structures with honeycomb cores of various length scales and shapes are of interest to engineers, primarily because of their high bending stiffness to weight ratio, energy absorption and relative ease of manufacturing due to their intrinsic periodicity. The dynamic characteristics of honeycomb cores have been significantly investigated within the structural dynamics research and development community within the last four decades. The spatial periodicity in these lattice structures causes band effects in wave propagation phenomena, and recently a significant body of research has been devoted to engineer the geometry of the lattices to create favorable band structure properties. Auxetic materials refer to a special class of solids exhibiting negative Poisson's ratio behaviour. Harnessing such property allows the engineering of different applications and opens up newer technological paradigms because of its potential. Improved stents in artificial medical prosthetics, auxetic fabric yarns in the textile industry, auxetic based morphed wings in aerospace applications, blast protection, body impact toughening and auxetic seat belts in automobile industry are only few examples of auxetics in engineering.

The works of Lakes [1] and Evans [2] have pioneered the field of research on auxetics. Natural and synthetic materials like long-fibre composites [4], micro-porous polymers [3], and types of open cell foams [2] are some of the examples of solids possessing negative Poisson's ratio behavior. A significant body of subsequent research has described the development of different topological configurations in periodic structures showing auxetic behavior. These topologies include for example the reentrant, chiral and their variants- multi-reentrant and anti-chiral structures [5, 6, 7, 8, 9, 10, 11]. During the last decade technological capabilities in fabricating structures with complicated geometries at smaller scales and different substrates have enabled the research on auxetics to push its boundaries further. The significant majority of the works dedicated to the development of cellular topologies with auxetic characteristics is focused on evaluating fully periodic lattice patterns. Little activity has however been focused on the effects of combining auxetic and conventional cores to achieve desired behavior, and this combination of hybrid cores has been developed so far only along one dimension of the lattice [29]. However, honeycomb lattices are intrinsically two-dimensional in terms of topology, and limiting the combination of different cores along one direction can only constitute a constraint in terms of dynamics characteristics. On the other hand, there is also a lack of a systematic methodology to design the geometry of these core combinations, probably because of the difficulty in establishing rigorously tessellating topologies that are seamlessly matching each other. These continuous tessellating cores also need to minimize the dynamic impedance at the junctions, thereby avoiding the localization of vibrational 'hot spots'. The continuity of the tessellation order is therefore an indispensable approach to assure a smoother transition between cores. This paper describes this new type of novel tessellating configuration and how it affects the dynamics behaviour of the core.

The main emphasis of this paper is to investigate some of the aspects of the wave propagation behaviour of these novel two-dimensional hybrid core geometries. Spatial periodicity in fully periodic structures made from single cell topologies is associated with the phenomenon of band effects in wave propagation. This phenomenon is characterized by the presence of pass-bands and stop-bands, which respectively allow or prohibit the propagation of waves in the corresponding frequency ranges. The analysis of band-gap behaviour in periodic structures was primarily driven by research activities in solid state physics through the discovery of the crystal structure and the modelling of electronic wave propagation in periodic potential nuclei. Some of the seminal contributions in these fields were made by Bloch [12, 13] and Brillouin [14]. In the field of engineering, Mead [15, 16, 17] has evaluated the behaviour of mono and multi-coupled periodic structures. Subsequent works in this field carried out during the last decade include those by Ruzzene [19], Phani [20], Gonella [21], Farzbod [22, 23] and Leamy [24]. Latter works have illustrated techniques directed towards engineering the geometry of the lattices to achieve a more favorable band structure [25, 26]. Some honeycomb geometries possess a complete band gap, and usual strategies of tuning the band gap involve the tailoring of the material properties and the adjustment of the slenderness ratio of the constituent frame elements. In this work we show that a combination of auxetic and conventional cores can be effectively used to design honeycomb lattice structures with improved band properties that show a complete band gap at a frequency level far lower than the one of the original pristine (i.e., with no hybridization) honeycomb geometries. It will also be shown that even when undergoing a pass-band behaviour these geometries possess very flat

bands, therefore indicating some significant improvements with respect to group and phase velocities when compared to their pristine counterparts. Part of the rational and concept design illustrated in this paper has been introduced in [37].

Design Method

This paragraph is to serve as a prelude to the design considerations described further in the analysis section of the paper. A periodic structure as depicted in Fig. 1(a) is characterised by a repetitive unit (as shown in Fig. 1(b)). A unit cell, which forms the entire structure through tessellation may be associated with the repetitive unit. In the case of a 2D lattice structure, the unit cell may be mathematically represented using a basis of two vectors. In the Fig. 1(c) the basis vectors are labelled with $\mathbf{a}_1$ and $\mathbf{a}_2$.

It is in general convenient to consider a dual or reciprocal space of the surface spanned by the aforesaid vectors, and to associate it with the wavenumber domain, This operation facilitates the analytical formulations and facilitates the graphical presentation of the wave propagation results. Therefore, if the basis vectors spanning the reciprocal space are $\mathbf{b}_1$ and $\mathbf{b}_2$, the former set of vectors can be related to the latter by the relation $\mathbf{a}_i \cdot \mathbf{b}_j = \delta_{ij}$. The dual space basis vectors are shown Fig. 1(d). Adopting the same mathematic representation used for the structural lattice space, the reciprocal space may also be considered to have a unit cell (formed by $\mathbf{b}_1$ and $\mathbf{b}_2$). It is also common practice in wave propagation analysis to define an area named Brillouin zone that encompasses all points closer to the cutting point of $\mathbf{b}_1$ and $\mathbf{b}_2$ than any other vertex (as shown in Fig. 1(d) by the hexagon with grey dashed line). The Brillouin zone is important in wave propagation analysis, because its depiction is sufficient to represent the wave propagation characteristics anywhere in the wavenumber space due to exact repetition. Due to the presence of any underlying symmetry in the lattice structure, a further reduction of the depiction zone may be made, still retaining the complete information about the wave propagation characteristics in that lattice. This minimal zone in the wavenumber space is called irreducible Brillouin zone [13] or IBZ.

Some of the most common topologies of auxetic lattices are shown in Fig. 2. These geometries include the reentrant, multi-reentrant, chiral and anti-chiral structures [5, 6, 7, 8, 9, 10, 11]. Different variations of the chiral and antichiral lattices like hexachiral, tetrachiral etc. have also been shown to demonstrate an auxetic behaviour.

Lattices with periodic alternate tessellations have also been developed, and example is the zero Poisson's ratio lattice developed by Grima et al. [27].

State-of-the-art hybrid lattices present in open literature involve the use of geometrical combinations of auxetic and conventional honeycombs, mostly distributed along only a single dimension. An example of such model is shown in Fig. 4, which is based on the core structure illustrated in the works of Ruzzene et al. [28] and Hou [29]. However, this topology is difficult to be developed into a 2D lattice with the hybrid cores distributed along two orthogonal directions.

The design presented in this paper consists in a combination of a finite number of periodic unit elements included in a wider periodic assembly of cells with different topologies. The different periodicity of cells at granular level and the periodicity of the wider lattice both influence the formation of the band gap. An example of these topologies is schematically shown in Fig. 5.

The combination can be made either of auxetic inclusions in a conventional lattice, or vice versa. We define in that sense a parameter called 'Degree of Auxeticity', which is a measure of concavity. This parameter could be used to identify different grades of auxetic inclusions. It is worth to notice that there could be various possible ways to quantify this parameter - for example, the ratio betweenf the area of the reentrant hexagon to the one of a regular hexagon having equal length of one side or the cell, the angle of concavity etc. The parameter definition adopted here for the reentrant core is related to the ratio of the distance between the reentrant opposite vertices and the length of one side of the regular hexagon used as the conventional core in the combined structure, Thus, as illustrated with the help of Fig. 6 below, the Degree of Auxeticity may be written as:

Figure 1: (a) A part of an arbitrary periodic structure (b) a choice of its repetitive unit is shown in (c). The correspondent unit cell with basis vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ (d) of the reciprocal lattice and the Brillouin zone are shown as grey dashed hexagon

Figure 2: Common auxetic lattices: (a) reentrant honeycomb (b) multi-reentrant honeycomb (c) chiral and (d) antichiral

Figure 3: lattice having zero Poisson's ratio

Figure 4: Hybrid lattice through combination of conventional and auxetic honeycomb in 1D

Figure 5: schematic diagram of periodicity at different level to cause band gap augmentation, both periodic 1 and periodic 2 components make the whole unit which is a new periodic unit cell

$$DoA = 1 - (X/L)$$

where X = distance between reentrant nodes

and L = Average length of the sides not connected to the reentrant nodes

(1)

The lattice turns auxetic for $1 - (X/L) > 0$, whereas the ratio may go up to 1 when the opposite vertices are in contact with each other. On the other hand, negative values correspond to conventional honeycombs, with a value of -1 for a regular hexagonal configuration. Therefore, the range $0 < (L - X)/L < 1$ implies an inclusion within the auxetic range.

Figure 6: schematic diagram illustrating degree of auxeticity with X/L having a real value varying between 1.0 and 0.0

All three lattices shown in Fig. 7 include two horizontally contiguous auxetic inclusions in a 3x3 unit cell, and are otherwise similar except having varying degree of auxeticity. Here, 3x3 signifies the 3x3 configuration of the Y shaped repetitive unit of a regular hexagonal honeycomb. The lattices have been positioned in an increasing degree of auxeticity of 0.0, 0.5 and 0.9 respectively following the auxeticity metrics previously defined.

Figure 7: three 3x3 unit cell with two horizontally contiguous auxetic inclusion of DoA 0, 0.5 and 0.9 respectively

It is evident from Figs. 8(a) and 8(b) that the shape of the unit cell of the regular hexagonal lattice is different from the one of the re-entrant cell. It is however possible to develop cells following isovolumic criteria, i.e., the geometric aspect ratio of the unit cell may be suitably altered to match that of a regular hexagon (Figs. 8(c) and 8(d)). The major advantage in using an isovolumic honeycomb is that it may readily replace a conventional hexagonal honeycomb. Fig. 9(a) shows a 3x3 configuration of a purely auxetic lattice unit cell and Fig. 9(b) shows the same with two horizontally contiguous conventional core inclusion having DoA -0.7. Figs. 10(a) and 10(b) show the above lattices, having been made isovolumic to a regular hexagonal honeycomb.

As the shape of the lattice unit cell changes the form of the reciprocal cell is also modified, thereby rendering the comparison of the wave propagation characteristics point-to-point significantly more cumbersome. Therefore, by retaining the shape of unit cell one can conserve the geometry of the reciprocal cell

Figure 8: Depiction of isovolumicity. (a) unit cell of a conventional hexagonal core (b) that of a reentrant core (c) unit cell of reentrant lattice made isovolumic to that of a regular hexagonal core (d) two lattices, the original regular hexagon and the isovolumic reentrant, shown to have identical unit cell by superimposing them in the same unit cell

Figure 9: A 3x3 unit cell configuration of purely auxetic unit cell (-20 degree) (b) a 3x3 auxetic unit cell with two horizontally contiguous conventional core inclusion of DoA -0.7

Figure 10: Isovolumic version Fig. 9. (a) of Fig. 9(a) and (b) of Fig. 9(b)

and hence of the Brillouin zone. It is thus possible to perform an easier comparison of the wave propagation behavior between lattices.

Analytical Methods

The computation of the band structure of the proposed periodic configurations and their pure counterparts is performed using a spectral finite element based methodology formulated by Mukherjee and Gopalakrishnan [30]. An outline of the relevant portion of the computational methodology is provided below.

Spectral Finite Element

A 2D honeycomb lattice may be considered to be constituted by frame elements having three degrees of freedom at each node (longitudinal, transverse displacements and in-place rotation). The frame element is modelled as a combination of an elementary rod and a Timoshenko beam. Here, a spectrally formulated finite element representing each frame as a 1D waveguide is used. Frequency dependent exponential solutions are taken as the shape functions of the finite element, which correspond to the exact solutions of the waveguide representation. As a result, the spectral finite element allows to obtain accurate solutions with only one element per frame, eliminating the necessity of using dense meshes of Finite Elements with Hermite or Lagrangian interpolation functions, which are necessary to perform simulations at high frequencies. These spectral finite element can be used to simulate the dynamic behaviour of frames following classical assembling techniques used in finite element of dynamic frame analysis, that result in global system matrices with frequency-dependent coefficients. Consequently, the spectral finite element method is a very efficient tool to perform wave propagation analysis, in particular at high frequency bandwidths. Because the spectral finite element methodology is formulated in the Fourier domain of wavenumbers and frequencies, the displacement degrees of freedom (dofs) and the corresponding forces are the wavenumber-dependent spectral components. The spectral finite element formulation for the frame used in this work is described below.

Spectral element based dynamic stiffness of Elementary Rod

An elementary rod with longitudinal displacement as the only dof at a node is governed by the following differential equation of motion:

$$\frac{\partial}{\partial x} \left[EA \frac{\partial u}{\partial x} \right] = \rho A \ddot{u} \quad (2)$$

After performing a discrete Fourier transformation of the variable $\hat{u}(x, t)$ one obtains [31]:

$$u(x, t) = \sum_n \hat{u}(x, \omega_n) e^{i\omega_n t} \quad (3)$$

Substitution of Eqn (3) into Eqn (2) leads to

$$\sum_n \left[EA \frac{\partial^2 \hat{u}}{\partial x^2} + \rho A \omega_n^2 \hat{u} \right] e^{i\omega_n t} = 0 \quad (4)$$

The function $e^{i\omega_n t}$ does not depend upon n , and therefore the term inside the square bracket in the equation is zero for all n . This condition gives the complete solution as:

$$\hat{u}(x, \omega_n) = M e^{-ik_1 x} + N e^{ik_1 x} \quad (5)$$

Where:

$$k_1 = \omega_n \sqrt{\frac{\rho A}{EA}} \quad (6)$$

Here, M, N are frequency dependent amplitudes. It is possible to express M and N in terms of the nodal displacement $\hat{u}(0)$ $\hat{u}(L)$. The longitudinal displacement at any arbitrary point may therefore be obtained as:

$$\hat{u}(x) = \hat{g}_1(x) \hat{u}(0) + \hat{g}_2(x) \hat{u}(L) \quad (7)$$

Where:

$$\begin{aligned}\hat{g}_1(x) &= [e^{-ik_1x} - e^{-ik_1(2L-x)}] / [1 - e^{-i2k_1L}] \\ \hat{g}_2(x) &= [-e^{-ik_1(L+x)} - e^{-ik_1(L-x)}] / [1 - e^{-i2k_1L}]\end{aligned}\quad (8)$$

The internal forces can be expressed as:

$$\hat{F}(x) = EA \frac{\partial \hat{u}}{\partial x} = EA[\hat{g}'_1(x)\hat{u}(0) + \hat{g}'_2(x)\hat{u}(L)] \quad (9)$$

where $\hat{g}'_1$ and $\hat{g}'_2$ indicate spatial derivative of $\hat{g}_1$ and $\hat{g}_2$ respectively.

Upon substitution of the nodal values with the spatial variable one may obtain the dynamic stiffness matrix that describes the relationship between the nodal forces $\hat{F}(0)$, $\hat{F}(L)$ and nodal displacement $\hat{u}(0)$, $\hat{u}(L)$ as:

$$\begin{Bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{Bmatrix} = \frac{EA}{L} \frac{ik_1L}{(1 - e^{2ik_1L})} \begin{Bmatrix} 1 - e^{2ik_1L} & -2e^{2ik_1L} \\ -2e^{2ik_1L} & 1 - e^{2ik_1L} \end{Bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix} \quad (10)$$

0.0.1. Spectral Element based dynamic stiffness of Timoshenko Beam

A 2D Timoshenko Beam model involves two dependent variables v and ϕ , the transverse displacement and the rotation respectively. One may obtain the governing equation of motion for the transverse vibration as:

$$\begin{aligned}GAK_1 \frac{\partial}{\partial x} \left[\frac{\partial v}{\partial x} - \phi \right] &= \rho A \ddot{v} \\ EI \frac{\partial^2 \phi}{\partial x^2} + GAK_1 \left[\frac{\partial v}{\partial x} - \phi \right] &= \rho IK_2 \ddot{\phi}\end{aligned}\quad (11)$$

The generalised forces associated have the form:

$$\begin{aligned}V &= GAK_1 \left[\frac{\partial v}{\partial x} - \phi \right], \\ M &= EI \frac{\partial \phi}{\partial x}\end{aligned}\quad (12)$$

A harmonic solution in the form $v = v_0 e^{-i(kx - \omega t)}$ and $\phi = \phi_0 e^{-i(kx - \omega t)}$ would lead to: [31]

$$\begin{aligned}GAK_1 k^2 v_0 e^{-i(kx - \omega t)} - ikGAK_1 \phi_0 e^{-i(kx - \omega t)} &= \rho A \omega^2 v_0 e^{-i(kx - \omega t)} \\ ikGAK_1 v_0 e^{-i(kx - \omega t)} + (EI k^2 + GAK_1) \phi_0 e^{-i(kx - \omega t)} &= \rho IK_2 \omega^2 \phi_0 e^{-i(kx - \omega t)}\end{aligned}\quad (13)$$

The exponential term in Eqn. (13), being nonzero in general may be cancelled. Therefore, it is possible to express the resulting set of equations in the matrix form as:

$$\begin{bmatrix} GAK_1 k^2 - \rho A \omega^2 & -ikGAK_1 \\ ikGAK_1 & EI k^2 + GAK_1 - \rho IK_2 \omega^2 \end{bmatrix} \begin{Bmatrix} v_0 \\ \phi_0 \end{Bmatrix} = 0 \quad (14)$$

A nontrivial solution leads to the characteristic equation:

$$\begin{aligned}[GAK_1 EI] k^4 - [GAK_1 \rho IK_2 \omega^2 + EI \rho A \omega^2] k^2 \\ + [\rho IK_2 \omega^2 - GAK_1] \rho A \omega^2 = 0\end{aligned}\quad (15)$$

Eqn. (15) is a standard quadratic equation in k^2 . Its solution gives the four wavenumbers as $\pm k_1$ and $\pm k_2$ where k_1 and k_2 may be expressed as:

$$k_1 = \sqrt{\frac{-Y + \sqrt{Y^2 - 4XZ}}{2a}} \quad k_2 = \sqrt{\frac{-Y - \sqrt{Y^2 - 4XZ}}{2a}} \quad \text{where}$$

$$X = [GAK_1EI] \quad (16)$$

$$Y = -[GAK_1\rho IK_2\omega^2 + EI\rho A\omega^2]$$

$$\text{and}$$

$$Z = [\rho IK_2\omega^2 - GAK_1]\rho A\omega^2$$

The equation above represents the spectral relationship for the beam and is conducive to the complete solution versus the dependent variables:

$$\begin{aligned} \hat{v}(x) &= R_1 M e^{-ik_1 x} + R_2 N e^{-ik_2 x} + R_1 P e^{ik_1 x} + R_2 Q e^{ik_2 x} \\ \hat{\phi}(x) &= M e^{-ik_1 x} + N e^{-ik_2 x} + P e^{ik_1 x} + Q e^{ik_2 x} \end{aligned} \quad (17)$$

In the Eqn. (17) M, N, P and Q are the frequency dependent spectral amplitudes. $R_i, i = 1, 2$ are the amplitude ratios defined as:

$$R_i = \frac{ik_i GAK_1}{GAK_1 k_i^2 - \rho A\omega^2} \quad (18)$$

The nodal displacements at the two end nodes of the beam may thus be expressed as:

$$\begin{Bmatrix} \hat{v}_1 \\ \hat{\phi}_1 \\ \hat{v}_2 \\ \hat{\phi}_2 \end{Bmatrix} = \begin{Bmatrix} \hat{v}(0) \\ \hat{\phi}(0) \\ \hat{v}(L) \\ \hat{\phi}(L) \end{Bmatrix} = [T_1] \begin{Bmatrix} M \\ N \\ P \\ Q \end{Bmatrix} \quad (19)$$

With the help of Eqn (19), one may write the constants M, N, P, Q in terms of the nodal degrees of freedom, i.e. $\hat{v}(0), \hat{\phi}(0), \hat{v}(L), \hat{\phi}(L)$. The generalised forces at the nodal points, i.e. $V(0), M(0), V(L), M(L)$ may be obtained upon substituting Eqn (17) into Eqn (12), as:

$$\begin{Bmatrix} \hat{V}_1 \\ \hat{M}_1 \\ \hat{V}_2 \\ \hat{M}_2 \end{Bmatrix} = \begin{Bmatrix} -\hat{V}(0) \\ -\hat{M}(0) \\ \hat{V}(L) \\ \hat{M}(L) \end{Bmatrix} = [T_2] \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} \quad (20)$$

Using Eqn. (20), one may obtain the relationship between the generalised forces and the associated degrees of freedom:

$$\begin{Bmatrix} \hat{V}_1 \\ \hat{M}_1 \\ \hat{V}_2 \\ \hat{M}_2 \end{Bmatrix} = [T_2][T_1]^{-1} \begin{Bmatrix} \hat{v}_1 \\ \hat{\phi}_1 \\ \hat{v}_2 \\ \hat{\phi}_2 \end{Bmatrix} = [\hat{K}] \begin{Bmatrix} \hat{v}_1 \\ \hat{\phi}_1 \\ \hat{v}_2 \\ \hat{\phi}_2 \end{Bmatrix}, \quad [\hat{K}] = [T_2][T_1]^{-1} \quad (21)$$

The Eqn. (21) represent in matrix form a set of frequency dependent relations linking the nodal spectral (generalised) forces to nodal spectral (generalised) displacements. The 4x4 coefficient matrix $[\hat{K}]$ thus obtained is termed as the dynamic stiffness matrix.

The elements of $[\hat{K}]$ are given below [31, 35]

$$\begin{aligned}
\hat{k}_{11} &= (k_2^2 - k_1^2) (r_2 z_{22} - r_1 z_{21}) / \Delta \\
\hat{k}_{13} &= (k_1^2 - k_2^2) (r_1 z_{21} - r_2 z_{22}) / \Delta \\
\hat{k}_{12} &= [-ik_2 (r_1 z_{11} + r_2 z_{12}) + ik_1 (r_1 z_{11} - r_2 z_{12})] / \Delta \\
\hat{k}_{14} &= [-ik_1 (r_1 z_{11} - r_2 z_{12}) + ik_2 (r_1 z_{11} + r_2 z_{12})] / \Delta \\
\hat{k}_{22} &= (-ik_1 R_2 + iK_2 R_1) (r_1 z_{22} - r_2 z_{21}) / \Delta \\
\hat{k}_{24} &= (ik_1 R_2 - iK_2 R_1) (r_1 z_{21} - r_2 z_{22}) / \Delta \\
\hat{k}_{23} &= -\hat{k}_{14}, \quad \hat{k}_{33} = \hat{k}_{11}, \quad \hat{k}_{34} = -\hat{k}_{12}, \quad \hat{k}_{44} = \hat{k}_{22}
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
r_1 &= (R_1 - R_2) z_{11}, \quad r_2 = (R_1 + R_2) z_{12}, \quad \Delta = (r_1^2 - r_2^2) / EI \\
z_{11} &= 1 - e^{-ik_1 L} e^{-ik_2 L}, \quad z_{12} = e^{-ik_1 L} - e^{-ik_2 L} \\
z_{22} &= 1 + e^{-ik_1 L} e^{-ik_2 L}, \quad z_{12} = e^{-ik_1 L} + e^{-ik_2 L}
\end{aligned}$$

The dynamic stiffness matrix of the frame characterised by three nodal dof, i.e. the longitudinal displacement, transverse displacement and the rotation can be obtained by suitably placing the elements from the dynamic stiffness matrix of the beam and the rod.

Bloch Theorem

A periodic structure is composed of repetitive structural elements with a specific spatial periodicity. The Bloch theorem, which had its origin in the description of the wavefunction of an electron within the periodic potential of a nucleus has been a standard tool in the analysis of wave propagation in periodic structures. This theorem states that the eigenstates $\mathbf{d}_{\mathbf{k}}$ of a periodic Hamiltonian may be expressed as the product of a plane wave with the associated wavenumber $\mathbf{k}$ and a periodic function $u_{\mathbf{k}}$ that has the same periodicity of the underlying system. For the elastic vibration of the honeycomb-like periodic structures, the eigenstates refer to the deformation eigenstates:

$$\mathbf{d}_{\mathbf{k}}(\mathbf{r}') = e^{i(m \cdot \mu_1 + n \cdot \mu_2)} \mathbf{d}_{\mathbf{k}}(\mathbf{r}) \tag{23}$$

where

$$\mathbf{r}' = \mathbf{r} + m \cdot \mathbf{a}_1 + n \cdot \mathbf{a}_2 \tag{24}$$

Here, $\mathbf{a}_1$ and $\mathbf{a}_2$ are the basis vectors that define a spatial periodicity, and μ_1 and μ_2 are the components of the wavenumber in the dual space along the basis vectors $\mathbf{b}_1$ and $\mathbf{b}_2$. The latter basis vectors are related to the former basis- $\mathbf{a}_1$ and $\mathbf{a}_2$ through $\mathbf{a}_i \cdot \mathbf{b}_j = \delta_{ij}$. Therefore, in a structure spatially spreading to infinity the Bloch theorem gives, for any given wavenumber, the relationship between deformation components of points differering from each other in terms of positionr by any combination of the integer multiples of these basis vectors. It is thus sufficient to analyse a single cell from a wave propagation point of view because the Bloch theorem enables to perform a complete kinematic description of the whole system only by means of the repeatitive unit cell, by reducing the problem size.

Figure 11 shows the schematic diagram of a unit cell, where the nodal displacements corresponding to any wavenumber are represented as $\hat{\mathbf{q}}$. The two series suffixes indicate the position within the periodic lattice (the first one represent top (t), middle (m) or bottom (b), and second one presenting left (l), middle (m) or right (r) - see Fig. 11). The Bloch conditions relate the deformations in the following form:

$$\hat{\mathbf{q}}_{mr} = e^{i\mu_1} \hat{\mathbf{q}}_{ml}, \quad \hat{\mathbf{q}}_{tm} = e^{i\mu_2} \hat{\mathbf{q}}_{bm}, \quad \hat{\mathbf{q}}_{br} = e^{i\mu_1} \hat{\mathbf{q}}_{bl}, \quad \hat{\mathbf{q}}_{tr} = e^{i(\mu_1 + \mu_2)} \hat{\mathbf{q}}_{bl}, \quad \hat{\mathbf{q}}_{tl} = e^{i\mu_2} \hat{\mathbf{q}}_{bl} \tag{25}$$

The nodal components of the internal forces corresponding to any wavenumber $\mathbf{k}$ are also related through the Bloch theorem as:

Figure 11: Schematic diagram of a generic unit cell. Deformation vectors associated to it have been shown as beads. Solid beads have been taken as independent degrees of freedom and the hollow beads are the dependent ones

$$\hat{\mathbf{f}}_{mr} = e^{i\mu_1} \hat{\mathbf{f}}_{ml}, \quad \hat{\mathbf{f}}_{tm} = e^{i\mu_2} \hat{\mathbf{f}}_{bm}, \quad \hat{\mathbf{f}}_{br} = e^{i\mu_1} \hat{\mathbf{f}}_{bl}, \quad \hat{\mathbf{f}}_{tr} = e^{i(\mu_1+\mu_2)} \hat{\mathbf{f}}_{bl}, \quad \hat{\mathbf{f}}_{tl} = e^{i\mu_2} \hat{\mathbf{f}}_{bl} \quad (26)$$

The equilibrium at a node dictates:

$$\hat{\mathbf{f}}_{tr} + e^{i\mu_1} \hat{\mathbf{f}}_{tl} + e^{i\mu_2} \hat{\mathbf{f}}_{br} + e^{i(\mu_1+\mu_2)} \hat{\mathbf{f}}_{bl} = 0 \quad (27)$$

Eigenfrequency Analysis

From the formulation of the frequency domain spectral finite element method, the spectral displacements $\hat{\mathbf{u}}$ and spectral forces $\hat{\mathbf{f}}$ corresponding to all the d.o.f. associated with a unit cell are related through the dynamic stiffness matrix $\hat{\mathbf{K}}$ of the unit cell. Therefore:

$$\hat{\mathbf{K}}\hat{\mathbf{q}} = \hat{\mathbf{f}} \quad (28)$$

It follows, after enforcing the Bloch condition that:

$$\begin{aligned} \hat{\mathbf{q}} &= \mathbf{T}\hat{\mathbf{q}}_r \\ \text{where} \\ \mathbf{T} &= \begin{bmatrix} I & 0 & 0 & 0 \\ Ie^{i\mu_1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & Ie^{i\mu_2} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & Ie^{i\mu_1} & 0 \\ 0 & 0 & Ie^{i\mu_2} & 0 \\ 0 & 0 & Ie^{i(\mu_1+\mu_2)} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \text{ and } \hat{\mathbf{q}}_r = \begin{Bmatrix} \hat{\mathbf{q}}_{ml} \\ \hat{\mathbf{q}}_{bm} \\ \hat{\mathbf{q}}_{bl} \\ \hat{\mathbf{q}}_{mm} \end{Bmatrix} \end{aligned} \quad (29)$$

The result is:

$$\hat{\mathbf{K}}_r \hat{\mathbf{q}}_r = \hat{\mathbf{f}}_r \quad \text{where} \quad \hat{\mathbf{K}}_r = \mathbf{T}^H \hat{\mathbf{K}} \mathbf{T} \quad \text{and} \quad \hat{\mathbf{q}}_r = \mathbf{T}^H \hat{\mathbf{q}} \quad (30)$$

For the free vibration analysis one therefore obtains the dynamic stiffness matrix relation:

$$\hat{\mathbf{K}}_r \hat{\mathbf{q}}_r = 0 \quad (31)$$

Wittrick-Williams Method

The solution of the constitutive equation related to the unit cell dynamic stiffness matrix involves two wavenumbers and a frequency component. The formulation of the dynamic stiffness is an implicit function of the frequency that forbids the development of a direct solution to express the frequency in terms of the wavenumbers. Any iterative ad-hoc method used to solve the problem might miss certain frequencies, particularly for the case of the closely space ones. The Wittrick-Williams method effectively circumvents this difficulty and guarantees the identification of all roots below a particular set frequency and desired accuracy [32]. The Wittrick-Williams method states that:

$$J(\omega^*) = J_0(\omega^*) + S(K(\omega^*)) \quad (32)$$

In the Eqn. (32)

$J(\omega^*)$ is the count of the natural frequencies below ω^*

$J_0(\omega^*)$ is the count of natural frequencies of the constrained system below ω_*

$S(K(\omega^*))$ is the sign count measuring the number of the principal minors $K_1(\omega^*), K_2(\omega^*), K_3(\omega^*) \dots K_r(\omega^*)$ change sign

The determinant of the matrix having the first m rows and columns of the dynamic stiffness is the principle minor of order m . The sign count can be calculated by bringing the matrix to upper diagonal form and then calculating the number of the negative diagonal entries.

Computation of Band Structure

The eigenfrequency analysis of the assembled dynamic stiffness matrix obtained provides the wave propagation frequencies of a given lattice corresponding to a two-component wavenumber vector. The computed data for a set of wavenumbers lying within the IBZ is, as per convention [13, 20], presented along the IBZ boundary. This generally, for a lattice endowed with certain symmetry, correctly depicts the band gaps. In the present work, the band gap values obtained from calculation along IBZ boundaries have been cross-verified from a simulation involving the whole of the IBZ.

Figure 12: (a) A regular hexagonal lattice with an hexagonal Brillouin zone and a triangular IBZ; (b) a isovolumic reentrant honeycomb with hexagonal Brillouin zone and quadrilateral IBZ

While a regular hexagonal lattice due to all of its rotational and reflective symmetries possesses a triangular IBZ covering one twelfth of the hexagonal Brillouin zone, the combined lattice does not however possess this symmetry. The lattices described in the paper have two sets of rotational and reflective symmetry, thereby making the IBZ one fourth of the area of the Brillouin zone and of a quadrilateral shape. An hexagonal lattice, its Brillouin with a triangular IBZ, and an isovolumic auxetic lattice along with its Brillouin zone with a quadrilateral IBZ are shown in Fig. 12. Despite this difference, the Brillouin zone has remained the same, and one-to-one comparisons between lattices are still possible.

Computation of Isofrequency Contours and Phase Speed

Another tool used to represent the behaviour of the wave propagation in 2D lattices is the use of isofrequency contours [20, 21]. Isofrequency curves for a particular mode are the contours of the respective spectrum relation surface corresponding to different values of frequencies, drawn in the wavenumber space. Thus, an isofrequency curve corresponding to a particular frequency indicates all the admissible wavenumber vectors associated with that frequency. It suffices to depict isofrequency curves only within the Brillouin zone. The isofrequency contours describe the nature of the wave propagation based on different directions. Hence, any anisotropy existing in the lattice becomes clear.

Phase speed is the speed at which individual frequencies travel, and therefore it also depicts characteristics of wave propagation, in particular the directionality. It can be represented as:

$$C_p = \frac{\omega}{|\mathbf{k}|} \quad (33)$$

where $\mathbf{k} = \mu_1 \mathbf{k}_1 + \mu_2 \mathbf{k}_2 = \xi_1 \mathbf{i}_1 + \xi_2 \mathbf{i}_2$

Where $\mathbf{i}_1$ and $\mathbf{i}_2$ are the Cartesian unit vectors and (μ_1, μ_2, ω) is any solution tuple obtained upon solving equation Eqn (31). Phase speed is depicted for a particular $\mathbf{k}$ by its radial span [21].

Results

Band Structure

Auxetic Inclusion in a Conventional regular hexagonal Honeycomb

The material properties and the geometric parameters used for the lattices consist in a Young's modulus of $210 * 10^9 \text{ GPa}$, Poisson's ratio of 0.25, density of 25000 kg/m^3 , width of 0.0125 m and depth of 0.0125 m . The frame elements forming the regular hexagonal lattice have a length of 0.125 m . The dimensions of the other lattice members forming the inclusions may be obtained from the details specifying the geometry of each single particular case.

Phononic band structures of grillages typical of periodic structures with different types of lattices (conventional and auxetic) have been widely described in open literature. A number of these structures present single or multiple complete band gap of different magnitudes. Here, the band structures for the regular hexagonal honeycomb and its auxetic counterpart (a reentrant hexagonal lattice) have their band gaps indicated for comparison against those of the combined lattices proposed in this work. It may be noted that for a given constituent material the shape of the band structure varies with the varying slenderness ratio of the frame elements. However, the band structure shape remains unchanged, and the whole set of frequencies in the band structure are scaled up/down according to the element length for a given fixed slenderness.

Fig. 13 shows the band structure of a regular hexagonal honeycomb with the presence of a band gap only within the range between 28000-32500 rad/sec. This finding is consistent with what observed in [24].

Fig. 14 shows the band structure of a pure auxetic honeycomb. At the origin the honeycomb has an internal cell angle of -29 degrees, and the dimensions of its cell ribs are subsequently modified to make the unit cell isovolumic to the one of a regular hexagon. This lattice shows a band gap within the range of 14800 rad/sec -17800 rad/sec. Although a significant band gap occurs at frequencies lower than the ones observed in the previous case, the band gap magnitude is also lower, and therefore this auxetic isovolumetric design does not appear to offer a superior dynamic behaviour compared to the conventional regular one.

The band structure of a 3x3 honeycomb cell with two contiguous horizontal auxetic inclusions (degree of auxeticity of 0.8) is shown in Fig. 15. This band structure shows a clean band gap between 3700 rad/sec and 6300 rad/sec. Compared to the previous cases in which the lattices were made by uniform distributions of unit cells, it is possible to notice a clear improvement in terms of band gap width at a lower frequency range. Because of its excellent filtering behaviour and possible reductions of modal densities at lower frequencies, this lattice formulation may be appealing for structures governed by low frequencies global modes.

Figure 13: (a) Band structure of a pure regular hexagonal honeycomb lattice (specification given in the text) (b) the unit cell used for calculation (top) and the first Brillouin zone having triangle OAB depicting the irreducible Brillouin zone(bottom)

Figure 14: (a) Band structure of a pure auxetic hexagonal honeycomb with inclined member angle -29 degree, and subsequently made isovolumic (b) unit cell used for calculation (top) and the first Brillouin zone having quadrilateral OABC depicting the irreducible Brillouin zone(bottom)

Figs. 16(a) , 16(c), and 16(e) show the band gap features of 3x3 , 4x4, 5x5 cells respectively, with two auxetic inclusions of varying degree of auxeticity along with a representative unit cell on the right. These figures clearly show a consistent trend of increasing the band gap for higher degrees of auxeticity. It is also apparent the presence of a veering of bandgaps (second and third for degrees of auxeticity between 0.1 and 0.2 in Figure 16(a), from first to fourth in Figure 16(c)). It may also be noted that the increase of volume fraction of the auxetic inclusion (at least up to two auxetic cores in nine cells) enlarges the band gap for any given value of Degree of Auxeticity.

To obtain a quantitative measure of the impact of the DoA on the wave propagation characteristics for each of these lattices, the magnitude of the band gap for a particular DoA may be presented as a percentage of the maximum band gap amongst all the DoA values for a particular configuration. For the 3x3 configuration with two auxetic inclusions the bandgap at DoA 0.0 is only about 6 percent of the maximum value, and it steadily rises with increasing DoA with the maximum value occurring as the DoA value approaches 1 (Fig. 17). For the 4x4, 5x5 configurations this trend remains the same, but in these cases the rise in band gap percentage with DoA is not monotonic as the presence of drops of values represent the case of closely occurring compound band gaps in which the band gap gets fragmented. Subsequent increases indicate the presence of bigger band gaps shifting from one zone to the other.

Conventional hexagonal inclusion in an auxetic honeycomb

The material properties used for the simulations related to these particular cases are the same reported before. The length of the vertical member of the auxetic lattice, prior to being made isovolumic, is 0.125 m and the inclined members have an internal cell angle of 70 degrees. A 3x3 unit of the same configuration with two inclusions of conventional lattice with DoA of -0.6 and subsequently made isovolumic is the first case considered here, and its band structure is shown in Fig. 18 along with the unit cell used and the respective Brillouin zone within the IBZ. It is possible to observe a clean band gap present in the frequency range of 3000 rad/sec-5200 rad/sec.

The band structure has then been computed for the baseline configuration described above with varying DoA. The results are shown in Fig. 19(a). The majority of the hybrid configurations proposed exhibit a clear band gap in the frequency range of 3000 rad/sec-4500 rad/sec. For DoAs ranging between -0.45 to -0.65 the band gap is raised to 3000 rad/sec-5200 rad/sec, the highest gap being the interval 2800 rad/sec-5300 rad/sec for a DoA of -0.55. Although this bandgap width is lower than the one observed for the case of a regular lattice with auxetic inclusion, it is still significant. While in the case of conventional honeycombs with auxetic inclusion the largest band gaps occurred for the highest magnitudes of the degree of auxeticity, in the current examples the largest band gaps are happening in the middle of the negative DoA zone.

Similar results are also been observed for the 4x4 auxetic lattice with two conventional inclusions with varying DoA (Fig. 19(c)). In this case the highest band gap values are between the 2800 rad/sec-4600 rad/sec for a DoA of 0.55.

Like the case of the auxetic inclusion, Fig. 20 shows the impact of varying the DoA on the magnitude of the band gap by plotting the band gap for a DoA as the percentage of the maximum band gap for across all degrees of auxeticity for a particular configuration. It may be readily seen that middle range of the DoA interval appears to provide the best bandgap performance for the lattices.

It may also be noted that the energy bands in the combined lattices are much flatter compared to their pristine counterparts. This phenomenon indicates a lower speed of wave propagation in the lattice attributed to different types of interfering waves scattered by the two different sets of lattices that impede the propagation.

Isofrequency Curves

Hereafter, isofrequency curves are drawn for the regular hexagonal and both type of combined lattice.

The isofrequency curves for the first two modes of a 3x3 regular hexagonal honeycomb are shown in Fig. 21 and compared with those belonging to the combined lattice. The isofrequency curves related to both the first and second band show an intrinsic sixfold symmetry. However, the second mode shows a strong anisotropy in terms of directionality, particularly at higher frequencies. The isofrequency curves for the auxetic inclusion in a conventional honeycomb for the first two modes are shown in Fig. 22, and exhibit a strong directional dependency and orthotropic nature of the wave propagation. The isofrequency curves for the inverted configuration, i.e. a conventional inclusion within an auxetic lattice (subsequently made isovolumic), is shown in Fig. 23 for the first two modes. Also in this case, the anisotropy of the wave propagation is evident like in the previous illustrated case.

Phase Speed

The phase speeds have been calculated both for the regular hexagonal honeycomb and combined structures lattices (Figs. 24, 25 and 26). Unlike the case of the hexagonal honeycomb, the variation of the speed directivity is much more prominent for the combined cases. As an example, the maximum speed to minimum speed ratio varies from 1.02 to 1.11 for low and high frequency ranges of the first band respectively, while the same values for the combined lattice- conventional with auxetic inclusion is 1.72 to 1.87, and for the reversed combination is 1.90 to 2.06. This behavior shows very a strong chirality, with the magnitude of the phase speed in the second mode being significantly lower for the combined lattices compared to the regular one. The maximum phase speed in both the combined cases is less than one third of the one of the regular hexagonal lattice. This peculiar behaviour can also be observed in Figs. 15 and 18 as the stooped down second mode. Another feature that may be seen in Figs. 25 and 26 is the multi-valued phase velocity.

Phononic crystals tend to show multivalued phase speeds, and these would indicate negative group velocity [33]. Therefore, the isofrequency plots and the phase speed plots of the combined lattices shows very exclusive features because of the presence of the inclusions arising out of skewed Bragg's scattering.

1. CONCLUSIONS

A novel design method of hybrid honeycomb lattices has been proposed by implementing a seamless combination of auxetic and conventional cores for band gap design. The band gap features of the novel lattices have been computationally analysed by using a spectral finite element approach combined with Bloch wave boundary conditions and the use of a Wittrick-Williams method for the solution of the eigenstates. The results from these simulations clearly demonstrate that the concept of using cell inclusions inside lattices with different topologies leads to tailored and enlarged band gap behaviour in specific frequency bandwidths compared to the pristine periodic lattices. Moreover, these designs could be adopted to engineer cellular structures with specific modal densities and eigenstates with isovolumic configurations compared to the regular hexagonal lattice, which is considered as the benchmark standard for wave propagation applications in periodic structures.

Figure 15: (a) Band structure of the combined hexagonal honeycomb using a 3x3 regular hexagonal cell with 2 auxetic inclusions of degree of auxeticity 0.8. (b) the unit cell of the combined hexagonal honeycomb used for the calculation (top) and the first Brillouin zone with the irreducible Brillouin zone marked with green line (bottom)

(a)

(b)

(c)

(d)

(e)

(f)

Figure 16: Band gaps of a combined lattice consisting (a)3x3 (c)4x4 (e) 5x5 hexagonal honeycomb with two consecutive, horizontal auxetic inclusion having varying degree of auxeticity. Representative unit cell shown on the right in (b) (d) and (f) respectively

Figure 17: Largest band gap for a particular DoA as a percentage of the largest band gap across the whole set DoA, for three different configurations of lattice with all of them having two auxetic units in a unit cell of varying number of regular hexagons. Frequency range under consideration is upto 10000 rad/sec

Figure 18: (a) Band structure of the combined isovolumic hexagonal honeycomb using 3x3 reentrant cell having pre-isovolumic angle 70 degree, with two conventional inclusion of degree of auxeticity -0.6 (b) the unit cell of the combined hexagonal honeycomb used for calculations (top) and the first Brillouin zone with the irreducible Brillouin zone marked with green line (bottom)

(a)

(b)

(c)

(d)

Figure 19: Band gaps of a combined lattice consisting a (a)3x3, (c) 4x4 reentrant hexagonal honeycomb having an angle of -20 degree with two consecutive, horizontal conventional inclusion having varying degree of auxeticity- subsequently made isovolumic to a regular hexagonal lattice. Representative unit cell shown on the right (b) and (d)

Figure 20: Largest band gap for a particular DoA as a percentage of the largest band gap across the whole set DoA, for three different configurations of lattice with all of them having two conventional inclusion in a isovolumic auxetic unit cell of various configuration . Frequency range under consideration is upto 10000 rad/sec

Figure 21: Isofrequency curves for the 3x3 hexagonal lattice for the first and the second mode respectively.

Figure 22: Isofrequency curves for the 3x3 hexagonal lattice with two 0.95 DoA auxetic inclusion for the first and the second mode respectively.

Figure 23: Isofrequency curves for the 3x3 isovolumic -20 degree reentrant lattice with 2 conventional inclusion of DoA -0.6 for the first and the second mode respectively.

Figure 24: Phase speed of a 3x3 regular hexagonal lattice for the first two modes, on the left, first mode for the frequency range of 20 rad/sec to 1000 rad/sec and the right the second mode frequency range of 20 rad/sec to 1200 rad/sec

Figure 25: Phase speed of a 3x3 regular hexagonal lattice with two 0.95 DoA auxetic inclusion for the first two modes, on the left, first mode for the frequency range of 200 rad/sec to 1000 rad/sec and the right the second mode frequency range of 200 rad/sec to 1200 rad/sec

Figure 26: Phase speed of a 3x3 isovolumic -20 degree reentrant lattice with 2 conventional inclusion of DoA -0.6 for the first two modes, on the left, first mode for the frequency range of 200 rad/sec to 600 rad/sec and the right the second mode frequency range of 20 rad/sec to 800 rad/sec

2. ACKNOWLEDGEMENT

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