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EFFECTIVE RATNER THEOREM FOR $\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ AND GAPS IN $\sqrt{n}$ MODULO 1

TIM BROWNING AND ILYA VINOGRADOV

ABSTRACT. Let $G = \mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ and $\Gamma = \mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$. Building on recent work of Strömbergsson we prove a rate of equidistribution for the orbits of a certain 1-dimensional unipotent flow of $\Gamma \backslash G$, which projects to a closed horocycle in the unit tangent bundle to the modular surface. We use this to answer a question of Elkies and McMullen by making effective the convergence of the gap distribution of $\sqrt{n} \bmod 1$.

1. INTRODUCTION

Results of Ratner on measure rigidity and equidistribution of orbits [11, 12] play a fundamental role in the study of unipotent flows on homogeneous spaces. They have many applications beyond the world of dynamics, ranging from problems in number theory to mathematical physics. This paper is concerned with the problem of obtaining *effective* versions of results that build on Ratner's theorem and is inspired by recent work of Strömbergsson [15].

Let $G = \mathrm{ASL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ be the group of affine linear transformations of $\mathbb{R}^2$. We define the product on G by

$$(M, \mathbf{x})(M', \mathbf{x}') = (MM', \mathbf{x}M' + \mathbf{x}'),$$

and the right action is given by $\mathbf{x}(M, \mathbf{x}') = \mathbf{x}M + \mathbf{x}'$. We always think of $\mathbf{x} \in \mathbb{R}^2$ as a row vector. Put $\Gamma = \mathrm{ASL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$ and let $X = \Gamma \backslash G$ be the associated homogeneous space. The group G is unimodular and so the Haar measure μ on G projects to a right-invariant measure on X . The space X is non-compact, but it has finite volume with respect to the projection of μ . Following the usual abuse of notation, we denote the projected measure by μ and normalize it so that $\mu(X) = 1$.

Let

$$a(y) = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix},$$

and write $A^+ = \{a(y) : y > 0\}$. In what follows we will use the embedding $\mathrm{SL}(2, \mathbb{R}) \hookrightarrow G$, given by $M \mapsto (M, \mathbf{0})$, which thereby allows us to think of $\mathrm{SL}(2, \mathbb{R})$ as a subgroup of G . Strömbergsson [15] works with the unipotent flow on X generated by right multiplication by the subgroup

$$U_0 = \left\{ \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, (0, 0) \right) : x \in \mathbb{R} \right\}.$$

He considers orbits of a point $(\mathrm{Id}_2, (\xi_1, \xi_2))$ subject to a certain Diophantine condition. In [15, Thm. 1.2], effective rates of convergence are obtained for the equidistribution of such orbits under the flow $a(y)$ as $y \rightarrow 0$. The goal of the present paper is to extend the methods of Strömbergsson to handle the orbit generated by right multiplication by the subgroup $U = \{u(x) : x \in \mathbb{R}\}$, where

$$u(x) = \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \left(\frac{x}{2}, \frac{x^2}{4} \right) \right).$$

As noted by Strömbergsson [15, §1.3], *any* Ad-unipotent 1-parameter subgroup of G with non-trivial image in $\mathrm{SL}(2, \mathbb{R})$ is conjugate to either U_0 or U .

With this notation we will study the rate of equidistribution of the closed orbit $\Gamma \backslash \Gamma U$ under the action of $a(y)$, as $y \rightarrow 0$. Geometrically this orbit is a lift of a closed horocycle in $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathrm{SL}(2, \mathbb{R})$ to $\Gamma \backslash G$, and the behaviour of horocycles under the flow A^+ on $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathrm{SL}(2, \mathbb{R})$ is very well understood. The main obstruction to treating the problem of horocycle lifts with the usual techniques of ergodic theory (such as thickening) is the fact that U is *not* the entire unstable manifold of the flow $a(y)$, but only a codimension 1 submanifold. Elkies and McMullen [8] used Ratner's measure classification theorem [11] to prove that the horocycle lifts equidistribute, but their method is ineffective. In [8, §3.6] they ask whether explicit error estimates can be obtained. The following result answers this affirmatively.

Theorem 1.1. *There exists $C > 0$ such that for every $f \in C_b^8(X)$ and $y > 0$ we have*

$$\left| \frac{1}{2} \int_{-1}^1 f(u(x)a(y)) dx - \int_X f d\mu \right| < C \|f\|_{C_b^8} y^{\frac{1}{4}} \log^2(2 + y^{-1}).$$

Here $C_b^k(X)$ denotes the space of k times continuously differentiable functions on X whose left-invariant derivatives up to order k are bounded (see equation (2.7) for the exact definition of the norm).

Our next result shows that we can replace dx by a sufficiently smooth absolutely continuous measure. Let $\rho: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be a compactly supported function that has $1 + \varepsilon$ derivatives in L^1 . For simplicity we follow [15] and interpolate between the Sobolev norms $\|\rho\|_{W^{1,1}}$ and $\|\rho\|_{W^{2,1}}$, which give the L^1 norms of first and second derivatives, respectively. This interpolation allows us to treat the case of piecewise constant functions with an ε -loss in the rate.

Theorem 1.2. *Let $\eta \in (0, 1)$. There exists $K > 1$ and $C(\eta) > 0$ such that for every $f \in C_b^8(X)$ and $y > 0$ we have*

$$\left| \int_{\mathbb{R}} f(u(x)a(y)) \rho(x) dx - \int_X f d\mu \int_{\mathbb{R}} \rho(x) dx \right| < C(\eta) \|\rho\|_{W^{1,1}}^{1-\eta} \|\rho\|_{W^{2,1}}^{\eta} \|f\|_{C_b^8} y^{\frac{1}{4}} \log^{K-1}(2 + y^{-1}).$$

The constant K in this result is absolute and does not depend on η . The proof of Theorems 1.1 and 1.2 builds on the proof of [15, Thm. 1.2]. It relies on Fourier analysis and estimates for complete exponential sums which are essentially due to Weil. Let us remark that while we strive to obtain the best possible decay in y , we take little effort to optimize the norms of f and ρ that appear in the estimates. The exponent $\frac{1}{4}$ in the error term is optimal for our method, but we surmise it can be improved by exploiting additional cancellation in certain two dimensional exponential sums. The natural upper limit is $\frac{1}{2}$, which holds for horocycles on $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathrm{SL}(2, \mathbb{R})$ due to work of Sarnak [13].

We may apply Theorem 1.1 to study gaps between the fractional parts of $\sqrt{n}$. Consider the sequence $\sqrt{n} \bmod 1 \subset \mathbb{R}/\mathbb{Z} \cong S^1$. It is easy to see from Weyl's criterion that this sequence is uniformly distributed on the circle. This means that for every interval $J \subset S^1$, we have

$$\lim_{N \rightarrow \infty} \frac{\#\{\sqrt{n} \bmod 1 : 1 \leq n \leq N\} \cap J}{N} = |J|,$$

where $|\cdot|$ denotes length. The statistic we focus on is the *gap distribution*. For each $N \in \mathbb{N}$, we consider the set $\{\sqrt{n} \bmod 1\}_{1 \leq n \leq N}$ and we allow $0 \in \mathbb{R}/\mathbb{Z}$ to be included for each perfect square. This set of N points divides the circle into N intervals (a few of which could be of zero length) which we refer to as *gaps*. For $t \geq 0$, we define the

gap distribution $\lambda_N(t)$ to be the proportion of gaps whose length is less than t/N . This function satisfies $\lambda_N(0) = 0$ and $\lambda_N(\infty) = 1$, and it is left-continuous.

The behaviour of $\lambda_N(t)$, as $N \rightarrow \infty$, has been analyzed by Elkies and McMullen [8] and later also by Sinai [14]. It is shown in [8] that there exists a function $\lambda_\infty(t)$ such that $\lambda_N(t) \rightarrow \lambda_\infty(t)$ for each t . We have

$$\lambda_\infty(t) = \int_0^t F(\xi) d\xi,$$

where F is given in [8, Thm. 1.1]. It is defined by analytic functions on three intervals, but it is not analytic at the endpoints joining these intervals. Moreover, it is constant on the interval $[0, 1/2]$.

The key input in [8] comes from Ratner's theorem [11], which is used in [8, Thm. 2.2] to find the limiting distribution of $\lambda_N(t)$ and therefore cannot give a rate of convergence. Armed with Theorem 1.1, we will refine this approach to get the following result.

Corollary 1.3. *Let $\lambda_N(t), \lambda_\infty(t)$ be as above. Then for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$|\lambda_N(t) - \lambda_\infty(t)| < C_\varepsilon(1/t^2 + t)N^{-\frac{1}{68} + \varepsilon}$$

for any $N \geq 2$ and $t > 0$.

The sequence $\sqrt{n} \bmod 1$ has also been studied from the perspective of its *pair correlation function*. This is a useful statistic for measuring randomness in sequences and, in this setting, it has been shown to converge to that of a Poisson point process by El-Baz, Marklof, and the second author [7]. In the light of Theorem 1.1, although we will not carry out the details here, by developing effective versions of the results in [7] it would be possible to conclude that the pair correlation function converges *effectively*. By way of comparison, we remark that Strömbergsson [15, §1.3] indicates how one might make effective the convergence of the pair correlation function in the problem of *visible lattice points* (see [6]).

The plan of the paper is as follows. In Section 2, we embark on the proof of Theorem 1.1 by developing f into a Fourier series in the torus coordinate. Section 3 is dedicated to estimating certain complete exponential sums that are required in Section 4 to control the error terms. Corollary 1.3 is proved in Section 5 and, finally, the proof of Theorem 1.2 is sketched in Section 6.

Notation. Given functions $f, g : S \rightarrow \mathbb{R}$, with g positive, we will write $f \ll g$ if there exists a constant c such that $|f(s)| \leq cg(s)$ for all $s \in S$.

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2. FOURIER DECOMPOSITION

In this section we develop the tools necessary to prove Theorem 1.1 and decompose f into a Fourier series on the torus. We proceed exactly as in [15]. To begin with we note that

$$f((1, \boldsymbol{\xi})M) = f((1, \boldsymbol{\xi} + \mathbf{n})M)$$

for $\mathbf{n} \in \mathbb{Z}^2$. So for M fixed, f is a well defined function on $\mathbb{R}^2/\mathbb{Z}^2$ and we can expand it into a Fourier series as

$$f((1, \boldsymbol{\xi})M) = \sum_{\mathbf{m} \in \mathbb{Z}^2} \hat{f}(M, \mathbf{m}) e(\mathbf{m} \cdot \boldsymbol{\xi}), \quad (2.1)$$

where

$$\hat{f}(M, \mathbf{m}) = \int_{\mathbb{T}^2} f((1, \boldsymbol{\xi}')M) e(-\mathbf{m} \cdot \boldsymbol{\xi}') d\boldsymbol{\xi}'.$$

Note that

$$\hat{f}(TM, \mathbf{m}) = \hat{f}(M, \mathbf{m}(T^{-1})^t), \quad (2.2)$$

for $T \in \mathrm{SL}(2, \mathbb{Z})$. Set $\tilde{f}_n(M) = \hat{f}(M, (n, 0))$. These functions of $M \in \mathrm{SL}(2, \mathbb{R})$ are left-invariant under the group $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$ by (2.2).

Now it follows from (2.2) that

$$\begin{aligned} \tilde{f}_n \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} M \right) &= \hat{f} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} M, (n, 0) \right) = \hat{f} \left(M, (n, 0) \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \right) \\ &= \hat{f}(M, (nd, -nc)). \end{aligned}$$

Therefore we can rewrite (2.1) with $\boldsymbol{\xi} = (x/2, -x^2/4)$ as

$$f((1, (x/2, -x^2/4))M) = \tilde{f}_0(M) + \sum_{n \geq 1} \sum_{(c,d)=1} \tilde{f}_n \left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} M \right) e \left(n \left(\frac{dx}{2} + \frac{cx^2}{4} \right) \right), \quad (2.3)$$

where $\begin{pmatrix} * & * \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any matrix in $\mathrm{SL}(2, \mathbb{Z})$ with c and d in the second row as specified.

Integrating (2.3) over x , we obtain

$$\frac{1}{2} \int_{-1}^1 f(u(x)a(y)) dx = M(y) + E(y),$$

where

$$M(y) = \frac{1}{2} \int_{-1}^1 \tilde{f}_0 \left(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) dx \quad (2.4)$$

and

$$E(y) = \sum_{\substack{n \geq 1 \\ (c,d)=1}} \frac{1}{2} \int_{-1}^1 e \left(n \left(\frac{dx}{2} + \frac{cx^2}{4} \right) \right) \tilde{f}_n \left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) dx. \quad (2.5)$$

The main term in this expression is $M(y)$ and, as is well-known (cf. [13, 9, 4, 15]), we have

$$M(y) = \int_X f d\mu + O(\|f\|_{C_b^4} y^{1/2-\epsilon}).$$

This statement is nothing more than effective equidistribution of horocycles under the geodesic flow on $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathrm{SL}(2, \mathbb{R})$. We need not seek the best error term for this problem, since there will be larger contributions to the error term in Theorem 1.1.

It remains to estimate $E(y)$ as $y \rightarrow 0$, which we do in Section 4.

We end this section with a pair of technical results that will help us to estimate $E(y)$. First, however, we give a precise definition of $\|\cdot\|_{C_b^m}$ for functions on G and hence also on X . Following [15], we let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}^2$ be the Lie algebra of G and fix

$$\begin{aligned} X_1 &= \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{0} \right), & X_2 &= \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{0} \right), & X_3 &= \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{0} \right), \\ X_4 &= \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (1, 0) \right), & X_5 &= \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0, 1) \right) \end{aligned} \quad (2.6)$$

to be a basis of $\mathfrak{g}$. Every element of the universal enveloping algebra $U(\mathfrak{g})$ corresponds to a left-invariant differential operator on functions on X . We define

$$\|f\|_{C_b^m} = \sum_{\deg D \leq m} \|Df\|_{L^\infty}, \quad (2.7)$$

where the sum runs over monomials in $X_1, \dots, X_5$ of degree at most m .

The following result is [15, Lemma 4.2].

Lemma 2.1. *Let $m \geq 0$ and $n > 0$ be integers. Then*

$$\tilde{f}_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ll_m \frac{\|f\|_{C_b^m}}{n^m (c^2 + d^2)^{m/2}}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

Passing to Iwasawa coordinates in $\mathrm{SL}(2, \mathbb{R})$, we write

$$\tilde{f}_n(u, v, \theta) = \tilde{f}_n \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{v} & 0 \\ 0 & 1/\sqrt{v} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \right). \quad (2.8)$$

for $u \in \mathbb{R}$, $v > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{R}$. The following is [15, Lemma 4.4].

Lemma 2.2. *Let $m, k_1, k_2, k_3 \geq 0$ and $n > 0$ be integers, and let $k = k_1 + k_2 + k_3$. Then*

$$\partial_u^{k_1} \partial_v^{k_2} \partial_\theta^{k_3} \tilde{f}_n(u, v, \theta) \ll_{m,k} \|f\|_{C_b^{m+k}} n^{-m} v^{m/2 - k_1 - k_2}.$$

3. COMPLETE EXPONENTIAL SUMS

In this section we make a detailed examination of the exponential sum

$$T_q(A, B) = \sum_{\substack{n \bmod q \\ (n, q) = 1}} e_q(A n^2 + B \bar{n}), \quad (3.1)$$

for $A, B \in \mathbb{Z}$ and $q \in \mathbb{N}$. Here $e_q(\cdot) = e(\frac{\cdot}{q})$ and $\bar{n}$ is the multiplicative inverse of n modulo q . Our main tool is Weil's resolution of the Riemann hypothesis for function fields in one variable (see Bombieri [3]), together with some general results due to Cochrane and Zheng [5] about exponential sums involving rational functions and higher prime power moduli. The procedure we follow is very general and could easily be adapted to handle other exponential sums of similar type.

We begin by recording the easy multiplicativity property

$$T_{q_1 q_2}(A, B) = T_{q_1}(\bar{q}_2 A, \bar{q}_2 B) T_{q_2}(\bar{q}_1 A, \bar{q}_1 B), \quad (3.2)$$

whenever $q_1, q_2 \in \mathbb{N}$ are coprime and $\bar{q}_1, \bar{q}_2 \in \mathbb{Z}$ satisfy $q_1 \bar{q}_1 + q_2 \bar{q}_2 = 1$. This renders it sufficient to study $T_{p^m}(A, B)$ for a prime power p^m . We may write $T_{p^m}(A, B)$ in the form

$$\sum_{n \bmod p^m}^* e_{p^m} \left(\frac{f_1(n)}{f_2(n)} \right), \quad (3.3)$$

where $f_1(x) = Ax^3 + B$ and $f_2(x) = x$. The symbol $\sum^*$ means that n is only taken over values for which $p \nmid f_2(n)$, in which scenario $f_1(n)/f_2(n)$ means $f_1(n)\overline{f_2(n)}$. We proceed by establishing the following result, which deals with the odd prime powers.

Lemma 3.1. *Let $p > 2$ and $m \in \mathbb{N}$. Then we have*

$$|T_{p^m}(A, B)| \leq \begin{cases} 3p^{m/2} (p^m, A, B)^{1/2}, & \text{if } p > 3, \\ 3^{1+3m/4} (3^m, A, B)^{1/4}, & \text{if } p = 3. \end{cases}$$

Proof. When $m = 1$ the sum in which we are interested is a classical exponential sum over a finite field and we may use the Weil bound, in the form developed by Bombieri [3] for rational functions. This leads to the satisfactory estimate

$$|T_p(A, B)| \leq 2p^{1/2}(p, A, B)^{1/2}. \quad (3.4)$$

Our investigation of the case $m \geq 2$ is founded on work of Cochrane and Zheng [5, §3], with $f(x) = f_1(x)/f_2(x)$. Note that

$$f'(x) = \frac{2Ax^3 - B}{x^2}.$$

Following [5, Eq. (1.8)] and recalling that p is odd, we put

$$\begin{aligned} t = \text{ord}_p(f') &= \text{ord}_p(2Ax^3 - B) - \text{ord}_p(x^2) \\ &= v_p((A, B)). \end{aligned}$$

Here, if $\text{ord}_p(h)$ is the largest power of p dividing all of the coefficients of a polynomial $h \in \mathbb{Z}[x]$, then $\text{ord}_p(f_1/f_2) = \text{ord}_p(f_1) - \text{ord}_p(f_2)$. Next, we put

$$\mathcal{A} = \{\alpha \in \mathbb{F}_p^* : 2A'\alpha^3 \equiv B' \pmod{p}\},$$

where $A' = p^{-t}A$ and $B' = p^{-t}B$. In particular $(p, A', B') = 1$ and $\#\mathcal{A} \leq 3$. The elements of $\mathcal{A}$ are called the *critical points*. If $p \mid A'$ or $p \mid B'$ then $\mathcal{A}$ is empty since $(p, A', B') = 1$. We therefore suppose that $p \nmid A'B'$.

The strength of our estimate for $T_{p^m}(A, B)$ depends on the multiplicity ν_α of each $\alpha \in \mathcal{A}$. Suppose first that $p > 3$ and write $r(x) = 2A'x^3 - B'$. Any root of multiplicity exceeding 1 must also be a root of $r'(x) = 6A'x^2$. Hence any $\alpha \in \mathcal{A}$ satisfies $\nu_\alpha = 1$ if $p > 3$. When $p = 3$ we have $2A'\alpha^3 - B' = (2A'\alpha - B')^3$ in $\mathbb{F}_3$ and so $\mathcal{A}$ contains a single element α of multiplicity $\nu_\alpha = 3$.

Next, as in [5, §1], one writes

$$T_{p^m}(A, B) = \sum_{\alpha \in \mathbb{F}_p^*} S_\alpha,$$

with

$$S_\alpha = \sum_{\substack{n \pmod{p^m} \\ n \equiv \alpha \pmod{p}}}^* e_{p^m} \left(\frac{f_1(n)}{f_2(n)} \right).$$

We are now ready to establish Lemma 3.1. When $m \leq t$ we have $T_{p^m}(A, B) = \varphi(p^m)$, which is satisfactory. When $m = t + 1$ we have

$$T_{p^m}(A, B) = p^{m-1}T_p(A', B'),$$

which has absolute value at most $2p^{m-1/2} \leq 2p^{m/2+t/2}$, by (3.4). It remains to deal with the case $m \geq t + 2$. Then [5, Thm. 3.1(a)] implies that $S_\alpha = 0$ unless $\alpha \in \mathcal{A}$. If $\alpha \in \mathcal{A}$ then this same result yields

$$|S_\alpha| \leq \nu_\alpha p^{t/(\nu_\alpha+1)} p^{m(1-1/(\nu_\alpha+1))}.$$

We recall that $\#\mathcal{A} \leq 3$ and $\nu_\alpha = 1$ for each $\alpha \in \mathcal{A}$ if $p > 3$, while $\#\mathcal{A} = 1$ and $\nu_\alpha = 3$ if $p = 3$. Substituting this into our expression for $T_{p^m}(A, B)$, this therefore concludes the proof of the lemma. $\square$

We complement our analysis of $T_{p^m}(A, B)$ for odd p by studying the exponential sum

$$T_{2^m}(A, B; \delta) = \sum_{\substack{n \pmod{2^m} \\ 2 \nmid n}} e_{2^{m+\delta}}(An^2 + 2^\delta B\bar{n}), \quad (3.5)$$

for $A, B \in \mathbb{Z}$ and $\delta \in \{0, 1\}$. When $\delta = 0$ we have $T_{2^m}(A, B; 0) = T_{2^m}(A, B)$, in our earlier notation. Furthermore, on writing $x = u + 2^m v$ for $u \in (\mathbb{Z}/2^m \mathbb{Z})^*$ and $v \in \mathbb{Z}/2\mathbb{Z}$, it is easy to check that

$$\sum_{x \bmod 2^{m+1}}^* e_{2^{m+1}} \left(\frac{Ax^3 + 2B}{x} \right) = 2T_{2^m}(A, B; 1).$$

Hence we have

$$T_{2^m}(A, B; \delta) = \frac{1}{2^\delta} \sum_{x \bmod 2^{m+\delta}}^* e_{2^{m+\delta}} \left(\frac{Ax^3 + 2^\delta B}{x} \right),$$

for $\delta \in \{0, 1\}$, which brings our sum in line with the exponential sums considered by Cochrane and Zheng [5]. We proceed to establish the following result.

Lemma 3.2. *Let $\delta \in \{0, 1\}$ and $m \in \mathbb{N}$. Then we have*

$$|T_{2^m}(A, B; \delta)| \leq 6 \cdot 2^{3m/4} (2^m, A, B)^{1/4}.$$

Proof. Let us put $t = v_2((2A, 2^\delta B))$. Then $u \leq t \leq 1 + u$, with $u = v_2((A, B))$. Suppose first that $m \leq t + 2$. Then the trivial bound gives

$$|T_{2^m}(A, B; \delta)| \leq \varphi(2^m) = 2^{m-1} \leq 2^{(m+\min\{m, u\}+1)/2},$$

which is satisfactory for the lemma. We henceforth assume that $m \geq t + 3$.

We are interested in a complete exponential sum modulo $2^{m+\delta}$. Arguing as in the proof Lemma 3.1 we have

$$f'(x) = \frac{2Ax^3 - 2^\delta B}{x^2}$$

and $\mathrm{ord}_2(f') = t$. Next, we put

$$\mathcal{A} = \{\alpha \in \mathbb{F}_2^* : A'\alpha^3 \equiv B' \bmod 2\},$$

where $A' = 2^{1-t}A$ and $B' = 2^{\delta-t}B$. In particular, A', B' are integers which cannot both be even and $\mathcal{A}$ consists of at most 1 element and it has multiplicity at most 3. It therefore follows from [5, Thm. 3.1(b)] that $S_\alpha = 0$ unless $\alpha \in \mathcal{A}$, in which case $|S_\alpha| \leq 3 \cdot 2^{t/4+3(m+\delta)/4}$. But then

$$|T_{2^m}(A, B; \delta)| \leq 3 \cdot 2^{(1+u)/4+3(m+\delta)/4} = 6 \cdot 2^{3m/4+u/4}.$$

This too is satisfactory for the lemma and so completes its proof. $\square$

For any $q \in \mathbb{N}$ we will henceforth write $q = q_0 q_1$, where

$$q_1 = \prod_{\substack{p^j \parallel q \\ p > 3}} p^j. \tag{3.6}$$

That is, q_0 is not divisible by primes other than 2 and 3, while q_1 is coprime to 6. Using the multiplicativity property (3.2), we may combine Lemma 3.1 and Lemma 3.2 with $\delta = 0$ to arrive at the following result.

Lemma 3.3. *Let $q \in \mathbb{N}$ and let $A, B \in \mathbb{Z}$. Then we have*

$$|T_q(A, B)| \leq 18 \cdot 3^{\omega(q_1)} q_0^{3/4} q_1^{1/2} (q_0, A, B)^{1/4} (q_1, A, B)^{1/2},$$

where $\omega(q_1)$ is the number of distinct prime factors of q_1 .

4. ERROR TERMS

The purpose of this section is to estimate $E(y)$ in (2.5). We begin with the case $c = 0$. Then $d = \pm 1$ by coprimality, and [15, Eq. (25)] yields

$$\frac{1}{2} \int_{-1}^1 \tilde{f}_n \left(\pm \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) dx \ll \|f\|_{C_b^2} \frac{y}{n^2}.$$

After summing over n , the contribution from this term is clearly much smaller than that claimed in Theorem 1.1.

Next we consider the effect of shifting the interval of integration by 2 in (2.5). For this it will be convenient to note that

$$\begin{aligned} \tilde{f}_n \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & (x-2)/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) &= \tilde{f}_n \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) \\ &= \tilde{f}_n \left(\begin{pmatrix} a & b-2a \\ c & d-2c \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) \end{aligned}$$

and

$$\begin{aligned} e \left(n \left(\frac{d(x-2)}{2} + \frac{c(x-2)^2}{4} \right) \right) &= e \left(n \left(\frac{dx}{2} + \frac{cx^2}{4} - cx \right) \right) \\ &= e \left(n \left(\frac{(d-2c)x}{2} + \frac{cx^2}{4} \right) \right). \end{aligned}$$

Bearing these in mind it follows that for any $D \in \mathbb{Z}$ and $s \in \mathbb{R}$ we have

$$\begin{aligned} &\sum_{\substack{(c,d)=1 \\ d \in [D, D+2c]}} \frac{1}{2} \int_s^{s+2} e \left(n \left(\frac{dx}{2} + \frac{cx^2}{4} \right) \right) \tilde{f}_n \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) dx \\ &= \sum_{\substack{(c,d)=1 \\ d \in [D, D+2c]}} \frac{1}{2} \int_{s+2}^{s+4} e \left(n \left(\frac{(d-2c)x}{2} + \frac{cx^2}{4} \right) \right) \tilde{f}_n \left(\begin{pmatrix} a & b-2a \\ c & d-2c \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) dx \\ &= \sum_{\substack{(c,d)=1 \\ d \in [D-2c, D]}} \frac{1}{2} \int_{s+2}^{s+4} e \left(n \left(\frac{dx}{2} + \frac{cx^2}{4} \right) \right) \tilde{f}_n \left(\begin{pmatrix} a & b-2a \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) dx. \end{aligned}$$

But the values of a and b are immaterial and so the contribution to (2.5) from terms with $c \neq 0$ is

$$\sum_{\substack{n \geq 1 \\ c \geq 1}} \frac{1}{2} \sum_{\substack{(c,d)=1 \\ d \bmod 2c}} \int_{\mathbb{R}} e \left(n \left(\frac{dx}{2} + \frac{cx^2}{4} \right) \right) \tilde{f}_n \left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) dx.$$

Next, we change to Iwasawa coordinates as in (2.8) (cf. [15, Lemma 6.1]). This leads to the expression

$$\begin{aligned} &\int_{\mathbb{R}} e \left(n \left(\frac{dx}{2} + \frac{cx^2}{4} \right) \right) \tilde{f}_n \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) dx \\ &= \int_0^\pi \tilde{f}_n \left(\frac{a}{c} - \frac{\sin 2\theta}{2c^2 y}, \frac{\sin^2 \theta}{c^2 y}, \theta \right) e \left(-\frac{nd^2}{4c} + \frac{ncy^2 \operatorname{ctg}^2 \theta}{4} \right) \frac{y d\theta}{\sin^2 \theta}, \end{aligned} \tag{4.1}$$

for positive c . For negative c , the limits on the integral are $-\pi$ and 0. Since $ad - bc = 1$ and a and b are otherwise arbitrary, we write $a = \bar{d}$ for any integer such that $\bar{d}d \equiv 1$

mod c . Combining the integrals for positive and negative c we get the contribution

$$\sum_{\substack{n \geq 1 \\ c \geq 1}} \int_{-\pi}^{\pi} \sum_{\substack{d \bmod 2c \\ (c,d)=1}} \tilde{f}_n \left(\frac{\bar{d}}{c} - \frac{\sin 2\theta}{2c^2 y}, \frac{\sin^2 \theta}{c^2 y}, \theta \right) e \left(-\frac{nd^2}{4c} + \frac{ncy^2 \operatorname{ctg}^2 \theta}{4} \right) \frac{y d\theta}{\sin^2 \theta}. \quad (4.2)$$

Recall that $\tilde{f}_n$ is left-invariant under $(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix})$, which in Iwasawa coordinates translates into having period 1 in the first coordinate. Therefore we can expand $\tilde{f}_n$ as a Fourier series to get

$$\tilde{f}_n \left(\frac{\bar{d}}{c} - \frac{\sin 2\theta}{2c^2 y}, \frac{\sin^2 \theta}{c^2 y}, \theta \right) = \sum_{l \in \mathbb{Z}} b_l^{(n,c)}(\theta) e \left(\frac{l\bar{d}}{c} \right) e \left(\frac{-l \sin 2\theta}{2c^2 y} \right), \quad (4.3)$$

whence the expression in (4.2) is at most

$$\ll \sum_{n,c,l} \int_{-\pi}^{\pi} \left| \sum_{\substack{d \bmod 2c \\ (c,d)=1}} e \left(-\frac{nd^2}{4c} + \frac{l\bar{d}}{c} \right) \right| |b_l^{(n,c)}(\theta)| \frac{y d\theta}{\sin^2 \theta}. \quad (4.4)$$

We need bounds for the Fourier coefficients and the exponential sum in (4.4). Beginning with the former we have the following result.

Lemma 4.1. *We have*

$$b_l^{(n,c)}(\theta) \ll \begin{cases} \|f\|_{C_b^m} \min \left\{ 1, \left(\frac{|\sin \theta|}{nc\sqrt{y}} \right)^m \right\} & \text{for any } m \geq 0, \\ l^{-2} \|f\|_{C_b^{m+2}} n^{-4} \min \left\{ 1, \left(\frac{|\sin \theta|}{nc\sqrt{y}} \right)^{m-4} \right\} & \text{for any } m \geq 4. \end{cases} \quad (4.5)$$

Proof. The first inequality follows from Lemma 2.1 by taking the smaller of the estimate for general m and $m = 0$. To obtain the second inequality we observe that

$$b_l^{(n,c)}(\theta) = \int_0^1 \tilde{f}_n \left(u, \frac{\sin^2 \theta}{c^2 y}, \theta \right) e(-lu) du. \quad (4.6)$$

We then apply integration by parts twice, followed by two applications of Lemma 2.2, one with $k_1 = 2$, $k_2 = k_3 = 0$, $m = 4$, and the other with $k_1 = 2$, $k_2 = k_3 = 0$, and m general. Taking the smaller of the two outcomes yields the result. $\square$

Next we turn to the exponential sum, with the following outcome.

Lemma 4.2. *We have*

$$\left| \sum_{\substack{d \bmod 2c \\ (d,c)=1}} e \left(-\frac{nd^2}{4c} + \frac{l\bar{d}}{c} \right) \right| \ll 3^{\omega(c_1)} c_0^{3/4} c_1^{1/2} (c_0, n, l)^{1/4} (c_1, n, l)^{1/2},$$

where $c = c_0 c_1$ and c_1 is given by (3.6).

Proof. Let $S(l, n; c)$ denote the exponential sum in the statement of the lemma. We need to relate $S(l, n; c)$ to the complete exponential sums considered in Section 3.

The sum over d runs modulo $2c$ in $S(l, n; c)$. Let us write

$$S(l, n; c) = S_1 + S_2,$$

where S_1 is the contribution to the sum from even d and S_2 is the remaining contribution. Writing $d = 2d'$, we see that

$$S_1 = \sum_{\substack{d' \bmod c \\ (2d', c)=1}} e \left(-\frac{nd'^2}{c} + \frac{l\bar{2d'}}{c} \right) = \begin{cases} T_c(-n, \bar{2}l), & \text{if } 2 \nmid c, \\ 0, & \text{if } 2 \mid c. \end{cases}$$

in the notation of (3.1). The desired estimate for S_1 is now a direct consequence of Lemma 3.3. Next, we note that

$$S_2 = \sum_{\substack{d \bmod 2c \\ (d, 2c)=1}} e_{2c} \left(-\frac{nd^2}{2} + 2l\bar{d} \right),$$

where $\bar{d}$ is now the multiplicative inverse of d modulo $2c$. If n is even then $S_2 = T_{2c}(-n/2, 2l)$, which can again be estimated using Lemma 3.3. If, on the other hand, n is odd we write $c = 2^{m-1}c'$ for odd $c' \in \mathbb{N}$. Then S_2 factorises as the product of an exponential sum modulo 2^m and an exponential sum modulo c' . A satisfactory estimate for the latter follows from Lemma 3.3. Using (3.2), the former is equal to $T_{2^m}(-n\bar{c}', 4l\bar{c}'; 1)$, in the notation of (3.5), where $\bar{c}' \in \mathbb{Z}$ is chosen to satisfy $c'\bar{c}' \equiv 1 \pmod{2^{m+1}}$. In this case $(2^m, -n, 4l) = 1$ since n is odd. This can be estimated using Lemma 3.2, which ultimately leads to a satisfactory estimate for $|S_2|$. This concludes the proof of the lemma. $\square$

We learn from [15, Eq. (35)] that

$$\frac{1}{a(1+a)} \ll \int_{-\pi}^{\pi} \min \left\{ 1, \left(\frac{|\sin \theta|}{a} \right)^2 \right\} \frac{d\theta}{\sin^2 \theta} \ll \frac{1}{a(1+a)}, \quad (4.7)$$

for $a > 0$. We will apply this with $a = nc\sqrt{y}$.

Returning to (4.4), we recall the factorisation $c = c_0c_1$, where c_0 is not divisible by primes greater than 3, and where c_1 given by (3.6). It will be useful to note that

$$\sum_{c_0} c_0^{-\gamma} = \sum_{\alpha, \beta \geq 0} 2^{-\alpha\gamma} 3^{-\beta\gamma} = O_{\gamma}(1), \quad (4.8)$$

for any $\gamma > 0$. We first consider the case $l = 0$. Combining the first line of (4.5) with $m = 2$ and Lemma 4.2, we obtain the contribution

$$\begin{aligned} &\ll \|f\|_{C_b^2} \sum_{\substack{n \geq 1 \\ c \geq 1}} \frac{y}{nc\sqrt{y}(1+nc\sqrt{y})} 3^{\omega(c_1)} c_0^{3/4} c_1^{1/2} (c_0, n)^{1/4} (c_1, n)^{1/2} \\ &\ll \|f\|_{C_b^2} \sqrt{y} \sum_{n, c_0, c_1} \frac{3^{\omega(c_1)} (c_0, n)^{1/4} (c_1, n)^{1/2}}{nc_0^{1/4} c_1^{1/2} (1+nc_0c_1\sqrt{y})} \\ &\ll \|f\|_{C_b^2} \sqrt{y} \sum_{n, c_1} \frac{3^{\omega(c_1)} (c_1, n)^{1/2}}{n^{3/4} c_1^{1/2} (1+nc_1\sqrt{y})}, \end{aligned}$$

by (4.8). The resulting sum is only made larger by summing over all positive integers c and so we freely replace c_1 by c . Let us denote the right hand side by J . Writing $h = (c, n)$ and $c = hc'$ and $n = hn'$, we see that

$$J \ll \|f\|_{C_b^2} \sqrt{y} \sum_{h, n', c'} \frac{3^{\omega(hc')}}{h^{3/4} n'^{3/4} c'^{1/2} (1+h^2 n' c' \sqrt{y})}.$$

To proceed further we recall (see Tenenbaum [17, Ex. I.3.4], for example) that there is an absolute constant $C > 0$ such that

$$\sum_{n \leq x} 3^{\omega(n)} = Cx \log^2 x + O(x \log x),$$

for any $x \geq 2$. The bounds

$$\sum_{c > x} \frac{3^{\omega(c)}}{c^{3/2}} \ll \frac{\log^2(2+x)}{x^{1/2}} \quad \text{and} \quad \sum_{c \leq x} \frac{3^{\omega(c)}}{c^{1/2}} \ll x^{1/2} \log^2(2+x) \quad (4.9)$$

now follow from this using partial summation and are valid for any $x > 0$. We therefore obtain

$$\sum_{c' > (h^2 n' \sqrt{y})^{-1}} \frac{3^{\omega(hc')}}{h^{3/4} n'^{3/4} c'^{1/2} (1 + h^2 n' c' \sqrt{y})} \ll \frac{3^{\omega(h)} \log^2(2 + (h^2 n' \sqrt{y})^{-1})}{h^{7/4} n'^{5/4} y^{1/4}},$$

and similarly,

$$\sum_{c' \leq (h^2 n' \sqrt{y})^{-1}} \frac{3^{\omega(hc')}}{h^{3/4} n'^{3/4} c'^{1/2} (1 + h^2 n' c' \sqrt{y})} \ll \frac{3^{\omega(h)} \log^2(2 + (h^2 n' \sqrt{y})^{-1})}{h^{7/4} n'^{5/4} y^{1/4}}.$$

It therefore follows that

$$\begin{aligned} J &\ll \|f\|_{C_b^8} \sqrt{y} \sum_{h, n'} \frac{3^{\omega(h)} \log^2(2 + (h^2 n' \sqrt{y})^{-1})}{h^{7/4} n'^{5/4} y^{1/4}} \\ &\ll \|f\|_{C_b^2} y^{1/4} \log^2(2 + y^{-1}). \end{aligned}$$

In the case $l \neq 0$, we apply the second inequality from (4.5) with $m = 6$, together with (4.7). The contribution of these terms is therefore found to be

$$\ll \|f\|_{C_b^8} \sum_{n, c, l} 3^{\omega(c_1)} c_0^{3/4} c_1^{1/2} (c_0, n, l)^{1/4} (c_1, n, l)^{1/2} y l^{-2} n^{-4} \frac{1}{nc \sqrt{y} (1 + nc \sqrt{y})}.$$

Using (4.8) and taking $(c_0, n, l)^{1/4} \leq l^{1/4}$ and $(c_1, n, l)^{1/2} \leq l^{1/2}$, we see that this is

$$\begin{aligned} &\ll \|f\|_{C_b^8} \sum_{n, c_0, c_1, l} 3^{\omega(c_1)} c_0^{3/4} c_1^{1/2} \frac{y}{l^{9/4} n^4} \frac{1}{nc_0 c_1 \sqrt{y} (1 + nc_0 c_1 \sqrt{y})} \\ &\ll \|f\|_{C_b^8} \sum_{n, c} 3^{\omega(c)} \frac{y^{1/2}}{n^4 c^{1/2} (1 + nc \sqrt{y})} \\ &\ll \|f\|_{C_b^8} \left\{ \sum_n \sum_{c > 1/(n\sqrt{y})} 3^{\omega(c)} y^{1/2} \frac{1}{n^5 c^{3/2} \sqrt{y}} + \sum_n \sum_{c \leq 1/(n\sqrt{y})} 3^{\omega(c)} y^{1/2} \frac{1}{n^4 c^{1/2}} \right\}, \end{aligned}$$

as before. Now we apply formulas (4.9), which shows that the latter expression is at most

$$\begin{aligned} &\ll \|f\|_{C_b^8} \left\{ \sum_n \frac{1}{n^5} \frac{\log^2(2 + (n\sqrt{y})^{-1})}{(n^{1/2} y^{1/4})^{-1}} + \sum_n \frac{y^{1/2}}{n^4} \frac{\log^2(2 + (n\sqrt{y})^{-1})}{n^{1/2} y^{1/4}} \right\} \\ &\ll \|f\|_{C_b^8} y^{1/4} \log^2(2 + y^{-1}). \end{aligned}$$

This therefore concludes the proof of Theorem 1.1.

5. PROOF OF COROLLARY 1.3

We adopt the notation of [8] for the most part. Define

$$\sigma_N(t) = \int_0^t \xi \, d\lambda_N(\xi).$$

For $c_+ > 0 > c_-$ let $\Delta_{c_-, c_+} \subset \mathbb{R}^2$ be the open triangle bounded by the lines $w_1 = 1$, $w_2 = 2c_+ w_1$, $w_2 = 2c_- w_1$ in the (w_1, w_2) -plane; its area is clearly $c_+ - c_-$. Also write Δ_c for $\Delta_{0, c}$, $\emptyset$, or $\Delta_{c, 0}$ according as c is positive, zero, or negative. For a lattice translate $\Gamma g \in \Gamma \backslash G$, let

$$L(\Gamma g) = \sup_{c_+ > 0 > c_-} \{c_+ - c_- : \Delta_{c_-, c_+} \cap \mathbb{Z}^2 g = \emptyset\},$$

with the convention that $L(\Gamma g) = 0$ if the set in the definition is empty and that $L(\Gamma g) = \infty$ if it is all of $\mathbb{R}^+$. $L(\Gamma g)$ is the area of the largest triangle in the family $\Delta_{c-, c+}$ that is disjoint from $\mathbb{Z}^2 g$.

Following [8], we establish a connection between homogeneous dynamics (embodied in the function L) and number theoretic quantities (embodied in σ_N and λ_N). This is achieved in Lemmas 5.2, 5.3, 5.4. For $y > 0$, define probability measures μ_y on $X = \Gamma \backslash G$ by

$$\int_X f(\Gamma g) d\mu_y(g) = \frac{1}{2} \int_{-1}^1 f(\Gamma u(x) a(y)) dx.$$

We also write $s = \lfloor N^{1/2} \rfloor$, and $\mathbb{1}(B)$ for the Boolean function

$$\mathbb{1}(B) = \begin{cases} 1, & \text{if } B = \text{TRUE}, \\ 0, & \text{if } B = \text{FALSE}. \end{cases}$$

The aim of this section is to prove the following result, of which Corollary 1.3 is a special case.

Proposition 5.1. *Let*

$$\begin{aligned} \sigma_\infty(t) &= \int_X \mathbb{1}(L(\Gamma g) < t) d\mu(g), \\ \lambda_\infty(t) &= \int_X \frac{1}{L(\Gamma g)} \mathbb{1}(L(\Gamma g) < t) d\mu(g). \end{aligned} \tag{5.1}$$

Then for every $\varepsilon > 0$ and every N we have

$$\sigma_N(t) - \sigma_\infty(t) \ll_\varepsilon (1 + t^2) N^{-\frac{1}{36} + \varepsilon}, \tag{5.2}$$

$$\lambda_N(t) - \lambda_\infty(t) \ll_\varepsilon (1/t^2 + t) N^{-\frac{1}{68} + \varepsilon}. \tag{5.3}$$

Our first step in the proof of this result is the following estimate.

Lemma 5.2 (Effective version of [8, Lemma 3.1]). *For $t > 0$ and s as above, we have*

$$\begin{aligned} \lambda_N(t) &= \frac{s^2}{N} \lambda_{s^2}(ts^2/N) + O(N^{-1/2}), \\ \sigma_N(t) &= \sigma_{s^2}(ts^2/N) + O(tN^{-1/2}). \end{aligned} \tag{5.4}$$

Proof. Consider (5.4). We have

$$\begin{aligned} \lambda_{s^2}(ts^2/N) &= \frac{\#\{\text{gaps from } s^2 \text{ points that are } < t/N\}}{s^2} \\ &= \frac{\#\{\text{gaps from } N \text{ points that are } < t/N\}}{s^2} + O(N^{-1/2}) \\ &= \frac{N \#\{\text{gaps from } N \text{ points that are } < t/N\}}{s^2 N} + O(N^{-1/2}) \\ &= \frac{N}{s^2} \lambda_N(t) + O(N^{-1/2}). \end{aligned}$$

This proves the first statement. The second statement can be obtained by partial integration of the first, or directly via a similar argument. $\square$

For $N \geq 2$, let $L_N(\alpha): \mathbb{R}/\mathbb{Z} \rightarrow [0, \infty)$ be N times the length of the gap containing α (and 0 if $\alpha \equiv \sqrt{n} \pmod{1}$ for some positive integer $n \in [1, N]$). Putting

$$r_t(a, b) = \sqrt{a^2 + b} - (a + t),$$

we can write

$$L_N(t) = N \left(\min_{r_t(a,b) \geq 0} r_t(a,b) - \max_{r_t(a,b) \leq 0} r_t(a,b) \right),$$

where a and b range over integers such that $0 < a < s$ and $0 \leq b \leq 2a + 1$. Let $I_N(t)$ denote the union of gaps that are less than t/N in length. Equivalently, we put $I_N(t) = \{\alpha \in \mathbb{R}/\mathbb{Z} : L_N(\alpha) < t\}$, and the sum of lengths of intervals comprising $I_N(t)$ equals $\sigma_N(t)$.

A brilliant move of Elkies and McMullen was to replace the function $r_t(a, b)$ by another function, thereby moving the points of the sequence $\sqrt{n} \bmod 1$ by a small amount and rendering the resulting point set amenable to techniques from homogeneous dynamics. Putting

$$\tilde{r}_t(a, b) = \frac{a^2 + b - (a + t)^2}{2(a + t)},$$

we write

$$\tilde{L}_N(t) = N \left(\min_{\tilde{r}_t(a,b) \geq 0} \tilde{r}_t(a,b) - \max_{\tilde{r}_t(a,b) \leq 0} \tilde{r}_t(a,b) \right),$$

with the same restrictions on a and b . Let $\tilde{I}_N(t) = \{\alpha \in \mathbb{R}/\mathbb{Z} : \tilde{L}_N(\alpha) < t\}$, and let $\tilde{\sigma}_N(t)$ denote the combined length of segments comprising $\tilde{I}_N(t)$. Then we have

Lemma 5.3 (Effective version of [8, Prop. 3.2 and Cor. 3.4]). *Let $\tilde{\sigma}_N(t)$ equal the length of the union of segments $\tilde{I}_N(t)$. Then*

$$\tilde{\sigma}_{s^2}((1 - s^{-1/3})t) + O(s^{-1/3}) \leq \sigma_{s^2}(t) \leq \tilde{\sigma}_{s^2}((1 + s^{-1/3})t) + O(s^{-1/3}).$$

Lemma 5.4 (cf. [8, Prop. 3.8]). *Let*

$$\tilde{\sigma}_{s^2}(t) = \int_X \mathbb{1}(L(\Gamma g) < t) d\mu_{1/s^2}(g).$$

Then we have $\tilde{\sigma}_{s^2}(t) = \tilde{\sigma}_{s^2}(t) + O(s^{-1})$.

Proposition 5.5. *For each $\varepsilon > 0$,*

$$\int_X \mathbb{1}(L(\Gamma g) \geq t) d\mu_{1/s^2}(g) = \int_X \mathbb{1}(L(\Gamma g) \geq t) d\mu(g) + O_\varepsilon \left((1 + t^2) N^{-\frac{1}{36} + \varepsilon} \right) \quad (5.5)$$

$$\int_X \frac{\mathbb{1}(L(\Gamma g) \geq t)}{L(\Gamma g)} d\mu_{1/s^2}(g) = \int_X \frac{\mathbb{1}(L(\Gamma g) \geq t)}{L(\Gamma g)} d\mu(g) + O_\varepsilon \left((1/t^2 + t) N^{-\frac{1}{68} + \varepsilon} \right). \quad (5.6)$$

Proof. To prove Proposition 5.5, it suffices to apply Theorem 1.1 to two bounded but glaringly discontinuous functions,

$$\Gamma g \mapsto \mathbb{1}(L(\Gamma g) \geq t), \quad (5.7)$$

$$\Gamma g \mapsto \frac{1}{L(\Gamma g)} \mathbb{1}(L(\Gamma g) \geq t). \quad (5.8)$$

(Boundedness is assured by reversing the sense of the inequality in (5.6) versus (5.1).) We therefore approximate functions by smooth analogues first. To this end let $\delta \in (0, 1/t)$; we will choose it later depending on N . Fix a left-invariant metric d on G coming from a Riemannian metric tensor, project it to X , and call the projected metric d , indulging the common abuse of notation. Also write U_δ for the δ -neighbourhood of $1 \in G$.

Following [16], for each δ we fix ψ_δ to be a non-negative smooth compactly supported test function on G so that

- (1) $\int_G \psi_\delta(g) d\mu(g) = 1$;
- (2) $\int_G |D\psi_\delta(g)| d\mu(g) \ll_k \delta^{-k}$ for every monomial $D \in U(\mathfrak{g})$ in the variables $X_1, \dots, X_5$ of order k ;

(3) $\text{supp } \psi_\delta \subset U_\delta$.

For $f_1: G \rightarrow \mathbb{R}$ and $f_2: X \rightarrow \mathbb{R}$, define their convolution by

$$f_1 * f_2(\Gamma g) = \int_G f_1(h) f_2(\Gamma g h^{-1}) d\mu(h).$$

Let $f: X \rightarrow \mathbb{R}$ be one of the two functions (5.7) or (5.8), and let $\text{Sing } f$ be the subset of X where f is not smooth or where $L(\Gamma g)$ is infinite. The following smoothing technique will work for any function that is *bounded*, has *controllable* derivatives outside its singular set, and its singular set is *thin*. For a subset S of a metric space, write $\partial_\delta S$ for the δ -neighbourhood of the set S , where the metric on the ambient space is implied. Write

$$f_\delta^\#(\Gamma g) = \begin{cases} \max f, & d(\Gamma g, \text{Sing } f) < 3\delta, \\ f(\Gamma g), & \text{otherwise,} \end{cases}$$

$$f_\delta^\flat(\Gamma g) = \begin{cases} \min f, & d(\Gamma g, \text{Sing } f) < 3\delta, \\ f(\Gamma g), & \text{otherwise,} \end{cases}$$

for f from (5.7). Let

$$E_\delta = [0, \delta^{1/2}] \times [-1, 1],$$

and write

$$f_\delta^\#(\Gamma g) = \begin{cases} \max f, & d(\Gamma g, \text{Sing } f) < 3\delta \text{ or } \mathbb{Z}^2 g \cap \partial_{3\delta} E_\delta \neq \emptyset, \\ f(\Gamma g), & \text{otherwise,} \end{cases}$$

$$f_\delta^\flat(\Gamma g) = \begin{cases} \min f, & d(\Gamma g, \text{Sing } f) < 3\delta \text{ or } \mathbb{Z}^2 g \cap \partial_{3\delta} E_\delta \neq \emptyset, \\ f(\Gamma g), & \text{otherwise,} \end{cases}$$

in the case of (5.8). For the two functions under consideration, maxima and minima are 1 and 0, and $1/t$ and 0, respectively.

To construct approximating functions $f_\delta^\pm: X \rightarrow \mathbb{R}$ we need to understand smoothness properties of the function L . The following result is needed for establishing (5.6).

Lemma 5.6. *Suppose g is such that $L(\Gamma g) \neq 0$, $L(\Gamma g) \neq \infty$, and $\mathbb{Z}^2 g \cap E_\delta = \emptyset$. Assume also that L is smooth at Γg . Then we have*

$$\begin{aligned} X_1.L(\Gamma g) &\ll 1, \\ X_2.L(\Gamma g) &\ll L^2(\Gamma g), \\ X_3.L(\Gamma g) &\ll L(\Gamma g), \\ X_4.L(\Gamma g) &\ll L(\Gamma g)\delta^{-1/2} + L^2(\Gamma g), \\ X_5.L(\Gamma g) &\ll \delta^{-1/2} + L(\Gamma g). \end{aligned}$$

Proof. We only prove the statement for X_4 ; the others are similar.

Since $L(\Gamma g) \neq \infty$, it follows that $\mathbb{Z}^2 g \cap ((0, 1) \times \mathbb{R})$ is an infinite set. We let (x, y) and (x', y') be contained in this set so that they lie on the boundary of the triangle $\Delta_{y'/(2x'), y/(2x)}$ with $\Delta_{y'/(2x'), y/(2x)}$ disjoint from $\mathbb{Z}^2 g$ and such that $y > 0 > y'$. This implies

that $L(\Gamma g) = \frac{y}{x} - \frac{y'}{x'}$. We assume first that there is only one such pair of points. Then

$$\begin{aligned} X_4.L(\Gamma g) &= \frac{d}{d\varepsilon} L(\Gamma g \exp \varepsilon X_4) \Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon} \left(\frac{y}{x+\varepsilon} - \frac{y'}{x'+\varepsilon} \right) \Big|_{\varepsilon=0} \\ &= \frac{y'}{x'^2} - \frac{y}{x^2}. \end{aligned}$$

Since $(x, y), (x', y') \notin E_\delta$, we get the desired conclusion.

If there are several points on the boundary of $\Delta_{y'/(2x'), y/(2x)}$, we obtain the inequality

$$X_4.L(\Gamma g) \leq \max \left(\frac{\tilde{y}'}{\tilde{x}'^2} - \frac{\tilde{y}}{\tilde{x}^2} \right),$$

where the maximum ranges over pairs

$$(\tilde{x}, \tilde{y}) \in \partial \Delta_{y'/(2x'), y/(2x)} \cap \mathbb{Z}^2 g \cap ((0, 1) \times \mathbb{R}_{>0}),$$

and

$$(\tilde{x}', \tilde{y}') \in \partial \Delta_{y'/(2x'), y/(2x)} \cap \mathbb{Z}^2 g \cap ((0, 1) \times \mathbb{R}_{<0}).$$

Since $(\tilde{x}, \tilde{y}), (\tilde{x}', \tilde{y}') \notin E_\delta$ by hypothesis, we arrive that the same conclusion as with one pair of points.

□

Let C be a sufficiently large constant that depends on the implied constants in Lemma 5.6. Then set

$$\begin{aligned} f_\delta^+(\Gamma g) &= (f_\delta^\sharp + C\delta^{1/2}) * \psi_\delta(\Gamma g), \\ f_\delta^-(\Gamma g) &= (f_\delta^\flat - C\delta^{1/2}) * \psi_\delta(\Gamma g) \end{aligned}$$

for f from (5.8). For f from (5.7), set

$$\begin{aligned} f_\delta^+(\Gamma g) &= f_\delta^\sharp * \psi_\delta(\Gamma g), \\ f_\delta^-(\Gamma g) &= f_\delta^\flat * \psi_\delta(\Gamma g) \end{aligned}$$

Lemma 5.7. *For each $m = 0, 1, 2, \dots$ we have $\|f_\delta^\pm\|_{C_b^m} \ll \|f\|_{L^\infty} \delta^{-m}$.*

Proof. Direct calculation.

□

For $p \in \mathbb{R}^2$ let $X(p) = \{\Gamma g \in X : p \in \mathbb{Z}^2 g\}$.

Lemma 5.8. *Let S be a subset of $\mathbb{R}^2$. Then*

$$\begin{aligned} \partial_\delta(X(p)) &\subset \bigcup_{d(p, p') \ll (1+|p|)\delta} X(p'), \\ \partial_\delta \left(\bigcup_{p \in S} X(p) \right) &\subset \bigcup_{d(p', S) \ll (1+|p'|)\delta} X(p'), \end{aligned}$$

where $d(\cdot, \cdot)$ denotes the Euclidean metric $\mathbb{R}^2$.

Proof. Direct calculation.

□

We need some estimates for the measure of lattices that have nodes in a small set and nodes in another small set, as well as the measure of lattices that have nodes in a small set but no nodes in another large set.

Lemma 5.9. *Let $S \subset \mathbb{R}^2$ be a measurable set. Then $\mu\{\Gamma g \in X : \mathbb{Z}^2 g \cap S \neq \emptyset\} \leq \text{Leb } S$.*

Proof. Follows from Markov's inequality. $\square$

Lemma 5.10. *Let T be a measurable subset of $\mathbb{R}^2$. Then*

$$\mu\{\Gamma g \in X : \mathbb{Z}^2 g \cap T = \emptyset\} \leq (1 + \text{Leb } T)^{-1}.$$

Proof. See [1, Theorem 1]. $\square$

Lemma 5.11. *Let S_1, S_2 be measurable subsets of $\mathbb{R}^2$. Then*

$$\mu\{\Gamma g : \mathbb{Z}^2 g \cap S_1 \neq \emptyset, \mathbb{Z}^2 g \cap S_2 \neq \emptyset\} \leq \text{Leb } S_1 \text{Leb } S_2 + \text{Leb}(S_1 \cap S_2).$$

Proof. Follows from [10, Propositions 7.10, 7.11]. $\square$

Lemma 5.12. *Let S_1, S_2 be measurable subsets of $\mathbb{R}^2$. Then*

$$\mu\{\Gamma g : \mathbb{Z}^2 g \cap S_1 \neq \emptyset, \mathbb{Z}^2 g \cap S_2 = \emptyset\} \leq \text{Leb } S_1 (1 + \text{Leb } S_2)^{-1}.$$

Proof. Follows from [10, Proposition 7.10] and [2, Theorem 2.2]. $\square$

We begin by studying the singular set of the function $\Gamma g \mapsto \mathbb{1}(L(\Gamma g) \geq t)$, which is obviously $\partial\{L(\Gamma g) \geq t\}$. We analyse five possibilities:

$$\begin{aligned} \partial\{L(\Gamma g) \geq t\} &= \partial\{L(\Gamma g) \geq t\} \cap \{L(\Gamma g) = \infty\} \\ &\quad \bigcup \partial\{L(\Gamma g) \geq t\} \cap \{\infty > L(\Gamma g) > t\} \\ &\quad \bigcup \partial\{L(\Gamma g) \geq t\} \cap \{L(\Gamma g) = t\} \\ &\quad \bigcup \partial\{L(\Gamma g) \geq t\} \cap \{0 < L(\Gamma g) < t\} \\ &\quad \bigcup \partial\{L(\Gamma g) \geq t\} \cap \{L(\Gamma g) = 0\}. \end{aligned}$$

In the first case we have

$$\begin{aligned} \partial_{2\delta}(\partial\{L(\Gamma g) \geq t\} \cap \{L(\Gamma g) = \infty\}) &\subset \partial_{2\delta}(\{L(\Gamma g) = \infty\}) \\ &\subset \partial_{2\delta}\{\mathbb{Z}^2 g \cap ((0, 1) \times \mathbb{R}) = \emptyset\} \\ &\subset \{\mathbb{Z}^2 g \cap Q_\delta = \emptyset\}, \end{aligned}$$

where Q_δ is the quadrilateral defined by inequalities

$$\begin{aligned} w_1 &\gg (1 + w_2)\delta \\ w_1 &\gg (1 - w_2)\delta \\ 1 - w_1 &\gg (1 + w_2)\delta \\ 1 - w_1 &\gg (1 - w_2)\delta \end{aligned}$$

in the (w_1, w_2) -plane by Lemma 5.8. Note that $L(\Gamma g) = \infty$ implies that the lattice has no nodes in the open strip $\{0 < w_1 < 1\}$ so that (both) c_+, c_- can be taken to be ∞ and $-\infty$, respectively. From Lemma 5.10, we have that

$$\mu(\partial_{2\delta}(\partial\{L(\Gamma g) \geq t\} \cap \{L(\Gamma g) = \infty\})) \ll 1/\text{Leb}(Q_\delta) \ll \delta.$$

In the second case we analyse

$$\partial\{L(\Gamma g) \geq t\} \cap \{\infty > L(\Gamma g) > t\}.$$

This set includes some lattices that contain the origin, as well as some lattices that contain a point on the line $\{w_1 = 1\}$. In this first subcase we include *all* lattices that contain the origin; their contribution is

$$\mu(\partial_{2\delta}\{0 \in \mathbb{Z}^2 g\}) \ll \delta^2$$

by Lemma 5.8.

The second subcase comprises lattices that contain $(1, 2h)$ from the segment $\{1\} \times [-2t, 2t]$ subject to the additional constraint that Δ_h does not meet the lattice. The 2δ -thickening of the set of such includes lattices with a node in

$$B_h^\delta = [1 - 5(|h| + 1)\delta, 1 + 5(|h| + 1)\delta] \times [h - 5, h + 5]$$

and no node in

$$\Delta_h^\delta = \Delta_h \cap \{p \in \mathbb{R}^2 : d(p, \mathbb{R}^2 \setminus \Delta_h) \ll 5(1 + |h|)\delta\} \quad (5.9)$$

for some integer h between $-t - 10$ and $t + 10$. By an application of Lemma 5.12 with $S_1 = B_h^\delta$ and $S_2 = \Delta_h^\delta$, the measure of a 2δ -thickening of this set is at most

$$\begin{aligned} \sum_{h=-[t]-10}^{[t]+10} \frac{\mathrm{Leb} B_h^\delta}{1 + \mathrm{Leb} \Delta_h^\delta} &\ll \sum_{h=1}^{[t]+10} \frac{(h+1)\delta}{1+h} \\ &\ll (t+1)\delta. \end{aligned} \quad (5.10)$$

For the third case we use a combination of Lemma 5.8 and Lemma 5.11. Any lattice in $\{L(\Gamma g) = t\}$ will contain a node in Δ_t ; call this node $(v, 2cv)$. If there are several such nodes, use one of the points with the smallest positive value of c . In addition to $(v, 2cv)$, the lattice must contain a node on the (open) segment joining $(0, 0)$ to $(1, 2(t - c)v)$. If $t \leq 10$, say, then we have

$$\mu(\partial_{2\delta}\{L(\Gamma g) = t\}) \ll 100 \times 100\delta \ll \delta$$

by Lemma 5.8. So assume that t is large, to wit $t > 10$. Then v is contained in $(0, 1/t) \cup (1 - 1/t, 1)$ by [8, Lemma 3.12]. Indeed, in order for the lattice to contain $(v, 2cv)$ and have no nodes inside $\Delta_{c-t, c}$, the quantity in [8, eq. (3.44)] must be positive, whence $v^2 - v + 1/(2t) > 0$. Since t is large, v must be away from the axis of symmetry of this critical parabola; that is, in, say, $(0, 1/t) \cup (1 - 1/t, 1)$. Similarly, the node $(v', 2c'v')$ of the lattice that lies on the open segment joining $(0, 0)$ to $(1, 2(t - c)v)$ must satisfy $v' \in (0, 1/t) \cup (1 - 1/t, 1)$. Thus,

$$\mu(\partial_{2\delta}\{L(\Gamma g) = t\}) \ll (t^2 + 1)\delta.$$

Case four does not arise.

Case five is a singular case for lattices that meet the open segment $(0, 1) \times \{0\}$. By Lemmas 5.8 and 5.9,

$$\mu(\partial_{2\delta}(\partial\{L(\Gamma g) \geq t\} \cap \{L(\Gamma g) = 0\})) \ll \mu(\partial_{2\delta}\{L(\Gamma g) = 0\}) \ll \delta.$$

We now have enough control to prove the first statement of Proposition 5.5; to prove the second statement it remains to understand the singular set of the function $\Gamma g \mapsto \frac{1}{L(\Gamma g)} \mathbb{1}(L(\Gamma g) \geq t)$, which, considering our analysis of $\Gamma g \mapsto \mathbb{1}(L(\Gamma g) \geq t)$, amounts to studying $\Gamma g \mapsto \frac{1}{L(\Gamma g)}$. If the latter function is not smooth, then $\mathbb{Z}^2 g$ contains a node in the set

$$([0, 1] \times \{0\}) \cup (\{1\} \times \mathbb{R}).$$

We distinguish three cases. If this node is contained in $([0, 1] \times \{0\}) \cup (\{1\} \times [-10, 10])$, we use the bound

$$\mu(\partial_{2\delta}\{\mathbb{Z}^2 g \cap (([0, 1] \times \{0\}) \cup (\{1\} \times [-10, 10])) \neq \emptyset\}) \ll \delta.$$

The other two cases are controlled by $h \in (5, o(1/\delta))$ to be chosen later, and we need to recall that Γg where L is not smooth are required not only to contain a node of the form

$(1, 2h')$, but also no nodes in the triangle $\Delta_{h'}$. (Without loss of generality assume $h' > 0$.) When $h' > h$, we keep only the latter condition, meaning that

$$\begin{aligned} \mu(\partial_{2\delta}\{\mathbb{Z}^2g \cap \Delta_h = \emptyset\}) &\ll \mu(\mathbb{Z}^2g \cap \Delta_h^\delta = \emptyset) \\ &\ll (1 + \text{Leb } \Delta_h^\delta)^{-1} \\ &\ll (1 + h)^{-1}. \end{aligned} \tag{5.11}$$

Here Δ_h^δ is a $2(1 + |h|)\delta$ -thinning of Δ_h as defined in (5.9). By virtue of the assumption $h < o(1/\delta)$, thinning barely modifies the original set, so that $\text{Leb } \Delta_h^\delta + 1 \gg \text{Leb } \Delta_h + 1 = h + 1$. The last line then follows by Lemma 5.10.

The final case is $5 < h' \leq h$. Here we use the method used to obtain (5.10) above to get

$$\begin{aligned} \mu(\partial_{2\delta}\{\mathbb{Z}^2g \cap \Delta_h = \emptyset, \mathbb{Z}^2g \cap (\{1\} \times [10, h])\}) \\ &\ll \sum_{h'=4}^{[h]+1} \mu\{\mathbb{Z}^2g \cap \Delta_{h'}^\delta = \emptyset, \mathbb{Z}^2g \cap B_{h'}^\delta \neq \emptyset\} \\ &\ll \sum_{h'=4}^{[h]+1} \frac{\text{Leb } B_{h'}^\delta}{1 + \text{Leb } \Delta_{h'}^\delta} \\ &\ll \sum_{h'=4}^{[h]+1} \frac{(h' + 1)t}{h' + 1} \\ &\ll (h + 1)\delta. \end{aligned} \tag{5.12}$$

The optimal choice for h is $\delta^{-1/2}$, making (5.11) and (5.12) equal to $\delta^{1/2}$. The measure of the thickened boundary is thus at most a constant times $(t^2 + 1)\delta$ or $(t^2 + 1)\delta + \delta^{1/2}$ for functions from (5.7) or (5.8), accordingly.

We now prove that

$$f_\delta^- \leq f \leq f_\delta^+$$

everywhere on X . By symmetry, it is enough to establish the right hand inequality. Consider the case of the function from (5.6). Suppose there exists g so that $f(\Gamma g) > f_\delta^+(\Gamma g)$. Then there exists $g' \in gU_\delta$ such that $f(\Gamma g) > f_\delta^\sharp(\Gamma g') + C\delta^{1/2}$. If $f_\delta^\sharp(\Gamma g') = f(\Gamma g')$, then $d(\Gamma g', \text{Sing } f) \geq 2\delta$ and $d(\mathbb{Z}^2g', E_\delta) \geq 2\delta$ (d denotes the Euclidean metric on $\mathbb{R}^2$ in the second case). In particular, Γg and $\Gamma g'$ are contained in a ball of size δ that does not intersect $\text{Sing } f$ or $\{\Gamma g'' : \mathbb{Z}^2g'' \cap E_\delta \neq \emptyset\}$, and $f_\delta^\sharp(\Gamma g') = f(\Gamma g')$ throughout this ball. By Lemma 5.6 and the Mean Value Theorem, $f(\Gamma g') - f(\Gamma g)$ cannot exceed a constant times

$$\begin{aligned} |\delta D(1/L(\Gamma g''))| &\ll \delta \left| \frac{D.L(\Gamma g'')}{L^2(\Gamma g'')} \right| \\ &\ll \delta \frac{1 + L^2(\Gamma g'') + \delta^{-1/2}(1 + L(\Gamma g''))}{L^2(\Gamma g'')} \\ &\ll \delta^{1/2} \left(\frac{1}{t^2} + 1 \right). \end{aligned}$$

Implied constants here are absolute. Choosing C to be a larger constant times $1 + 1/t^2$ will lead to a contradiction. If $f_\delta^\sharp(\Gamma g') = \max f$, then the contradiction is immediate. In the case of f from (5.5), the inequality is obvious.

In order to prove the error terms in the statement of the theorem, we apply Theorem 1.1 to $f_\delta^\pm$, which leads to the problem of optimising the expression

$$\int_X (f_\delta^+ - f_\delta^-) d\mu + N^{-1/4+\varepsilon} (\|f_\delta^+\|_{C_b^8} + \|f_\delta^-\|_{C_b^8})$$

as a function of δ . For the first term in the case of (5.6) we have

$$\begin{aligned} \int_X (f_\delta^+ - f_\delta^-) d\mu &= \int_X (f_\delta^\sharp - f_\delta^\flat + 2C\delta^{1/2}) * \psi_\delta d\mu \\ &= \int_X (f_\delta^\sharp - f_\delta^\flat) * \psi_\delta d\mu + O((1/t^2 + 1)\delta^{1/2}) \\ &\ll \int_{d(\Gamma g, \mathrm{Sing} f) < 4\delta \text{ or } \mathbb{Z}^2 g \cap \partial_{3\delta} E_\delta \neq \emptyset} (\max f - \min f) d\mu + O((1/t^2 + 1)\delta^{1/2}) \\ &\ll \|f\|_{L^\infty} \mu(\partial_{4\delta}(\mathrm{Sing} f)) + \|f\|_{L^\infty} \mu\{\Gamma g : \mathbb{Z}^2 g \cap E_\delta \neq \emptyset\} \\ &\quad + O((1/t^2 + 1)\delta^{1/2}) \\ &\ll \left(\frac{1+t^2}{t} + \frac{1}{t^2} + 1 \right) \delta^{1/2} \\ &\ll \left(\frac{1}{t^2} + t \right) \delta^{1/2}. \end{aligned}$$

In the case of (5.5) we have

$$\begin{aligned} \int_X (f_\delta^+ - f_\delta^-) d\mu &= \int_X (f_\delta^\sharp - f_\delta^\flat) * \psi_\delta d\mu \\ &\ll \mu(\partial_{4\delta}(\mathrm{Sing} f)) \\ &\ll \delta(1 + t^2). \end{aligned}$$

For (5.5), we minimise $\delta(t^2 + 1) + N^{-1/4+\varepsilon}\delta^{-8}$. On the other hand, for (5.6), we instead minimise $\delta^{1/2}(1/t^2 + t) + N^{-1/4+\varepsilon}\delta^{-8}$. In each case we view δ as a function of N only. The optimal choices are $\delta = N^{-\frac{1}{36}}$ and $\delta = N^{-\frac{1}{34}}$, respectively. This proves (5.5) and (5.6). $\square$

Lemma 5.13. *For $\lambda_\infty(t)$ and $\sigma_\infty(t)$ defined as in Proposition 5.1, we have*

$$\begin{aligned} \frac{d}{dt} \sigma_\infty(t) &\ll \min(t, t^{-2}), \\ \frac{d}{dt} \lambda_\infty(t) &\ll \min(1, t^{-3}), \end{aligned}$$

Proof. Follows from the explicit formulas for σ_∞ and λ_∞ from [8, Prop. 3.14]. $\square$

Proof of Proposition 5.1. We wish to establish control on σ_N first, so we write

$$\begin{aligned} \sigma_N(t) - \sigma_\infty(t) &\leq \sigma_N(t) - \sigma_{s^2}(ts^2/N) \\ &\quad + \sigma_{s^2}(ts^2/N) - \tilde{\sigma}_{s^2}((1 - s^{-1/3})ts^2/N) \\ &\quad + \tilde{\sigma}_{s^2}((1 - s^{-1/3})ts^2/N) - \tilde{\tilde{\sigma}}_{s^2}((1 - s^{-1/3})ts^2/N) \\ &\quad + \tilde{\tilde{\sigma}}_{s^2}((1 - s^{-1/3})ts^2/N) - \sigma_\infty((1 - s^{-1/3})ts^2/N) \\ &\quad + \sigma_\infty((1 - s^{-1/3})ts^2/N) - \sigma_\infty(t). \end{aligned}$$

Now using Lemmas 5.2, 5.3, 5.4, Proposition 5.5, and Lemma 5.13, we arrive at the upper bound

$$O(tN^{-1/2}) + O(s^{-1/3}) + O(s^{-1}) + O_\varepsilon((1+t^2)N^{-\frac{1}{36}+\varepsilon}) + O(s^{-1/3}).$$

For the lower bound on the difference, we write

$$\begin{aligned} \sigma_N(t) - \sigma_\infty(t) &\geq \sigma_N(t) - \sigma_{s^2}(ts^2/N) \\ &\quad + \sigma_{s^2}(ts^2/N) - \tilde{\sigma}_{s^2}((1+s^{-1/3})ts^2/N) \\ &\quad + \tilde{\sigma}_{s^2}((1+s^{-1/3})ts^2/N) - \tilde{\tilde{\sigma}}_{s^2}((1+s^{-1/3})ts^2/N) \\ &\quad + \tilde{\tilde{\sigma}}_{s^2}((1+s^{-1/3})ts^2/N) - \sigma_\infty((1+s^{-1/3})ts^2/N) \\ &\quad + \sigma_\infty((1+s^{-1/3})ts^2/N) - \sigma_\infty(t), \end{aligned}$$

which matches the upper bound upon application of Lemmas 5.2, 5.3, 5.4, Proposition 5.5, and Lemma 5.13. The proof of (5.2) is thus complete.

To get at $\lambda_N(t)$ we start with the following bounds.

$$\begin{aligned} \lambda_N(t) - \lambda_\infty(t) &= \lambda_N(t) - \frac{s^2}{N} \lambda_{s^2}(ts^2/N) \\ &\quad + \frac{s^2}{N} (\lambda_{s^2}(ts^2/N) - 1) - (\lambda_\infty(t) - 1) + O(N^{-1/2}). \end{aligned} \tag{5.13}$$

The first line is controlled by Lemma 5.2, while for the second we need a more complicated argument. Write the second line without the error term as

$$\begin{aligned} \frac{s^2}{N} (\lambda_{s^2}(ts^2/N) - 1) - (\lambda_\infty(t) - 1) &= \int_t^\infty \frac{d\sigma_\infty(\xi)}{\xi} - \frac{s^2}{N} \int_{ts^2/N}^\infty \frac{d\sigma_{s^2}(\xi)}{\xi} \\ &= \int_t^\infty \frac{d\sigma_\infty(\xi)}{\xi} \\ &\quad - \frac{s^2}{N} \left(\int_{ts^2/N}^\infty \frac{\sigma_{s^2}(\xi) d\xi}{\xi^2} - \frac{\sigma_{s^2}(ts^2/N)}{ts^2/N} \right). \end{aligned} \tag{5.14}$$

The terms containing σ_{s^2} are controlled above using

$$\begin{aligned} \sigma_{s^2}(\xi) &= \sigma_\infty(\xi) \\ &\quad + \sigma_{s^2}(\xi) - \tilde{\sigma}_{s^2}((1-s^{-1/3})\xi) \\ &\quad + \tilde{\sigma}_{s^2}((1-s^{-1/3})\xi) - \tilde{\tilde{\sigma}}_{s^2}((1-s^{-1/3})\xi) \\ &\quad + \tilde{\tilde{\sigma}}_{s^2}((1-s^{-1/3})\xi) - \sigma_\infty((1-s^{-1/3})\xi) \\ &\quad + \sigma_\infty((1-s^{-1/3})\xi) - \sigma_\infty(\xi) \end{aligned}$$

and below using

$$\begin{aligned} \sigma_{s^2}(\xi) &= \sigma_\infty(\xi) \\ &\quad + \sigma_{s^2}(\xi) - \tilde{\sigma}_{s^2}((1+s^{-1/3})\xi) \\ &\quad + \tilde{\sigma}_{s^2}((1+s^{-1/3})\xi) - \tilde{\tilde{\sigma}}_{s^2}((1+s^{-1/3})\xi) \\ &\quad + \tilde{\tilde{\sigma}}_{s^2}((1+s^{-1/3})\xi) - \sigma_\infty((1+s^{-1/3})\xi) \\ &\quad + \sigma_\infty((1+s^{-1/3})\xi) - \sigma_\infty(\xi) \end{aligned}$$

Using Lemmas 5.3, 5.4, Proposition 5.5, and Lemma 5.13 and the splittings above, we get

$$\begin{aligned} \frac{s^2}{N} \left(\int_{\frac{ts^2}{N}}^{\infty} \frac{\sigma_{s^2}(\xi) d\xi}{\xi^2} - \frac{\sigma_{s^2}(\frac{ts^2}{N})}{ts^2/N} \right) &\leq \frac{s^2}{N} \int_{ts^2/N}^{\infty} \frac{d\sigma_{\infty}(\xi)}{\xi} \\ &+ \int_{ts^2/N}^{\infty} \frac{O(s^{-1/3}) d\xi}{\xi^2} - \frac{O(s^{-1/3})}{ts^2/N} \\ &+ \int_{ts^2/N}^{\infty} \frac{O(N^{-1/2}) d\xi}{\xi^2} - \frac{O(N^{-1/2})}{ts^2/N} \\ &+ \left(\left(\frac{N}{ts^2} \right)^2 + \frac{ts^2}{N} \right) O_{\varepsilon} \left(N^{-\frac{1}{68} + \varepsilon} \right) \\ &- \frac{\left(1 + \left(\frac{ts^2}{N} \right)^2 \right) O_{\varepsilon} \left(N^{-\frac{1}{36} + \varepsilon} \right)}{ts^2/N} + O(s^{-1/3}). \end{aligned}$$

The lower bound is the same except for implied constants. Therefore we can rewrite (5.14) as

$$\begin{aligned} \frac{s^2}{N} (\lambda_{s^2}(ts^2/N) - 1) - (\lambda_{\infty}(t) - 1) &= \lambda_{\infty}(t) - \frac{s^2}{N} \lambda_{\infty}(ts^2/N) + O_{\varepsilon} \left((1/t^2 + t) N^{-\frac{1}{68} + \varepsilon} \right) \\ &\ll_{\varepsilon} (1/t^2 + t) N^{-\frac{1}{68} + \varepsilon} \end{aligned}$$

by Lemma 5.13. Substituting this result into (5.13), we finally arrive at the required bound. $\square$

6. SKETCH OF THE PROOF OF THEOREM 1.2

The proof of Theorem 1.2 proceeds analogously to that of Theorem 1.1. The main term in equation (2.4) becomes

$$\int_{\mathbb{R}} \tilde{f}_0 \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \rho(x) dx.$$

From [15, Eqs. (23), (24)] this equals

$$\int_X f d\mu \int_{\mathbb{R}} \rho(x) dx + O(\|f\|_{C_b^4} \|\rho\|_{W^{1,1}} y^{1/2} \log^3(2 + 1/y)),$$

which is satisfactory for the theorem.

Turning to the new error terms, we treat the term $c = 0$ following [15, Eq. (25)]. Our analogue of the remaining contribution to the error term $E(y)$ in (2.5) is

$$E_{\rho}(y) = \sum_{\substack{c, n \geq 1 \\ (c, d) = 1}} \int_{\mathbb{R}} e \left(n \left(\frac{dx}{2} + \frac{cx^2}{4} \right) \right) \tilde{f}_n \left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) \rho(x) dx.$$

Now we proceed directly to the change of variables (4.1), following [15, Lemma 6.1]. This gives

$$\int_{\mathbb{R}} e \left(n \left(\frac{dx}{2} + \frac{cx^2}{4} \right) \right) \tilde{f}_n \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) \rho(x) dx = \int_0^{\pi} g(\theta) d\theta,$$

for $c > 0$, where

$$g(\theta) = \tilde{f}_n \left(\frac{a}{c} - \frac{\sin 2\theta}{2c^2 y}, \frac{\sin^2 \theta}{c^2 y}, \theta \right) e \left(-\frac{nd^2}{4c} + \frac{ncy^2 \operatorname{ctg}^2 \theta}{4} \right) \rho \left(-\frac{d}{c} + y \operatorname{ctg} \theta \right) \frac{y}{\sin^2 \theta}.$$

We have the same integral with limits $-\pi$ and 0 if $c < 0$. Combining terms with positive and negative c gives

$$E_\rho(y) = \sum_{c,n \geq 1} \int_{-\pi}^{\pi} \sum_{(c,d)=1} g(\theta) d\theta.$$

We periodise ρ with period 2, by setting $P(z) = \sum_{m \in \mathbb{Z}} \rho(z+2m)$. Now the latter expression can be rewritten using only periodic functions as

$$E_\rho(y) = \sum_{c,n \geq 1} \int_{-\pi}^{\pi} \sum_{\substack{d \bmod 2c \\ (c,d)=1}} \tilde{f}_n \left(\frac{\bar{d}}{c} - \frac{\sin 2\theta}{2c^2 y}, \frac{\sin^2 \theta}{c^2 y}, \theta \right) e \left(-\frac{nd^2}{4c} + \frac{ncy^2 \operatorname{ctg}^2 \theta}{4} \right) \\ \times P \left(-\frac{d}{c} + y \operatorname{ctg} \theta \right) \frac{y d\theta}{\sin^2 \theta}.$$

Exploiting periodicity of $\tilde{f}_n$ and P , we replace them by their Fourier series leaving an exponential sum and two Fourier coefficients to control. Thus

$$E_\rho(y) \ll \sum_{c,n \geq 1} \int_{-\pi}^{\pi} \sum_{k,l \in \mathbb{Z}} \left| \sum_{\substack{d \bmod 2c \\ (c,d)=1}} e \left(-\frac{nd^2}{4c} + \frac{l\bar{d}}{c} - \frac{kd}{2c} \right) \right| |b_l^{(n,c)}(\theta) a_k| \frac{y d\theta}{\sin^2 \theta}, \quad (6.1)$$

where a_k are the Fourier coefficients of P and $b_l^{(n,c)}(\theta)$ are as in (4.3) and (4.6). In particular, we have

$$a_k \ll_\eta (1 + |k|)^{-1-\eta} \|\rho\|_{W^{1,1}}^{1-\eta} \|\rho\|_{W^{2,1}}^\eta \quad (6.2)$$

for all $\eta \in (0, 1)$, as can be seen using integration by parts.

The exponential sum in (6.1) can be estimated using the tools developed in Section 3. Note that the case $k = 0$ corresponds precisely to the sum considered in Lemma 4.2. We say $u \in \mathbb{N}$ is *square-free* if $p \mid u$ implies $p^2 \nmid u$ for every prime p , and similarly $v \in \mathbb{N}$ is *square-full* if $p \mid v$ implies $p^2 \mid v$.

Lemma 6.1. *Write $c = c_0 c_1 = c_0 uv$ with u square-free, v square-full, and furthermore $(uv, 6) = (u, v) = 1$. Then there exists an absolute constant $K \in \mathbb{N}$ such that*

$$\left| \sum_{\substack{d \bmod 2c \\ (c,d)=1}} e \left(-\frac{nd^2}{4c} + \frac{l\bar{d}}{c} - \frac{kd}{2c} \right) \right| \leq K^{\omega(c)} c_0^{3/4} u^{1/2} v^{2/3} (c_0, k, n, l)^{1/4} (u, k, n, l)^{1/2} (v, k, n, l)^{1/3}.$$

Proof. We will sketch the proof of this result based on the methods of Section 3. In doing so we will not pay heed to the particular value of K . Arguing as in the proof of Lemma 4.2, the main task is to estimate the exponential sum

$$T_{p^m}(A, B, C) = \sum_{n \bmod p^m}^* e_{p^m} (An^2 + B\bar{n} + Cn),$$

for $A, B, C \in \mathbb{Z}$ and a prime power p^m . Note that $T_{p^m}(A, B, 0) = T_{p^m}(A, B)$ in the notation of (3.1). We may now write $T_{p^m}(A, B, C)$ in the form (3.3), with $f_1(x) = Ax^3 + B + Cx^2$ and $f_2(x) = x$. When $m = 1$ it follows from Bombieri [3] that

$$|T_p(A, B, C)| \leq 2p^{1/2} (p, A, B, C)^{1/2}.$$

When $m \geq 2$, we apply Cochrane and Zheng [5], as before. We see that

$$f'(x) = \frac{2Ax^3 - B + Cx^2}{x^2},$$

whence $t = \mathrm{ord}_p(f') = v_p((2A, B, C))$, in the notation of [5, Eq. (1.8)]. This time we have

$$\mathcal{A} = \{\alpha \in \mathbb{F}_p^* : A'\alpha^3 - B' + C'\alpha^2 \equiv 0 \pmod{p}\},$$

where $A' = p^{-t}2A$, $B' = p^{-t}B$ and $C' = p^{-t}C$. In particular $(p, A', B', C') = 1$. We may henceforth assume that $m \geq t + 3$ since the desired conclusion follows from the trivial bound otherwise. One finds that any critical point $\alpha \in \mathcal{A}$ has multiplicity $\nu_\alpha \leq 2$ when $p \neq 3$ and multiplicity $\nu_\alpha \leq 3$ when $p = 3$. Next, one applies [5, Thm. 3.1] to deduce that

$$T_{p^m}(A, B, C) \ll \begin{cases} p^{2m/3 + \min\{m, t\}/3}, & \text{if } p \neq 3, \\ 3^{3m/4 + \min\{m, t\}/4}, & \text{if } p = 3, \end{cases}$$

for $m \geq 2$.

Once combined with our treatment of the case $m = 1$, one arrives at the statement of the lemma on invoking multiplicativity. $\square$

We have four regimes in (6.1) to consider, according to whether or not k or l vanish. The cases with $k = 0$ are identical to those already dealt with in Section 4. We proceed to present the argument needed to handle the case $l = 0$ and $k \neq 0$. After using (4.5) with $m = 2$ in (6.1), followed by (4.7), Lemma 6.1 and (6.2), we arrive at the bound

$$\begin{aligned} E_\rho(y) &\ll \|f\|_{C_b^2} \|\rho\|_{W^{1,1}}^{1-\eta} \|\rho\|_{W^{2,1}}^\eta \\ &\quad \times \sum_{c=c_0 uv, n \geq 1} \sum_{k \neq 0} \frac{y}{|k|^{1+\eta}} \cdot \frac{K^{\omega(c)} c_0^{3/4} u^{1/2} v^{2/3} (c_0, k, n)^{1/4} (u, k, n)^{1/2} (v, k, n)^{1/3}}{nc\sqrt{y}(1 + nc\sqrt{y})} \\ &= \|f\|_{C_b^2} \|\rho\|_{W^{1,1}}^{1-\eta} \|\rho\|_{W^{2,1}}^\eta S(y), \end{aligned} \quad (6.3)$$

say. We will apply the upper bounds $(c_0, k, n)^{1/4} \leq c_0^{1/4 - \eta/4} |k|^{\eta/4}$, $(u, k, n)^{1/2} \leq (u, k)^{1/2}$, and $(v, k, n)^{1/3} \leq v^{1/3 - \eta/4} |k|^{\eta/4}$ in order to simplify this expression. In particular the resulting sum over c_0 is absolutely convergent by (4.8). Next we divide the sum so that uv belongs to the dyadic intervals $[2^{j-1}, 2^j)$ for $j \in \mathbb{N}$. In this way we deduce that

$$\begin{aligned} S(y) &\ll \sum_{n \geq 1} \sum_{k \neq 0} \frac{y^{1/2}}{|k|^{1+\eta/2}} \sum_{j \geq 1} \sum_{\substack{v \leq 2^j \\ v \text{ sq.-full}}} \sum_{2^{j-1}/v \leq u < 2^j/v} \frac{K^{\omega(uv)}(u, k)^{1/2} v^{1/2 - \eta/4}}{n\sqrt{uv}(1 + nuv\sqrt{y})} \\ &\ll \sum_{n \geq 1} \frac{1}{n} \sum_{k \neq 0} \frac{y^{1/2}}{|k|^{1+\eta/2}} \sum_{j \geq 1} \frac{1}{2^{j/2}(1 + n\sqrt{y}2^j)} \sum_{\substack{v \leq 2^j \\ v \text{ sq.-full}}} K^{\omega(v)} v^{1/2 - \eta/4} \sum_{u \leq 2^j/v} K^{\omega(u)}(u, k)^{1/2}. \end{aligned}$$

Now, we have $\sum_{n \leq x} K^{\omega(n)} \ll x \log^{K-1} x$ for any $K > 1$ and $x \geq 2$ (see [17, Thm. II.6.1], for example). Hence it follows that

$$\begin{aligned} \sum_{n \leq x} K^{\omega(n)}(n, k)^{1/2} &\leq \sum_{h|k} h^{1/2} \sum_{\substack{n \leq x \\ h|n}} K^{\omega(n)} \\ &\ll x \log^{K-1} x \sum_{h|k} h^{-1/2} K^{\omega(h)}, \end{aligned}$$

for $x \geq 2$. The remaining sum over h is at most $\ll_K \tau(k)$, where τ denotes the divisor function, whence

$$S(y) \ll y^{1/2} \sum_{n \geq 1} \frac{1}{n} \sum_{k \neq 0} \frac{\tau(k)}{|k|^{1+\eta/2}} \sum_{j \geq 1} \frac{2^{j/2} \log^{K-1} 2^j}{1 + n\sqrt{y}2^j} \sum_{\substack{v \leq 2^j \\ v \text{ sq.-full}}} \frac{K^{\omega(v)}}{v^{1/2 + \eta/4}}.$$

The sum over k is convergent. Furthermore, since square-full integers have square root density, the sum over v is also convergent. Hence

$$S(y) \ll y^{1/2} \sum_{n \geq 1} \frac{1}{n} \sum_{j \geq 1} \frac{2^{j/2} \log^{K-1} 2^j}{1 + n\sqrt{y}2^j}$$

Once substituted into (6.3), this leads to the satisfactory contribution

$$\ll \|f\|_{C_b^s} \|\rho\|_{W^{1,1}}^{1-\eta} \|\rho\|_{W^{2,1}}^\eta y^{1/4} \log^{K-1}(2 + y^{-1}),$$

to $E_\rho(y)$, for any $\eta \in (0, 1)$.

In a similar manner, using instead the second estimate from (4.5) with $m = 6$, the contribution from $kl \neq 0$ is found to be

$$\ll \|f\|_{C_b^s} \|\rho\|_{W^{1,1}}^{1-\eta} \|\rho\|_{W^{2,1}}^\eta y^{1/4} \log^{K-1}(2 + y^{-1}),$$

for any $\eta \in (0, 1)$. This completes the sketch of the proof of Theorem 1.2.

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