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Exit from sliding in piecewise-smooth flows: deterministic vs. determinacy-breaking

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The collapse of flows onto hypersurfaces where their vector fields are discontinuous creates highly robust states called sliding modes. The way flows *exit* from such sliding modes can lead to complex and interesting behaviour about which little is currently known. Here we examine the basic mechanisms by which a flow exits from sliding, either along a switching surface, or along the intersection of two switching surfaces, with a view to understanding sliding and exit when many switches are involved. On a single switching surface, exit occurs via tangency of the flow to the switching surface. Along an intersection of switches, exit can occur at a tangency with a lower codimension sliding flow, or by a spiralling of the flow that exhibits geometric divergence (infinite steps in finite time). Determinacy-breaking can occur where a singularity creates a set-valued flow in an otherwise deterministic system, and we resolve such dynamics as far as possible by blowing up the switching surface into a switching *layer*. We show preliminary simulations exploring the role of determinacy-breaking events as organizing centres of local and global dynamics.

Switching is found in dynamical models of wide-ranging applications, from mechanics and geophysics to biological growth and ecology. Switches occur between different dynamical laws whenever certain thresholds are encountered. In this paper we consider how systems behave when they exit from highly constrained states *sliding* along those thresholds or intersections thereof. For one or two switches we examine the basic mechanisms of exit. In particular we show that exit from sliding is not always deterministic, and we describe the main features of *determinacy-breaking* exit points. Example simulations that illustrate the theoretical results as novel dynamical phenomena are given.

I. INTRODUCTION

Many physical and biological systems are a mixture of smooth steady change and sudden transitions. A transition may occur as a *switching surface* is crossed in phase space. Perhaps surprisingly, and despite substantial progress in local (see e.g. [2, 14]) and global (see e.g. [10, 11]) dynamical theory with switching, we are still only beginning to understand the potential effects of switches on dynamical systems.

Consider the piecewise smooth dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \boldsymbol{\lambda}), \quad \lambda_i = \text{sign}(h_i(\mathbf{x})), \quad (1)$$

for some $i = 1, 2, \dots, r$, where $\mathbf{f}$ is a vector field with smooth dependence on the variables $\mathbf{x} = (x_1, \dots, x_n)$, and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r)$ is a vector of *switching parameters*. The dot over $\mathbf{x}$ denotes differentiation with respect to time. Each h_i is an independent scalar function, and the sets $h_i = 0$ are the *switching surfaces*.

Early piecewise-smooth models arose in electronics and mechanics, but are increasingly a feature of the life sciences and an array of other physical problems, from superconductors [4] to predator-prey strategies [5, 26]. For

example take the three systems

$$\ddot{x}_i = \sum_j k_{ij}(x_j - x_i) - c_i \dot{x}_i - N_i \text{sign}(\dot{x}_i - v), \quad (2)$$

$$\dot{x}_i = \mathcal{B}(z_1, z_2, \dots, z_n) - \gamma_i x_i, \quad z_i = H(x_i - v_i), \quad (3)$$

$$\dot{x}_i = r_i x_i (1 - x_i) - \sum_{j=1}^m k_{ij} x_j \text{step}(x_i - v_i), \quad (4)$$

over $i = 1, 2, \dots, m$. The first represents a network of oscillators with displacements x_i , coupled via spring constants k_{ij} and damping coefficients c_i . The oscillators have slipping speeds $\dot{x}_i - v$ relative to a surface with speed v , resulting in Coulomb friction forces $N_i \text{sign}(\dot{x}_i - v)$ with coefficients N_i . The second system represents a genetic regulatory network with gene product concentrations x_i , degradation rates γ_i , and a production rate function $\mathcal{B}$. Genes contribute to production if above a threshold v_i , regulated by a Hill function H [17]. The Hill function is often approximated as a step. The third system represents logistic growth at rates r_i , of m populations x_i . These are consumed by other species x_j at rates k_{ij} when they exceed abundance thresholds v_i , so feeding is turned on or off by a Heaviside step function. The cannibalistic coefficients k_{ii} are usually zero. Systems of these forms may be used to model how microscopic dry-friction leads to macroscale stick-slip or even earthquakes [3, 6], or to model networks of switching in electronic, genetic, or neural circuitry (see e.g. [15, 27]).

These are typical examples of high-dimensional systems with transverse switching surfaces $h_i = 0$ for some $i = 1, 2, \dots, m$ (with $h_i = \dot{x}_i - v$ in (2) and $h_i = x_i - v_i$ in (3)-(4)), across which discontinuities occur in the differential equations. Our aim here is to show how principles learned from low-dimensional discontinuous systems provide insight into such high-dimensional systems. We make only preliminary steps here, studying the key features that will form the basis for future study of local and global phenomena, of which we give a few examples.

In general, systems like (2)-(4) are the subject of

piecewise-smooth dynamical systems theory [10, 14, 24]. In the piecewise-smooth approach to dynamics, changes that take place abruptly at a threshold are modelled as discontinuities at an *event* or *switching* surface. The event surface becomes a new topological object in the qualitative theory of dynamical systems, with its own associated attractivity, singularities, and bifurcations, which comprise the growing theory of piecewise-smooth dynamical systems [7, 10].

Entry and exit points from a switching surface (see figure 1 for a few examples) are of particular interest for studying high-dimensional problems, because trajectories can become constrained to one or more of the surfaces $h_i(\mathbf{x}) = 0$ as in figure 1(i). Each entry/exit point onto/off of a different switching surface can therefore decrease/increase the degrees of freedom. The only well-studied exit points so far are exit points from sliding via tangencies to a single switching surface (the second exit point in figure 1(iii)), which have been studied as the organizing centres of limit cycle bifurcations in low-dimensional systems [10], a study which becomes rapidly more complex in higher dimensions [16, 28].

FIG. 1. Examples of: (i) entry to sliding first on a single switching surface (“codimension one sliding”), then on the intersection of two switching surfaces (“codimension two sliding”); (ii) deterministic exit from codimension one sliding induced by an intersection, (iii) deterministic exit from codimension two sliding to codimension one and then to ‘free’ flow, both induced by tangencies; (iv) determinacy-breaking exit induced by a double tangency. The symbols $\mathbf{f}^{\pm\pm}$ denote vector fields $\mathbf{f}(\mathbf{x}; \pm 1, \pm 1)$ that apply in different regions.

Exit from high codimension sliding (figure 1(iii-iv)) has hardly been studied as yet, though substantial steps in this direction are starting to be made, for example in [13] where the problem of computability of solutions at exit points is raised in particular. Our aim here is to open up this problem by demonstrating basic but non-trivial behaviours induced by exit from sliding. A complete classification of exit points is not possible, as new topologies of exit points will appear with each higher dimension and

each extra switching surface. Our aim here is instead to highlight the different forms that exit may take, and to reveal their common properties and means of study.

Ideally we should seek normal forms for the exit points we present, but there is presently no normal form theory for systems of the kind we will study. (Even in the simplest nonsmooth systems, claims of normal forms and completeness of classifications have proven misleading, see [19]). We therefore provide prototypes, or *structural models*, for the exit points currently known. It is not the precise form of the vector field expressions, but the qualitative behaviours possible and the means to study them, that concerns us.

An important feature of exit points is whether or not they are deterministic. Determinacy-breaking (figure 1(iv)) occurs when a deterministic trajectory reaches an exit point in finite time, then generates a multi-valued flow at the exit point, with determinism still maintained elsewhere. This poses obvious conceptual problems: a numerical computation may select one of many possible exit trajectories depending on the numerical method, while an application may require more detailed modeling to resolve the ambiguity. We shall focus only on the extent to which mathematics can resolve such points, and treat all trajectories permitted by the vector field as equally valid. Nevertheless, we shall see that in certain cases the geometry of the flow alone favours certain trajectories over others, and this is reflected in simulations.

To create simulations at a determinacy-breaking point, we could use an event detection method, followed by a decision either to: 1) simulate an ensemble of possible on-ward trajectories, or 2) introduce a criterion for selecting between the possible values by introducing discretisation, stochasticity, hysteresis, smoothing, or other modeling factors. The best understood of these is regularization by smoothing, in which the discontinuity is replaced by a steep sigmoid transition, and for which basic results exist describing how such systems approximate discontinuous systems [25, 30]. Therefore when simulating examples of exit point behaviour for illustrative purposes only, we shall use smoothed out approximations of the discontinuous vector field as described in the text, and let the numerical integrator choose the path through the intersection as a numerical experiment. Specifically we use Mathematica’s NDSolve, which for sufficiently high precision and accuracy goals yields repeatable results. We then approximate any term $\text{sign}(h_i)$ by a smooth sigmoid function $\phi(h_i/\varepsilon)$ such that $\phi(h_i/\varepsilon) \rightarrow \text{sign}(h_i)$ as $\varepsilon \rightarrow 0$.

We begin by setting out some preliminaries of piecewise-smooth systems in section II. We then begin our study of exit points. Exit from codimension one sliding is discussed in section III. In section IV we begin the study of exit from higher codimension sliding.

Exit from sliding on an intersection of multiple switches can take place via simple tangencies as in section IV A, via multiple tangencies whose study we investigate in section IV B, or via a Zeno process as in section IV C. The latter involves a flow that spirals in to

wards an intersection, travels along it, and spirals back out, with a determinacy-breaking event in the middle. In each case we define a structural model for the scenario, examine its dynamics in the switching layer, and conclude with illustrative simulations. Some closing remarks are made in section VI.

II. PRELIMINARIES: RESOLVING THE DISCONTINUITY

Taking the system (1), let us assume that all of the gradient vectors ∇h_i are linearly independent. Then the manifolds $h_i = 0$ are transversal, so the number of regions N and number of switching surfaces m is related by $N = 2^m$ (assuming the number of spatial dimensions is $n \geq m$). The full switching surface is the zero set of the scalar function

$$h(\mathbf{x}) = h_1(\mathbf{x})h_2(\mathbf{x})\dots h_m(\mathbf{x}) ,$$

of which each set $h_i(\mathbf{x}) = 0$ is a sub-manifold. Each of the $\mathbf{f}$'s is a vector field that is smooth on an open region that extends across the local domain boundaries defined by the switching surface.

Throughout this paper we will use the following coordinates. At a point p where $r \leq m$ switching surfaces intersect, say the set where $h_1 = h_2 = \dots = h_r = 0$ without loss of generality, we can find coordinates $\mathbf{x} = (x_1, x_2, \dots, x_n)$ such that $x_i = h_i$ for $i = 1, 2, \dots, r$. The switching surface in the neighbourhood of p consists of the hypersurfaces $x_1 = 0, x_2 = 0, \dots, x_r = 0$, and their intersection is the set $x_1 = x_2 = \dots = x_r = 0$. The components of a vector field $\mathbf{f}$ are written as $\mathbf{f} = (f_1, f_2, \dots, f_n)$.

The system (1) gives a well defined dynamical system in each region outside the switching surface (for $h \neq 0$), but not on the switching surface $h = 0$. The next step is therefore to prescribe the dynamics on $h = 0$.

A. Vector field combination at the discontinuity

The system (1) is typically (see e.g. [14, 20]) extended across the discontinuity by letting

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \boldsymbol{\lambda}) \quad : \quad \begin{cases} \lambda_i = \text{sign}(h_i) & \text{if } h_i \neq 0, \\ \lambda_i \in [-1, +1] & \text{if } h_i = 0, \end{cases} \quad (5)$$

forming a differential inclusion which interpolates between the different values $\mathbf{f}$ can take in the neighbourhood of the discontinuity. A lot can be achieved with such a general statement, beginning with the proof that solutions to the discontinuous system do exist [14]. What those solutions look like, however, and how they behave, is still an active and very open field of research.

The set-valued vector field in (5) contains vector field values that are dynamically irrelevant in the sense that the flow cannot follow them for any non-vanishing interval of time. Those values the flow *can* follow may be

found by re-writing the vector as a *canopy* combination [20] of the values of $\mathbf{f}$ in the neighbourhood of a point on the switching surface,

$$\mathbf{f}(\mathbf{x}; \boldsymbol{\lambda}) = \sum_{u_1, u_2, \dots, u_m = \pm} \lambda_1^{(u_1)} \lambda_2^{(u_2)} \dots \lambda_m^{(u_m)} \mathbf{f}^{u_1 u_2 \dots u_m}(\mathbf{x}) , \quad (6)$$

using a shorthand $\lambda_i^{(\pm)} \equiv (1 \pm \lambda_i)/2$, and using hereon the more convenient index notation

$$\mathbf{f}^{u_1 u_2 \dots u_m}(\mathbf{x}) \equiv \mathbf{f}(\mathbf{x}; u_1 1, u_2 1, \dots, u_m 1) \quad (7)$$

with each u_i taking either a $+$ or $-$ sign corresponding to the sign of h_i . For two switching manifolds ($m = 2$), the combination (6) becomes (omitting arguments)

$$\begin{aligned} \mathbf{f} = & \frac{1}{2}(1 + \lambda_2) \left[\frac{1}{2}(1 + \lambda_1) \mathbf{f}^{++} + \frac{1}{2}(1 - \lambda_1) \mathbf{f}^{-+} \right] \\ & + \frac{1}{2}(1 - \lambda_2) \left[\frac{1}{2}(1 + \lambda_1) \mathbf{f}^{+-} + \frac{1}{2}(1 - \lambda_1) \mathbf{f}^{--} \right] , \end{aligned} \quad (8)$$

and for a single switching surface ($m = 1$) this reduces to Filippov's commonly used convex combination

$$\mathbf{f}(\mathbf{x}) = \frac{1}{2}(1 + \lambda_1) \mathbf{f}^+(\mathbf{x}) + \frac{1}{2}(1 - \lambda_1) \mathbf{f}^-(\mathbf{x}) . \quad (9)$$

For $m = 1$ the Filippov/Utkin [14, 31] criteria may then be used to determine the existence of sliding modes on $h_1 = 0$. More generally to find $\boldsymbol{\lambda}$ and any possible sliding modes on the thresholds $h_i = 0$, we need the switching layer methods outlined as follows.

B. Switching layer and sliding

To reveal the dynamics on λ_i that transports the flow across the discontinuity, we blow up each manifold $h_i = 0$ into a layer $\lambda_i \in [-1, +1]$ on $h_i = 0$. We review the main details of the method from [21, 22] here.

The dynamics on each λ_i is induced by the h_i component of the flow, and thus given by

$$\lambda'_i = \mathbf{f}(\mathbf{x}; \boldsymbol{\lambda}) \cdot \nabla h_i(\mathbf{x}) \quad \text{on } h_i = 0 , \quad (10)$$

where the prime denotes differentiation with respect to a dummy instantaneous timescale. One way to describe this is that $\dot{x}$ denotes $\frac{d}{dt}x$, while λ' denotes $\varepsilon \frac{d}{dt}\lambda$ for infinitesimal $\varepsilon > 0$, and while this particular interpretation permits singular perturbation analysis, see e.g. [22], in the piecewise smooth context here, only the singular limit $\varepsilon \rightarrow 0$ concerns us. Each switching surface $x_i = 0$ becomes a switching layer $\{x_i = 0, \lambda_i \in [-1, +1]\}$.

At a point where $r \leq m$ switching surfaces intersect, say where $h_1 = h_2 = \dots = h_r = 0$ and $h_{i>r} \neq 0$, take local coordinates $\mathbf{x} = (x_1, x_2, \dots, x_n)$ where each $h_i = 0$ coincides with a coordinate level set $x_i = 0$ for $i = 1, 2, \dots, r$. We then have the dynamics in the switching layer

$$\begin{cases} (\lambda'_1, \dots, \lambda'_r) = (f_1(\mathbf{x}; \boldsymbol{\lambda}), \dots, f_r(\mathbf{x}; \boldsymbol{\lambda})) , \\ (\dot{x}_{r+1}, \dots, \dot{x}_n) = (f_{r+1}(\mathbf{x}; \boldsymbol{\lambda}), \dots, f_n(\mathbf{x}; \boldsymbol{\lambda})) . \end{cases} \quad (11)$$

If the fast λ'_i subsystem has equilibria, where $\lambda'_i = 0$ for all $i = 1, \dots, r$, the resulting equations

$$\begin{cases} (0, \dots, 0) = (f_1(\mathbf{x}; \boldsymbol{\lambda}), \dots, f_r(\mathbf{x}; \boldsymbol{\lambda})), \\ (\dot{x}_{r+1}, \dots, \dot{x}_n) = (f_{r+1}(\mathbf{x}; \boldsymbol{\lambda}), \dots, f_n(\mathbf{x}; \boldsymbol{\lambda})), \end{cases} \quad (12)$$

describe states that evolve inside the switching surfaces $x_1 = \dots = x_r = 0$ on the main timescale, because $\lambda'_i = 0$ implies $\dot{x}_i = \mathbf{f} \cdot \nabla h_i = 0$. These are *sliding modes* (an extension of Filippov's sliding modes [14, 20] for $r = 1$). The values of the λ_i 's corresponding to sliding modes are then given by

$$\mathcal{S}(\boldsymbol{\lambda}) := \left\{ (\lambda_1, \dots, \lambda_r) \in [-1, +1]^r : x_i = 0 \right. \\ \left. \& f_i(\mathbf{x}; \boldsymbol{\lambda}) = 0 \text{ for } i = 1, \dots, r \right\}. \quad (13)$$

In the absence of sliding modes, when (13) has no solutions, the system (11) facilitates an instantaneous transition from one boundary of $\lambda_i \in [-1, +1]$ to another, and the flow crosses through the switching surface.

When solutions $(\lambda_1, \dots, \lambda_r) = \mathcal{S}(\lambda_1, \dots, \lambda_r)$ do exist, they form invariant manifolds of the switching layer system (11), given by

$$\mathcal{M}^S = \left\{ (\lambda_1, \dots, \lambda_r) \in [-1, +1]^r : \boldsymbol{\lambda} = \mathcal{S}(\boldsymbol{\lambda}) \right\} \quad (14)$$

on which the system obeys the sliding dynamics (12). We call $\mathcal{M}^S$ the *sliding manifold*. Examples are illustrated in figure 2 for one or two switches. If it exists, $\mathcal{M}^S$ may be comprised of many connected or disconnected branches on which the conditions (14) hold, and on which $\mathcal{M}^S$ is normally hyperbolic. The normal hyperbolicity of $\mathcal{M}^S$, as an equilibrium of the $\boldsymbol{\lambda}$ subsystem, requires

$$\det \left| \frac{\partial(\lambda'_1, \dots, \lambda'_r)}{\partial(\lambda_1, \dots, \lambda_r)} \right|_{\mathcal{M}^S} \neq 0. \quad (15)$$

Provided (14) and (15) hold then the manifold $\mathcal{M}^S$ so defined is invariant except at its boundaries. The theory of invariant manifolds can be found for example in [18, 23], and we emphasize that in the context of singular perturbations, the interest here is in the singular limit (where $\mathcal{M}^S$ is known as the critical manifold, and the fast timescale is infinitely fast) [22].

The boundaries of $\mathcal{M}^S$ are points where (14) or (15) break down, which respectively give rise to:

1. *end points*: where $\mathcal{M}^S$ passes through the boundary of $\lambda_i \in [-1, +1]$ for some $i \in \{1, \dots, r\}$; or
2. *turning points*: where two branches of $\mathcal{M}^S$ meet (in a fold or higher catastrophe) and normal hyperbolicity of $\mathcal{M}^S$ is lost.

If trajectories exit from sliding they will typically do so at boundaries of $\mathcal{M}^S$ given, therefore, by these conditions.

In both cases 1 and 2 above, the number of modes $\mathcal{S}(\boldsymbol{\lambda})$ changes, typically by unity in the former case (because one root leaves the domain of existence), and by two in

FIG. 2. Sketch showing the blow up of the switching surface into a switching layer at simple exit points. In (i) we see a crossing region (cr.), and an attracting sliding region (a.sl.) inside which an invariant manifold $\mathcal{M}^S$ exists. In (ii) we see an intersection of two switching surfaces where crossing and attracting sliding occur over two sections of the switching surface each. In each figure the lefthand portrait shows the piecewise-smooth flow in the (x_1, x_2) plane, the righthand portrait shows the switching layer where $x_i = 0$ blows up into $\lambda_i \in [-1, +1]$, with sliding on $\mathcal{M}^S$. The flow outside the switching surface is indicated with single arrows, the sliding flow on $\mathcal{M}^S$ is indicated with double arrows, and the fast flow is indicated with filled arrows.

the latter case (because pairs of solutions undergo fold bifurcations); for more details see [21]. We see interplay between these two types in the following sections. Examples of type 1 are illustrated in figure 2 for one or two switches.

An orbit is a piecewise-smooth continuous curve, along which the direction of time is preserved, formed by concatenating: solution trajectories of (5) outside the switching surface, with solution trajectories of (11) inside the switching layer. Solutions of (11) are themselves either ‘fast’ solutions of (10) that cross through the switching layer, or else are ‘fast’ solutions of (10) that collapse onto a sliding manifold $\mathcal{M}^S$, where they are concatenated with sliding solutions of (12). (In figure 2 only individual trajectories, including the fast switching layer solutions (filled arrows) are shown to illustrate the phase portrait. In figure 3 later in the paper such concatenated trajectories are shown, but the fast solutions are not shown.)

Orbits defined in this way may partially overlap, so that multiple orbits can pass through a single point. In an attractive sliding region, every point has a family of distinct orbits reach it in finite time. The converse is

also possible: a family of distinct orbits can depart from a point so that the flow through the point is set-valued in forward time. If a flow becomes set-valued in forward time at a specific point, we say determinacy has been broken there.

III. EXIT FROM CODIMENSION $r = 1$ SLIDING

We begin by considering how orbits may exit from sliding along a codimension one switching surface $h_1 = 0$.

We shall not consider points inside repelling sliding regions, occurring where $\mathbf{f}^+ \cdot \nabla h_1(\mathbf{x}) > 0$ and $\mathbf{f}^- \cdot \nabla h_1(\mathbf{x}) < 0$ on $h_1(\mathbf{x}) = 0$. The flow can exit from the switching surface at all such points, so they do not directly give rise to interesting dynamics. Moreover these are only the reverse time equivalent of attracting sliding regions (where $\mathbf{f}^+ \cdot \nabla h_1(\mathbf{x}) < 0$ and $\mathbf{f}^- \cdot \nabla h_1(\mathbf{x}) > 0$ on $h_1(\mathbf{x}) = 0$), which have been well studied.

Our interest henceforth will be how trajectories are able to exit from regions of attracting sliding, which, since attractive regions are invariants of the flow (given by $\mathcal{M}^S$), can only happen at their boundaries.

In a deterministic exit there is only one possible trajectory that an orbit can follow through the exit point. The two basic forms to be discussed in the following sections are shown in figure 3 (i) and (iii). In (i) exit occurs at a tangency (type 1 – endpoint), and in (iii) exit occurs at an intersection with a second switching manifold (type 2 – endpoint).

Multiple trajectories may be followed beyond the exit point at a determinacy-breaking exit, and the two basic forms to be discussed are triggered by a double tangency as shown in figure 3(ii), or again by an intersection as shown in figure 3(iv). The insets in figures (ii) and (iv) illustrate the set-valued flows through an exit point. These will be described in more detail throughout section III.

A. Exit via a tangency: deterministic

The simplest kind of exit point is that represented by figure 3(i), namely the boundary of a sliding region on a single switching manifold. Considering (13) for $r = 1$, we see that an end point of $\mathcal{M}^S$ occurs when $f_1(\mathbf{x}; \lambda_1) = 0$ is satisfied at the boundary of the switching layer, i.e. at $\lambda_1 = +1$ or $\lambda_1 = -1$. Hence $f_1^\pm(\mathbf{x}; \pm 1) \equiv f_1^\pm(\mathbf{x}) = 0$ at such a point, implying that it constitutes a tangency between the respective vector field $\mathbf{f}^\pm$ and the switching manifold $h_1(\mathbf{x}) = 0$.

If the flow curves away from the switching surface at such a tangency then the flow can exit from sustained sliding at that point, and we call it a *visible* tangency. A generic visible tangency is a point satisfying

$$0 = f_1^+ < \frac{d}{dt} f_1^+ \quad \text{or} \quad 0 = f_1^- < \frac{d}{dt} f_1^- . \quad (16)$$

FIG. 3. Exit from codimension one sliding via: (i) a simple tangency; (ii) a two-fold singularity; (iii-iv) a double-switch. The switching surface is made up of regions where the flow is attracted to the surface then slides (a.sl.), slides but is repelled from the surface (r.sl.), or crosses (cr.). The phase portraits indicate that in (ii) and (iv) determinacy is broken at the exit point (the resulting set-valued flow is shown inset).

for a tangency from the $h_1 > 0$ or $h_1 < 0$ side of the switching manifold, respectively.

The righthand sides of the switching layer system (11), the sliding system (12), and the discontinuous system (1), are equal precisely at points where $\mathcal{S}(\lambda_1) = +1$ or $\mathcal{S}(\lambda_1) = -1$. The dynamics at a non-degenerate tangency, i.e. a quadratic tangency of one flow only, where only one set of the conditions (16) hold, is therefore locally very simple. The flow actually transitions differentially from sliding on the switching surface into smooth motion outside it, and by implication, such a flow is deterministic.

Simple tangencies have been well studied. They are interesting for their role in global dynamics, as the instigators of so-called *sliding bifurcations* (see [10]), whereby limit cycles or stable/unstable manifolds lose or gain connections to the switching surface. They will therefore be of no further interest here.

A point where both conditions in (16) hold is non-trivial, since then $\mathcal{S}(\lambda_1) = +1$ and $\mathcal{S}(\lambda_1) = -1$ are both solutions of (13) and (12) is then singular. This is covered in the next section.

B. Exit via a two-fold singularity

In a system with one switching manifold, exit from sliding can happen where $\mathcal{S}(\lambda_1) = +1$ and $\mathcal{S}(\lambda_1) = -1$ are simultaneously solutions of (13). This constitutes a compound tangency as in figure 3(ii), when both vector fields are tangent to the switching surface. The flow through these compound tangencies can be set-valued in forward (as well as backward) time, which breaks the determinacy of the flow. The simplest example is the two-fold singularity, illustrated in figure 3(ii).

A tangency of either vector field that is non-degenerate can be described as a *fold* of the flow with respect to the switching surface. A double-tangency point, where both $f_1^\pm$ vanish, can be described as a *two-fold* if it is non-degenerate. The non-degeneracy conditions for a fold are $\partial f_1^\pm / \partial x_1 \neq 0$ where $f_1^\pm = 0$ (i.e. the inequalities (16)), and for a two-fold the conditions are that $\partial f_1^\pm / \partial x_1$ do not vanish locally, and that the vectors ∇x_1 , $\nabla(\partial f_1^+ / \partial x_1)$, and $\nabla(\partial f_1^- / \partial x_1)$, are linearly independent.

The canonical form of the two-fold singularity (see [8, 14, 29]) under these conditions is

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = \begin{cases} \mathbf{f}^+ = (-x_2, a_1, b_1) & \text{if } x_1 > 0, \\ \mathbf{f}^- = (+x_3, b_2, a_2) & \text{if } x_1 < 0, \end{cases} \quad (17)$$

in terms of constants $b_i \in \mathbb{R}$ and $a_i = \pm 1$. The singularity lies at $x_1 = x_2 = x_3 = 0$, and three dimensions are sufficient for a local analysis. The regions $x_2, x_3 > 0$ and $x_2, x_3 < 0$ on the switching surface are attracting and repelling sliding regions, respectively. There is a fold along $x_1 = x_2 = 0$, which is visible if $a_1 < 0$ (since then $\ddot{x}_1 = -a_1 > 0$), and a fold along $x_1 = x_3 = 0$, which is visible if $a_2 < 0$ (since then $\ddot{x}_1 = a_2 < 0$). To study exit points we are therefore interested in the case where one or both of a_1 and a_2 are negative.

The dynamics of (17) have been thoroughly studied (see [8] and references therein), we include it for completeness but shall review only the pertinent features here.

Different values of b_1 and b_2 give topologically different phase portraits. The cases which create exit points are those in which the flow traverses the singularity in finite time, from the attractive sliding region into the repelling sliding region. In all such cases, the flow can follow an infinite number of forward trajectories resulting in determinacy-breaking as illustrated in figure 4 (see [8]); the relevant parameter regimes are listed in the caption. The basic analysis proceeds as follows.

Filippov's convex combination, given by applying (9) to (17), is

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = \frac{1+\lambda_1}{2} (-x_2, a_1, b_1) + \frac{1-\lambda_1}{2} (x_3, b_2, a_2) \\ := (F_1, F_2, F_3),$$

however this is shown in [22] to be structurally unstable inside the switching layer. To obtain a structurally stable

FIG. 4. Determinacy breaking in three different kinds of two-fold. Left figures sketch the piecewise-smooth flow and sliding flow, right figures show a single trajectory exploding into a set-valued flow at the singularity. The set-value flow has 2 dimensions in (i) and 3 dimensions in (ii). The cases are: (i) $a_1 = a_2 = -1$ (visible two-fold) with $b_1 < 0$ or $b_2 < 0$ or $b_1 b_2 < 1$; (ii) $a_1 a_2 = -1$ (mixed two-fold) with $b_1 < 0 < b_2$ and $b_1 b_2 < -1$ or with $b_1 + b_2 < 0$ and $b_1 - b_2 < -2$.

system we can perturb this and write

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = (F_1, F_2, F_3) + (1 - \lambda_1^2)(\alpha, 0, 0) \\ := (f_1, f_2, f_3), \quad (18)$$

for some small constant α . This is consistent with (17) because the term $(1 - \lambda_1^2)\alpha$ vanishes for $\lambda_1 = \pm 1$.

The switching layer system on $x_1 = 0$, obtained by substituting (17) into (11) for $r = 1$, is

$$(\lambda_1', \dot{x}_2, \dot{x}_3) = (f_1, f_2, f_3). \quad (19)$$

The λ_1' subsystem has equilibria at $\lambda_1 = \mathcal{S}(\lambda_1) = (x_3 - x_2)/(x_3 + x_2) + \mathcal{O}(\alpha)$, which form the sliding manifold

$$\mathcal{M}^S = \{(\lambda_1, x_2, x_3) \in [-1, +1] \times \mathbb{R}^2 : \\ -\frac{1+\lambda_1}{2}x_2 + \frac{1-\lambda_1}{2}x_3 + \alpha(1 - \lambda_1^2) = 0\}, \quad (20)$$

illustrated in figure 5. On $\mathcal{M}^S$ the sliding dynamics is given by

$$(\lambda_1', \dot{x}_2, \dot{x}_3) = \frac{(0, b_2 x_2 + a_1 x_3, a_2 x_2 + b_1 x_3)}{x_3 + x_2} + \mathcal{O}(\alpha). \quad (21)$$

FIG. 5. The sliding manifolds $\mathcal{M}^S$ inside the switching layer for the two cases in figure 4. The curve $\mathcal{L}$ is the set where the vertical (λ) direction lies tangent to $\mathcal{M}$, where the attracting (a.sl.) and repelling (r.sl.) branches meet.

The invariance of $\mathcal{M}^S$ breaks down at the folds (on the boundaries of the switching layer where $\lambda_1 = \pm 1$), and also inside the switching layer where (15) (for $r = 1$) is violated, which simplifies to the condition $\frac{\partial \lambda'_1}{\partial \lambda_1} \neq -\frac{1}{2}(x_2 + x_3) - 2\alpha\lambda_1$. Combining this with (20), the invariance of $\mathcal{M}^S$ breaks down on the set

$$\mathcal{L} = \left\{ (\lambda_1, x_2, x_3) \in \mathcal{M}^S : \right. \\ \left. \lambda_1 = 2 \frac{2\alpha + x_3 - x_2}{x_3 + x_2} = -\frac{x_3 + x_2}{4\alpha} \in [-1, +1] \right\}, \quad (22)$$

Either side of $\mathcal{L}$, the two-dimensional curved surface $\mathcal{M}^S$ has an attracting branch in an α -neighbourhood of $x_2, x_3 > 0$, where $\frac{\partial \lambda'_1}{\partial \lambda_1} \Big|_{\mathcal{M}^S} < 0$, and a repelling branch in an α -neighbourhood of $x_2, x_3 < 0$, where $\frac{\partial \lambda'_1}{\partial \lambda_1} \Big|_{\mathcal{M}^S} > 0$. The non-hyperbolicity line $\mathcal{L}$ is a curve with tangent vector $e_{\mathcal{L}} = (1, 2\alpha(\lambda_1 - 1), -2\alpha(\lambda_1 + 1))$.

This means that the quantity α is vital. The set $\mathcal{L}$ is the continuation of the two-fold singularity through the switching layer, and the perturbation α ensures that $\mathcal{L}$ is in a generic position with respect to the flow. If $\alpha = 0$ then $\mathcal{L}$ aligns precisely with the λ'_1 dummy system (i.e. it is vertical in figure 5), constituting a degeneracy of infinite codimension since $\mathcal{L}$ aligns with λ_1 over infinitely many points on $[-1, +1]$, at which the sliding dynamics (12) is undefined. We therefore take $\alpha \neq 0$.

An isolated point singularity may exist along $\mathcal{L}$, where the flow's projection along the λ_1 -direction onto $\mathcal{M}^S$ is indeterminate, defined as the point where

$$f_1 = 0, \quad \frac{\partial f_1}{\partial \lambda_1} = 0, \quad (f_2, f_3) \cdot \frac{\partial f_1}{\partial (x_2, x_3)} = 0. \quad (23)$$

In two-timescale systems like (19), such singularities have been studied in general [32] (where they are called *folded singularities*, an unfortunate clash of nomenclature that we will not use further below). We can make a coordinate transformation that straightens out $\mathcal{L}$ and puts the point singularity at the origin, as derived in [22]. The switching layer system (19) then becomes

$$\begin{cases} y_1' = y_2 + y_1^2 + \mathcal{O}(y_1 y_3), \\ \dot{y}_2 = b y_3 + \tilde{c} y_1 + \mathcal{O}(y_3^2, y_1 y_3), \\ \dot{y}_3 = \tilde{a} + \mathcal{O}(y_3, y_1), \end{cases} \quad (24)$$

which is the canonical form of the singularity [32], where

$$\tilde{a} = f_{3s}, \quad \tilde{b} = -\left(f_{2s} + f_{3s} - 2\tilde{c}\sqrt{|\alpha|}\right)/4|\alpha|, \\ \tilde{c} = ((1 - \lambda_{1s})k_{3s} - (1 + \lambda_{1s})k_{2s})/2\sqrt{|\alpha|},$$

and $f_{2s} = l_{2s} + k_{2s}\lambda_{1s}$, $f_{3s} = l_{3s} + k_{3s}\lambda_{1s}$, $l_{2s} = \frac{1}{2}(a_1 + b_2)$, $l_{3s} = \frac{1}{2}(b_1 + a_2)$, $k_{2s} = \frac{1}{2}(a_1 - b_2)$, $k_{3s} = \frac{1}{2}(b_1 - a_2)$, and λ_{1s} is the solution to (23). As can be seen from the values of these constants, the transformation to obtain the canonical form is only nonsingular if α in (18) is non-vanishing.

The most important factor in determining the role of such exit points is the dimension of the set-valued flow

through the singularity. As shown in figure 4(i), if both tangencies are visible, then only a single sliding trajectory passes through the two-fold, and the flow generated is two-dimensional. This means that a typical orbit is unlikely to pass through the two-fold. In figure 6(i) we simulate an example system

$$\begin{aligned} \mathbf{f}^+ &= (-x_2, \frac{2}{5}x_1 + \frac{1}{10}x_2 - 1, \frac{3}{10}x_2 - \frac{1}{5}x_2x_3 - \frac{2}{5}), \\ \mathbf{f}^- &= (x_3, \frac{1}{5}x_2x_3 - \frac{3}{5}, \frac{2}{5}x_3 - 1 - x_1), \end{aligned} \quad (25)$$

with $\alpha = 1/5$, which has a two-fold at the origin formed by two visible tangencies. This system contains a re-injection to the neighbourhood of the two-fold, which creates periodic or chaotic dynamics as we vary the coefficients. This verifies that, despite intricate local dynamics, no trajectories pass through the two-fold singularity, so the exit point itself does not play a role in the dynamics, though it is the organizing centre of the surrounding attractor.

If one tangency is visible and the other invisible as shown in figure 4(ii), a whole family of sliding trajectories pass through the two-fold, generating a three-dimensional flow. This is therefore a significant feature in the local flow. As the flow passes through the exit point, its ensuing set-valuedness means that in simulations the system is highly sensitive to perturbations of the model itself, or the method of calculation. As an example take the system

$$\begin{aligned} \mathbf{f}^+ &= (-x_2 + \frac{1}{10}x_1, x_1 - c_1, x_1 - 2), \\ \mathbf{f}^- &= (x_3 + c_2x_1, x_1 + c_3, 1 - x_1), \end{aligned} \quad (26)$$

again with $\alpha = 1/5$. As in the last example, this contains a re-injection to the neighbourhood of the two-fold. For different coefficients this creates pseudo periodic or chaotic motion that persists over long times, but in this case the orbits pass through the exit point at the two-fold itself, and closed attractors may not exist. Small changes in parameters or the computational method can then result in very different quantitative behaviour due to determinacy-breaking at the two-fold, and figure 6(ii-iii) shows two examples for different parameters given in the caption. In (ii) a chaotic-like motion persists for long times (more than $t = 1500$ in this simulation), while in (iii), after some time $t > 400$ the orbit begins evolving along a canard trajectory that explores the repelling sliding region, and on the second such excursion diverges to infinity.

The numerical solutions in both examples are obtained by approximating $\lambda_1 = \text{sign}(x_1)$ by $\lambda_1 \approx \tanh(x_1/\varepsilon)$ with $\varepsilon = 10^{-7}$, taking an initial point $(x_1, x_2, x_3) = (0.4, 1, 1.4)$. Although the resulting simulations are highly sensitive (including high sensitivity to step sizes, numerical tolerances, and the choice of sigmoid function), different values result in qualitatively similar behaviour.

The implication of the singularity described by (24) existing inside the switching layer is that, at the heart of the two-fold, lies the discontinuous limit of a two-timescale

FIG. 6. Three examples of attractor organised around a two-fold singularity; examples based on those in [22]. Showing simulations of: (i) the system (25), (ii-iii) the system (26) with (i) $c_1 = 6/5$, $c_2 = 1/10$, $c_3 = 23/100$, and (ii) $c_1 = 11/10$, $c_2 = 1/20$, $c_3 = 21/100$.

singularity responsible for so-called *canard* phenomena [32]. A canard is a trajectory that travels from an attracting to a repelling branch of an invariant manifold, in this case $\mathcal{M}^S$, corresponding to traveling from the attracting to repelling regions of sliding in figure 4. This allows us to interpret determinacy-breaking at the two-fold singularity as the infinite crowding of trajectories that occurs in the singular limit of a deterministic slow-fast system. The different topologies of canards possible may be found in [9, 22, 32].

C. Exit via an intersection: deterministic

Sliding on one switching manifold can also be terminated by transversal intersection with another switching manifold. Even in the simplest example of a codimension $r = 1$ sliding region, terminated by meeting a second switching surface at a codimension $r = 2$ switching intersection (as in figure 3(iii-iv)), there are a huge number of scenarios by which exit can occur. No classification has been attempted to date. Here we describe the typical behaviour that characterises such exit, particularly whether it is deterministic (this section) or determinacy-breaking (in section III D).

Consider, without loss of generality, a sustained interval of sliding on $x_1 = 0 > x_2$, terminated by a second switching surface $x_2 = 0$. A trajectory may exit into one of the two regions $x_1, x_2 > 0$ or $x_2 < 0 < x_1$ (exit into $x_1 < 0$ is impossible because the flow is attracting towards $x_1 = 0 > x_2$ by assumption), or into one of the three switching surface regions $x_1 = 0 < x_2$, $x_2 = 0 < x_1$, $x_2 = 0 > x_1$. Provided that exit is possible into only one of these regions at $x_1 = x_2 = 0$, the system may remain deterministic, in the form represented by figure 3(iii), as we consider below. If exit is possible into more than one

such region then determinacy is broken, and we consider that in the following section III D.

As a structural model of deterministic exit at an intersection, consider the piecewise-constant system

$$(\dot{x}_1, \dot{x}_2) = \begin{cases} \mathbf{f}^{++} = \mathbf{f}^{--} = (1, 1) & \text{if } x_1 x_2 > 0, \\ \mathbf{f}^{-+} = \mathbf{f}^{+-} = (1, -1) & \text{if } x_1 x_2 < 0, \end{cases} \quad (27)$$

for which (8) simplifies to

$$(\dot{x}_1, \dot{x}_2) = \frac{1}{2}(1 - \lambda_1 + \lambda_2 + \lambda_1 \lambda_2, 1 + \lambda_1 - \lambda_2 + \lambda_1 \lambda_2).$$

Sliding occurs in the regions $x_2 = 0 > x_1$ and $x_1 = 0 > x_2$, and flows towards the switching intersection $x_1 = x_2 = 0$. Crossing occurs on $x_2 = 0 < x_1$ and $x_1 = 0 < x_2$. The result is that all trajectories flow eventually into the region $x_1, x_2 > 0$, and trajectories that slide initially and exit at the intersection do so along a common trajectory $\{x_1(t), x_2(t)\} = \{t, t\}$, $t \geq 0$.

A switching layer system (11) can be taken separately on each region of the switching surface, using $r = 1$ on $x_2 = 0 > x_1$, $x_2 = 0 < x_1$, $x_1 = 0 > x_2$, $x_1 = 0 < x_2$, and using $r = 2$ on the intersection $x_1 = x_2 = 0$. The invariant manifold $\mathcal{M}^S$ exists in the sliding regions on the codimension $r = 1$ switching surfaces. The switching layer system at the intersection is

$$(\lambda'_1, \lambda'_2) = \frac{1}{2}(1 - \lambda_1 + \lambda_2 + \lambda_1 \lambda_2, 1 + \lambda_1 - \lambda_2 + \lambda_1 \lambda_2),$$

in which the flow converges to the trajectory $\{\lambda_1(\tau), \lambda_2(\tau)\} = \{\tau, \tau\}$, $-1 < \tau < +1$, and the nearby flow carries trajectories from the sliding regions onto the exit trajectory $\{x_1(t), x_2(t)\} = \{t, t\}$, $t \geq 0$.

This is rather simple because the flow is single-valued. Various other scenarios may be studied, but they generate little of interest for deeper study here. In particular one may consider

$$\begin{aligned} \mathbf{f}^{++} &= (-1, 1), & \mathbf{f}^{--} &= (1, 1), \\ \mathbf{f}^{-+} &= (1, 1), & \mathbf{f}^{+-} &= (-1, 1), \end{aligned}$$

where trajectories slide along $x_1 = 0 > x_2$ into the intersection, and exit via sliding along $x_1 = 0 < x_2$, or

$$\begin{aligned} \mathbf{f}^{++} &= (2, -1), & \mathbf{f}^{--} &= (1, 1), \\ \mathbf{f}^{-+} &= (1, -1), & \mathbf{f}^{+-} &= (-1, 1), \end{aligned}$$

where trajectories slide along $x_1 = 0 > x_2$ and $x_2 = 0 > x_1$ into the intersection, and exit via sliding along $x_1 = 0 < x_2$. Both cases are deterministic. An attracting branch of a sliding manifold $\mathcal{M}^S$ exists in each sliding region, and the different branches are connected by trajectories passing through the intersection in finite time. The analysis of these is quite straightforward, and the steps are similar to those above.

D. Exit via a switching intersection: determinacy-breaking

As a structural model of determinacy-breaking exit from codimension $r = 1$ sliding at a codimension $r = 2$

intersection, illustrated in figure 3(iv), consider

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = \begin{cases} \mathbf{f}^{++} = \mathbf{f}^{--} = (1, x_3 + 1, 0) & \text{if } x_1 x_2 > 0, \\ \mathbf{f}^{-+} = \mathbf{f}^{+-} = (1, x_3 - 1, 0) & \text{if } x_1 x_2 < 0, \end{cases} \quad (28)$$

for $|x_3| < 1$. We will show that this exhibits determinacy-breaking, but that the lack of determinacy is partially resolved by the switching layer dynamics. The equality between diagonally opposite vector fields in (28) is for economy here, and has no bearing on the results (small constant, linear, or nonlinear terms can be added to any of the four vector fields without significant effect).

The canopy combination (8) applied to (28) simplifies to

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = (1, x_3 + \lambda_1 \lambda_2, 0) \quad (29)$$

where $\lambda_i = \text{sign}(x_i)$.

Substituting into (12) with $r = 1$, it is easily seen that trajectories in $x_1 < 0$ reach the intersection in finite time via sliding on $x_2 = 0 \geq x_1$. The trajectories lying on planes $x_2/x_1 = x_3 \pm 1$ reach the intersection directly without sliding. Similarly, trajectories in $x_1 > 0$ depart the intersection in finite time via sliding on $x_2 = 0 \leq x_1$, and trajectories on the planes $x_2/x_1 = x_3 \pm 1$ depart directly without sliding.

The line $x_1 = x_2 = 0$ is a determinacy-breaking singularity. From an inspection of the phase portrait outside the surfaces, and the sliding portrait on $x_1 = 0$, it appears that all trajectories in the region $x_3 - 1 \leq x_2/x_1 \leq x_3 + 1$ pass through the intersection $x_1 = x_2 = 0$ (see figure 7(i)). They form a continuum of trajectories all flowing into and out of the intersection in finite time. Any point in this set with $x_1 < 0$ is connected via the flow to any point in this set with $x_1 > 0$ with the same x_3 value. We shall have to inspect the switching layer dynamics to verify whether all of these orbits actually exist through the intersection. Outside the region $x_3 - 1 \leq x_2/x_1 \leq x_3 + 1$, at least, the system is deterministic.

The switching layer system on $x_1 = 0$ for $x_2 \neq 0$, given by (11) with $r = 1$, is

$$(\lambda'_1, \dot{x}_2, \dot{x}_3) = (1, x_3 + \lambda_1 \text{sign}(x_2), 0), \quad (30)$$

with $\lambda_1 \in [-1, +1]$. The λ'_1 equation is constant, so this system provides a simple transition between the surfaces $\lambda_1 = -1$ and $\lambda_1 = +1$ on the dummy (prime) timescale.

The switching layer system on $x_2 = 0$ for $x_1 \neq 0$, given again by (11) with $r = 1$ but adapted so that the switching surface is $x_2 = 0$, is

$$(\dot{x}_1, \lambda'_2, \dot{x}_3) = (1, x_3 + \lambda_2 \text{sign}(x_1), 0), \quad (31)$$

with $\lambda_2 \in [-1, +1]$. The λ'_2 equation has a set of x_3 -parameterized equilibria $\lambda_2 = -x_3 \text{sign}(x_1)$, which are normally hyperbolic since $\partial \lambda'_2 / \partial \lambda_2 = \text{sign}(x_1)$, forming simple planar invariant surfaces which are attracting for $x_1 < 0$ and repelling for $x_1 > 0$. These are the sliding

FIG. 7. Determinacy breaking at a switching intersection. (i) shows the discontinuous system, (ii) shows the blow up system in the plane $x_3 = 0$. The trajectory of any point p in $x_1 < 0$ becomes multi-valued as it exits the intersection, identifiable as the set $x_{1,2} = 0$ in (a) and $\lambda_{1,2} \in [-1, +1]$, in (b).

manifolds

$$\mathcal{M}^S = \left\{ (x_1, \lambda_2, x_3) : \begin{array}{l} x_1 \neq 0, |x_3| < 1, \\ \lambda_2 = -x_3 \text{sign}(x_1) \end{array} \right\} \quad (32)$$

of the dynamics on $x_2 = 0$. On $\mathcal{M}^S$ the system obeys the sliding dynamics

$$(\dot{x}_1, 0, \dot{x}_3) = (1, x_3 + \lambda_1 \lambda_2, 0). \quad (33)$$

This gives a constant drift in the positive x_1 direction on $\mathcal{M}^S$ inside $x_2 = 0$, with $\lambda_2 = -x_3 \text{sign}(x_1)$.

The switching layer system on the intersection $x_1 = x_2 = 0$, given by (12) with $r = 2$, is

$$(\lambda'_1, \lambda'_2, \dot{x}_3) = (1, x_3 + \lambda_2 \text{sign}(x_1), 0), \quad (34)$$

for $\lambda_1, \lambda_2 \in [-1, +1]$, which has solution trajectories satisfying

$$\lambda_2(\lambda_1) = e^{\lambda_1^2/2} \left(\lambda_{20} e^{-\lambda_{10}^2/2} + x_3 \sqrt{\frac{\pi}{2}} \times \left\{ \text{Erf} \left[\frac{\lambda_1}{\sqrt{2}} \right] - \text{Erf} \left[\frac{\lambda_{10}}{\sqrt{2}} \right] \right\} \right), \quad (35)$$

where Erf is the standard error function [1]. The λ'_2 equation in (34) has a nullcline $\lambda_1 \lambda_2 = -x_3$ on which $\frac{\partial \lambda'_2}{\partial \lambda_1} = \lambda_2 = -x_3/\lambda_1$. The nullcline diverges and leaves the region $\lambda_{1,2} \in [-1, +1]$, existing only for $|\lambda_{1,2}| > |x_3|$. The nullcline is structurally stable with respect to the flow,

having a gradient vector $\left(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \lambda_2}, \frac{\partial}{\partial x_3}\right) \lambda'_2 = (\lambda_2, \lambda_1, 1)$ throughout $\lambda_{1,2} \in [-1, +1]$.

The continuation of the attracting and repelling planes of $\mathcal{M}^S$ (given by $\lambda_2 = \pm x_3$ in $x_1 \lessgtr \mp 1$) into the region $\lambda_{1,2} \in [-1, +1]$ are given from (35) by

$$\lambda_2(\lambda_1) = x_3 e^{\lambda_1^2/2} \left(\pm e^{-1/2} + \sqrt{\frac{\pi}{2}} \times \left\{ \text{Erf} \left[\frac{\lambda_1}{\sqrt{2}} \right] \pm \text{Erf} \left[\frac{1}{\sqrt{2}} \right] \right\} \right). \quad (36)$$

This implies that the flow from the attracting plane of $\mathcal{M}^S$ curves towards negative λ_2 in $x_3 < 0$, and towards positive λ_2 in $x_3 > 0$, thus exiting either into the region $x_1, x_2 > 0$ in $x_3 < 0$ or into the region $x_2 < 0 < x_1$ in $x_3 > 0$. In fact, upon reaching either $\lambda_1 = +1$ or $\lambda_2 = +1$, the λ'_2 and λ'_1 dynamics respectively, given by (31) and (30), drive the flow into the corners where $\lambda_1 = +1$ and $\lambda_2 = \text{sign}(x_3)$.

The dynamics is illustrated in figure 8 for $x_3 < 0$. The splitting in the x_2 direction between the attracting and repelling manifolds (36) inside the intersection depends linearly on x_3 , given by $\Delta \lambda_2(\lambda_1) = x_3 e^{\lambda_1^2/2} \left(2e^{-1/2} + \sqrt{2\pi} \text{Erf} \left[\frac{1}{\sqrt{2}} \right] \right)$. There exists a unique solution trajectory given by

$$\lambda_1(t) = t, \quad \lambda_2(t) = 0, \quad x_3(t) = 0, \quad (37)$$

in the region $\lambda_2 \in [-1, +1]$, valid for all t and hence running along the λ_1 coordinate axis. This is a *canard* trajectory, meaning an orbit that passes from an attracting invariant manifold to a repelling invariant manifold, spending $\mathcal{O}(1)$ time on each. In this case the canard passes from the attracting plane $\lambda_2 = x_3$ for $\lambda_1 < -1$ to the repelling plane $\lambda_2 = x_3$ for $\lambda_1 > +1$. There is only one such trajectory, and it is structurally stable, because the attracting and repelling branches of $\mathcal{M}^S$ intersect transversally at $\lambda_1 = \lambda_2 = x_3 = 0$. Note that the existence of a single canard, rather than every trajectory on $x_2 = 0$ being a canard, is evident only from this switching layer analysis, and cannot be seen by inspecting the dynamics outside the switching surface (figure 7(i)) alone.

One trajectory therefore exists that passes through the intersection and remains asymptotic to $x_2 = 0$ as $x_1 \rightarrow \pm\infty$. All trajectories that enter the intersection are expelled via the point $\lambda_1 = +1, \lambda_2 = \text{sign}(x_3)$, depending on the value of x_3 along them. (Conversely, all trajectories that travel along the repelling sliding region can be followed back in time to the point $\lambda_1 = -1, \lambda_2 = -\text{sign}(x_3)$, depending on their x_3 values).

In the (x_1, x_2) plane with x_3 fixed, the structural model above shows that different values of x_3 give qualitatively different dynamics, and determinacy-breaking occurs only at $x_3 = 0$. In three dimensions the different scenarios unfold to create a structurally stable singularity, and at its heart, a canard trajectory (37) through the intersection, hidden inside the switching layer.

Numerous other scenarios exhibit similar behaviour and yield to similar analysis, consider for example $\mathbf{f}^{--} =$

FIG. 8. Sketch of the switching layer system for simple exit from a crossing of switching surfaces. In (ii) we show the layer of the intersection, as well as the switching layers along $x_1 = 0$ (for $x_2 \neq 0$) and $x_2 = 0$ (for $x_1 \neq 0$), and the dynamics outside the switching surfaces, the case shown is in a coordinate plane with constant $x_3 < 0$; and in (i) we show the invariant manifold $\mathcal{M}^S$ inside the switching layer of the intersection point $x_1 = x_2 = 0$.

$(1, x_3 + 1, 0)$, $\mathbf{f}^{+-} = (1, -1, 0)$, and $\mathbf{f}^{-+} = (-1, -2, 0)$, with either $\mathbf{f}^{++} = (1, 3, 0)$ or $\mathbf{f}^{++} = (-1, 2, 0)$, in both of which there is similar determinacy-breaking passage through the intersection, which can be resolved except at a special value of x_3 . In these examples there is also re-injection of the set-valued flow back into the singularity, resulting in complex oscillatory dynamics in the neighbourhood of the intersection.

We shall not look in detail at further examples, but conclude with a simulation to demonstrate the effect of such a determinacy-breaking exit point. Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 - \lambda_2 x_2 + \frac{1}{5} \lambda_1 \\ x_3 + \lambda_1 \lambda_2 - c \lambda_2 \\ -\frac{1}{10} x_3 - \frac{1}{5} x_2 \end{pmatrix} \quad (38)$$

where c is a constant in the range $0 < c < 1$. This provides an example of the global dynamics induced by a local singularity of the form (28), having the same qualitative phase portrait near the intersection $x_1 = x_2 = 0$.

First, observe that there is little qualitative difference between the phase portraits (figure 9) of (38) for different values of c . The simulations below will reveal that very different dynamics is seen depending on c , due to sensitivity in the flow's exit from the intersection.

In figure 10 we simulate (38) by approximating $\lambda_i = \text{sign}(x_i)$ with $\lambda_i \approx \phi(x_i/\varepsilon) = \tanh(x_i/\varepsilon)$ for $i = 1, 2$, with $\varepsilon = 10^{-4}$. The result is periodic or chaotic dynamics for different parameters. The flow enters the origin by sliding along $x_1 < 0 = x_2$, then exits into positive x_2 for $x_3 > 0$ and into negative x_2 for $x_3 < 0$ (this is verified from closer inspection of the simulations, not shown). This is as predicted from the switching layer analysis above. The result in (i) is a simple periodic orbit, and as we vary c the period of this attractor changes rapidly, becoming eventually the complex attractor in (ii). (In (ii-b) trajectories are also seen that cross the half-plane $x_2 = 0 < x_1$, which have strayed to large enough x_3

FIG. 9. The flow of (38) in a plane $x_3 = \text{constant}$ for small x_3 . The phase portrait does not change qualitatively outside the switching surfaces $x_1 x_2 = 0$ for different values of c .

FIG. 10. An attractor driven through an intersection exit point: a simulation of (38) with (i) $c = \frac{3}{10}$ and (ii) $c = \frac{2}{5}$. The full three-dimensional simulation and its projection into the (x_1, x_2) plane are shown.

that $x_2 = 0$ is no longer a sliding region, so the flow crosses through transversally). Any trajectories that pass through $x_1 = 0$ cross it transversally (in the positive x_1 direction near the intersection, but also in the negative x_1 direction at large x_3 values which allows the flow in (ii) to loop around more intricately). Any trajectories that hit the half-plane $x_2 = 0 > x_1$ do so at small enough x_3 that they then slide along $x_2 = 0$ into the singularity.

To verify that the dynamics observed is a result of the singularity geometry, and not of the choice of smoothing in the simulation, we can simulate the same system for the same parameters, but approximate the switch by different sigmoid functions (we could also take different values of $0 < \varepsilon \ll 1$, and introduce hysteresis, delay, or noise, with similar results). In figure 11 we repeat the simulation (showing only the three-dimensional

image) with the smooth rational function $\phi(x_i/\varepsilon) = (x_i/\varepsilon)/\sqrt{1+(x_i/\varepsilon)^2}$ in (i-ii.a), and the continuous but non-differentiable ramp function $\phi(x_i/\varepsilon) = \text{sign}(x_i)$ for $|x_i| > \varepsilon$ and $\phi(x_i/\varepsilon) = x_i/\varepsilon$ for $|x_i| \leq \varepsilon$ in (i-ii.b). These demonstrate that the choice of smoothing has no significant effect upon the dynamics, and is not responsible for the complex dynamics observed.

FIG. 11. The attractors in figure 10 for the smooth rational (a) and ramp (b) smoothings described in the text, with parameters and initial conditions corresponding to those in figure 10.

We have considered what happens when a codimension $r = 1$ sliding flow arrives either at a tangency or a codimension $r = 2$ intersection. A codimension $r = 1$ sliding trajectory will not generically encounter an intersection of codimension $r \geq 3$ (i.e. where three or more switching manifolds intersect). This therefore completes our study of the basic generic mechanisms for exit from codimension $r = 1$ sliding.

IV. EXIT FROM CODIMENSION TWO SLIDING

As we add more dimensions, and more switches, phenomena will occur at higher codimension that are analogous to the four kinds analysed above. For example, trajectories sliding along an intersection with codimension $r = 2$ may exit to codimension $r = 1$ sliding by intersection with a third switching manifold, analogous to the cases in sections III C-III D. In the section below we look at the less obvious scenario of how tangential exit points extend to higher codimensions, for which the principles above extend rather powerfully. We also consider a new case that is introduced, that of exit by spiralling around a codimension $r = 2$ sliding region.

A. Tangential exit from an intersection

To study exit from sliding via a simple tangency of the flow to an intersection, take as a structural model

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = \begin{cases} \mathbf{f}^{++} = (x_3 + 1, -1, 1) & \text{if } 0 < x_1, x_2, \\ \mathbf{f}^{-+} = (1, -1, 0) & \text{if } x_1 < 0 < x_2, \\ \mathbf{f}^{+-} = (-1, 1, 0) & \text{if } x_2 < 0 < x_1, \\ \mathbf{f}^{--} = (1, 1, 0) & \text{if } x_1, x_2 < 0, \end{cases} \quad (39)$$

whose geometry is sketched in figure 12, for $x_3 > -1$.

FIG. 12. Sketch of the system with vector fields (39), showing projections of the vector fields into three constant x_3 planes, and the sliding dynamics on three half-planes and on the intersection.

The canopy (6) of the component vector fields $\mathbf{f}^{\pm\pm}$ gives

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = \frac{1+\lambda_2}{2} \left(\frac{1+\lambda_1}{2} x_3 + 1, -1, \frac{1+\lambda_1}{2} \right) + \frac{1-\lambda_2}{2} (-\lambda_1, 1, 0) \quad (40)$$

with $\lambda_1, \lambda_2 \in [-1, +1]$, and $\lambda_i = \text{sign}(x_i)$ for $x_1, x_2 \neq 0$.

First let us find the dynamics of the codimension $r = 1$ surfaces, i.e. excluding the intersection. By applying (12)-(13) for $r = 1$ on $x_1 = 0$ and $x_2 = 0$ separately, to derive sliding modes if they exist, we find:

- $x_1 = 0 < x_2$ is a crossing region for $x_3 < 1$ since $f_1^{++} f_1^{-+} = 1 + O(x_3) > 0$;
- $x_1 = 0 > x_2$ is a sliding region since $f_1^{+-} f_1^{--} = -1 < 0$, the sliding modes satisfy $\lambda_1 = \mathcal{S}(\lambda_1) = 0$, giving a sliding system $(\lambda'_1, \dot{x}_2, \dot{x}_3) = (0, 1, 0)$;
- $x_2 = 0 \neq x_1$ is a sliding region since $f_2^{++} f_2^{+-} = -1 < 0$ on $x_2 = 0 < x_1$ and $f_2^{-+} f_2^{--} = -1 < 0$ on $x_2 = 0 > x_1$, the sliding modes in both regions satisfy $\lambda_2 = \mathcal{S}(\lambda_2) = 0$, giving sliding systems $(\dot{x}_1, \lambda'_2, \dot{x}_3) = (x_3, 0, 1)$ and $(\dot{x}_1, \lambda'_2, \dot{x}_3) = (1, 0, 0)$ respectively.

At the intersection $x_1 = x_2 = 0$, applying (12)-(13) with $r = 2$, sliding modes exist only for $x_3 < 0$, with

$(\lambda_1, \lambda_2) = \mathcal{S}(\lambda_1, \lambda_2) = ((x_3 + 2)/(x_3 - 2), 0)$, giving one-dimensional dynamics $(\lambda'_1, \lambda'_2, \dot{x}_3) = (0, 0, 1)/(2 - x_3)$.

The outcome of the sliding analysis is that trajectories in $x_3 < 0$ are attracted onto the sliding surfaces $x_1 = 0 > x_2$ and $x_2 = 0 \neq x_1$, and thence attracted onto the intersection $x_1 = x_2 = 0$ where they travel towards the origin. At the origin the intersection ceases to admit sliding, and trajectories exit along the sliding system $(\dot{x}_1, \lambda'_2, \dot{x}_3) = (x_3, 0, 1)$ on $x_2 = 0 < x_1$, which at the origin is tangent to the intersection as sketched in figure 12.

As for the visible tangency in section III A, here we have a visible tangency of the sliding flow to the intersection, and the exit is deterministic. Let us briefly analyse what happens inside the switching layer in analogy to section III A.

The switching layer system (11) on $x_1 = 0$ is

$$(\lambda'_1, \dot{x}_2, \dot{x}_3) = \begin{cases} \left(\frac{1}{2}(1 + \lambda_1)x_3 + 1, -1, \frac{1}{2}(1 + \lambda_1) \right) & \text{if } x_2 > 0, \\ (-\lambda_1, 1, 0) & \text{if } x_2 < 0, \end{cases} \quad (41)$$

with $\lambda_1 \in [-1, +1]$, illustrated in figure 13. For $x_2 < 0$ the set $\lambda_1 = 0$ forms an attracting sliding manifold $\mathcal{M}^S$, whose sliding vector field $(\lambda'_1, \dot{x}_2, \dot{x}_3) = (0, 1, 0)$, so all trajectories flow into the intersection in finite time. For $x_2 > 0$ there is no sliding, instead the dummy system $\lambda'_1 = 1 + O(x_3)$ carries the flow across the switching surface in the direction of increasing x_2 , at least for small x_3 .

The switching layer system on $x_2 = 0$ is

$$(\dot{x}_1, \lambda'_2, \dot{x}_3) = \begin{cases} \left(\frac{1}{2}(1 + \lambda_2)x_3 + \lambda_2, -\lambda_2, \frac{1}{2}(1 + \lambda_2) \right) & \text{if } x_1 > 0, \\ (1, -\lambda_2, 0) & \text{if } x_1 < 0, \end{cases} \quad (42)$$

with $\lambda_2 \in [-1, +1]$. The set $\lambda_2 = 0$ forms an attractive sliding manifold $\mathcal{M}^S$ for all $x_1 \neq 0$ and all x_3 , on which the sliding vector field is $(\dot{x}_1, \lambda'_2, \dot{x}_3) = (x_3/2, 0, 1/2)$ for $x_1 > 0$ and $(1, 0, 0)$ for $x_1 < 0$. The $\dot{x}_1$ component implies that the sliding flow enters the intersection from $\mathcal{M}^S$ for $x_3 < 0$, but for $x_3 > 0$ crosses through the intersection in the direction of increasing x_1 along $\mathcal{M}^S$.

The attraction of dynamics towards $x_1 = 0$ and $x_2 = 0$ implies that the switching layer there should possess a sliding manifold $\mathcal{M}^S$ for $x_3 < 0$. The switching layer system on the intersection $x_1 = x_2 = 0$, given by (11) with $r = 2$, is

$$(\lambda'_1, \lambda'_2, \dot{x}_3) = \frac{1+\lambda_2}{2} \left(\frac{1+\lambda_1}{2} x_3 + 1, -1, \frac{1+\lambda_1}{2} \right) + \frac{1-\lambda_2}{2} (-\lambda_1, 1, 0) \quad (43)$$

with $\lambda_1, \lambda_2 \in [-1, +1]$. For $x_3 < 0$ this has an attracting sliding manifold $\mathcal{M}^S$ consistent with (14) along the line $\lambda_1 = \frac{2+x_3}{2-x_3}$, $\lambda_2 = 0$, along which the flow follows the one-dimensional system $\dot{x}_3 = 1/(2 - x_3)$. When the flow enters the intersection in the region $x_3 < 0$ it collapses onto $\mathcal{M}^S$ and travels towards $x_3 = 0$, where $\mathcal{M}^S$ leaves the region $\lambda_1, \lambda_2 \in [-1, +1]$. Inside the intersection the

FIG. 13. Sketch of the switching layer dynamics of the system (39), showing an attracting sliding manifold $\mathcal{M}^S$ consisting of curves in the regions $x_2 = 0 < x_1$, $x_2 = 0 > x_1$ and $x_1 = 0 > x_2$, and a point inside $x_1 = x_2 = 0$ for $x_3 < 0$ (this refers to curves and points in $\mathbb{R}^2$, which are of course surfaces and curves respectively in the full $\mathbb{R}^3$).

flow is still attracted towards the line $\lambda_2 = 0$, on which $\lambda_1' = \frac{1}{2}(1 - \lambda_1) + \frac{1}{4}x_3(1 + \lambda_1)$ is strictly positive for $x_3 > 0$. This directs the flow out of the intersection, into sliding on the switching surface $x_2 = 0 < x_1$.

As in the previous cases, one may construct many other examples that exhibit similar behaviour, the only key features being that: a sliding mode exists on the intersection for some values of x_3 , the codimension $r = 1$ sliding flow has a visible tangency to the codimension $r = 2$ intersection, and the exit of the sliding mode corresponds to an equilibrium exiting from the switching layer of the intersection. The exit is deterministic.

One may build up a hierarchy of intersections and sliding modes of successively higher codimension r , and exit points from the intersections via tangency of the codimension $r - 1$ sliding vector field. By a series of such points a trajectory may cascade down from sliding along a high codimension intersection to lower and lower codimension, eventually releasing from the switching surface altogether. Each of these exit events should behave similar to that above, that is, deterministically, and each decreasing the sliding codimension by one. Coincidences of many such events could decrease the codimension by more than one, however, accompanied by determinacy-breaking, as in the following section.

B. Two-fold exit from an intersection

To study a double tangency to an intersection, consider the structural model

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4) = \begin{cases} \mathbf{f}^{++} = (1 + x_3, -1, a_1, b_1) & \text{if } 0 < x_1, x_2, \\ \mathbf{f}^{-+} = (+1, -1, 0, 0) & \text{if } x_1 < 0 < x_2, \\ \mathbf{f}^{+-} = (-1, +1, 0, 0) & \text{if } x_2 < 0 < x_1, \\ \mathbf{f}^{--} = (d - x_4, -d, b_2, a_2) & \text{if } x_1, x_2 < 0, \end{cases} \quad (44)$$

in terms of constants $d = \pm 1$, $a_i = \pm 1$ and $b_i \in \mathbb{R}$. It is necessary here to consider four dimensions, as multiple tangencies to a switching intersection do not occur generically in $\mathbb{R}^3$. We restrict attention to a neighbourhood of the origin $|x_3| < 1$, $|x_4| < 1$.

Figure 14 illustrates the basic dynamics in the (x_1, x_2) plane in different regions of (x_3, x_4) space. Of the four regions of the switching surface $\{x_1 = 0 < x_2\}$, $\{x_1 = 0 > x_2\}$, $\{x_2 = 0 < x_1\}$, $\{x_2 = 0 > x_1\}$, two exhibit crossing, and two exhibit sliding. For $d = -1$ the two sliding regions are coplanar (on $x_1 = 0$), for $d = +1$ they are orthogonal.

- The coplanar case $d = -1$:

At $x_1 = 0$ the flow crosses the switching surface, since $f_1^{++}f_1^{-+} = 1 + x_3 > 0$ in $x_2 > 0$ and $f_1^{+-}f_1^{--} = 1 + x_4 > 0$ in $x_2 < 0$.

The $x_2 = 0$ hyperplane is an attracting sliding region for all $x_2 \neq 0$ since $f_2^{++}f_2^{+-} = -1 < 0$ in $x_1 > 0$ and $f_2^{-+}f_2^{--} = -1 < 0$ in $x_1 < 0$. The sliding modes from (13) are given by $\mathcal{S}(\lambda_2) = 0$, and give dynamics

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4) = \begin{cases} (x_3, 0, a_1, b_1)/2 & \text{if } x_1 > 0, \\ (-x_4, 0, b_2, a_2)/2 & \text{if } x_1 < 0, \end{cases} \quad (45)$$

on $x_2 = 0$.

The intersection exhibits sliding for $x_3x_4 > 0$. By (13) the sliding modes are given by $\mathcal{S}(\lambda_1) = \frac{x_4 - x_3}{x_4 + x_3}$ and $\mathcal{S}(\lambda_2) = 0$ (recall by (13) these must both be inside $[-1, +1]$ hence they exist only for $x_3x_4 > 0$), giving dynamics

$$(\dot{x}_3, \dot{x}_4) = \frac{(a_1x_4 + b_2x_3, a_2x_3 + b_1x_4)}{j(x_3, x_4)} \quad (46)$$

on $x_1 = x_2 = 0$, where $j(x_3, x_4) = 2(x_3 + x_4)$ satisfies $x_3, x_4 > 0 \Rightarrow j(x_3, x_4) > 0$ and $x_3, x_4 < 0 \Rightarrow j(x_3, x_4) < 0$. For $x_3x_4 < 0$ the flow therefore crosses through the intersection, from one sliding region to another. There exists a singularity at $x_3 = x_4 = 0$ where these sliding modes are undefined.

FIG. 14. Dynamics in the (x_1, x_2) plane. As x_3 and x_4 change sign the fields $\mathbf{f}^{00}$ and $\mathbf{f}^{--}$ rotate, and their directions relative to $\mathbf{f}^{+-}$ and $\mathbf{f}^{++}$ change whether the sliding vector fields point towards or away from the intersection $x_1 = x_2 = 0$.

- The orthogonal case $d = +1$:

On $x_1 = 0$, for $x_2 > 0$ the flow crosses the switching surface since $f_1^{++}f_1^{--} = 1 + x_3 > 0$. For $x_2 < 0$ we have $f_1^{+-}f_1^{--} = x_4 - 1 < 0$, which by (13) has sliding modes $\mathcal{S}(\lambda_1) = \frac{x_4}{x_4 - 2}$, with dynamics

$$(\dot{x}_2, \dot{x}_3, \dot{x}_4) = \frac{(-x_4, b_2, a_2)}{2 - x_4} \quad (47)$$

on $x_1 = 0 > x_2$.

On $x_2 = 0$, for $x_1 < 0$ the flow crosses the switching surface since $f_2^{++}f_2^{--} = 1 > 0$. For $x_1 > 0$ we have $f_2^{+-}f_2^{--} = -1 < 0$, which by (13) has sliding modes $\mathcal{S}(\lambda_2) = 0$, giving dynamics

$$(\dot{x}_1, \dot{x}_3, \dot{x}_4) = (x_3, 0, a_1, b_1)/2$$

on $x_2 = 0 < x_1$.

Both of these sliding regions are attracting. The intersection exhibits sliding for $x_3x_4 > 0$, where the sliding modes satisfy $\mathcal{S}(\lambda_1) = 2x_4/j(x_3, x_4)$ and

$\mathcal{S}(\lambda_2) = -2x_3/j(x_3, x_4)$, giving

$$(\dot{x}_3, \dot{x}_4) = \frac{(a_1x_4 + b_2x_3, a_2x_3 + b_1x_4)}{j(x_3, x_4)}$$

on $x_1 = x_2 = 0 < x_3x_4$, where $j(x_3, x_4) = \frac{-4x_3x_4}{x_3 + x_4 + \sqrt{(x_3 + x_4)^2 + 4x_3x_4}}$ and $x_3x_4 > 0 \Rightarrow j(x_3, x_4) > 0$. For $x_3x_4 < 0$ the flow crosses through the intersection from one sliding region to another.

The curvature of the flow towards or away from the intersection is characterised by $\ddot{x}_1$ on $x_2 = 0$ or $\ddot{x}_2$ on $x_1 = 0$. Along the set $x_3 = 0$ we have $\ddot{x}_1 = a_1$ for $x_2 = 0 < x_1$. Along the set $x_4 = 0$ we have $\ddot{x}_1 = -a_2$ on $x_2 = 0 > x_1$ and $\ddot{x}_2 = -a_2$ on $x_1 = 0 > x_2$. The result is that both tangencies are of *visible* type for $a_1 = a_2 = +1$ (curving away from the intersection in both sliding regions), *invisible* type for $a_1 = a_2 = -1$ (curving towards the intersection in both sliding regions), and of mixed type for $a_1a_2 = -1$ (one curves towards and one away from the intersection in either sliding region). This curvature also implies, as seen in figure 14, that the intersection is attracting with respect to the sliding dynamics for $x_3, x_4 < 0$, repelling for $x_3, x_4 > 0$, and the flow crosses between sliding regions at the intersection for $x_3x_4 < 0$.

The switching between the two sliding regions, each of dimension three on (x_1, x_3, x_4) or (x_2, x_3, x_4) space, closely mimics the switching between two regions on (x_1, x_2, x_3) space in the two-fold of section III B; an example comparable to figure 4 is sketched in figure 15. In fact, the sliding vector field on the intersection given by (45) and (47) on (x_3, x_4) space, both expressible as

$$(\lambda'_1, \lambda'_2, \dot{x}_3, \dot{x}_4) \propto \frac{(0, 0, b_2x_3 + a_1x_4, a_2x_3 + b_1x_4)}{j(x_3, x_4)}, \quad (48)$$

are equivalent up to time scaling to (21), i.e. the canonical form sliding vector field of a two-fold singularity on the switching surface $x_1 = 0$ of a system in (x_1, x_2, x_3) space. Note we neglect the term of order α from (21) here; we will remark on this below.

The system of sliding resulting from (44) differs from the two-fold in one important aspect, the sign of the time scaling $j(x_3, x_4)$. That time scaling crucially changes the character of the singularity at $x_3 = x_4 = 0$. The singularity for the ‘coplanar’ case $d = +1$ may be called a *bridge point*, forming a bridge between attracting and repelling sliding regions, while for the ‘orthogonal’ case $d = -1$ it may be called a *jamming point*, an equilibrium that the flow may reach or depart in *finite* time. This is shown as follows.

The phase portraits of (48) are that of a linear equilibrium at $x_3 = x_4 = 0$, which takes the form of a node, focus, or saddle depending on a_i and b_i . Because of the time scaling this is actually a false equilibrium, and we must consider how $j(x_3, x_4)$ affects the dynamics nearby. For $d = +1$, similar to the two-fold singularity, the time

FIG. 15. Sketch of x_3 - x_4 sliding dynamics. The visible-visible case of folds is shown with a saddle-like case of sliding dynamics. A complete catalogue of the possible two-dimensional sliding topologies in this codimension $r = 2$ surface is equivalent to the two-dimensional sliding phase portraits for the codimension $r = 1$ two-folds in [8].

scaling is positive in the attracting sliding region and negative in the repelling sliding region. This time scaling becomes zero at the origin, such that the vector field remains finite and nonzero there, permitting the flow to pass in finite time from one sliding region to another. For $d = -1$ the time scaling is strictly negative in both sliding regions, becoming zero at the origin such that the vector field remains finite and nonzero, so if they are attracted to/repelled from the singularity, they reach/depart it in finite time.

This comparison to the two-fold singularity reveals the basic character of the singularity at the origin of the system above. Firstly the singularity exhibits determinacy-breaking. Secondly, the system is degenerate, and to obtain structural stability requires the addition of a nonlinear term $(1 - \lambda_1^2)(\alpha, 0, 0, 0)$ for some small α . For brevity we refer the reader to [22] for the straightforward steps to obtain the switching layer on the intersection for the system above, obtaining the invariant sliding manifolds $\mathcal{M}^S$ that connect at $x_3 = x_4 = 0$.

More intriguingly, this opens the way to considering k -fold singularities for $k \geq 2$. In an n -dimensional system, at the intersection of r switching surfaces, there will generically occur sets of dimension $d = n - k$ where $k \leq 2r$ codimension $r - 1$ sliding flows are tangent to the intersection. We present an example in section V, and the complex dynamics that results.

C. Zeno exit from an intersection

Exit without tangency is also possible. Filippov discussed a planar piecewise constant example in [14], stating that it exhibited geometric convergence, or the Zeno phenomenon, meaning that infinite switches occur as the switching intersection is reached in finite time. (The system is so simple yet compelling that it has no doubt been considered elsewhere in literature this author is un-

aware of). Filippov also noted that this constituted a form of determinacy-breaking when the intersection is repelling. In [12] the scenario was studied for perhaps the first time in three dimensions, highlighting the computational problem raised by spiralling exit from an intersection.

We bring together these observations here, showing that the Zeno phenomenon continues to apply as the Zeno set (the intersection) changes stability in a three-dimensional system, creating first a codimension two sliding attractor, followed by a determinacy-breaking exit.

Again in three dimensions and with two switching surfaces, consider the structural model

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = \begin{cases} \mathbf{f}^{++} = (x_3 + 1, -1, 1) & \text{if } 0 < x_1, x_2, \\ \mathbf{f}^{-+} = (+1, +1, 0) & \text{if } x_1 < 0 < x_2, \\ \mathbf{f}^{+-} = (-1, -1, 0) & \text{if } x_2 < 0 < x_1, \\ \mathbf{f}^{--} = (-1, +1, 0) & \text{if } x_1, x_2 < 0 \end{cases} \quad (49)$$

restricted to $x_3 > -1$. The simplicity of this has no qualitative bearing on the results, but greatly simplifies the calculations.

There is no sliding on the surfaces $x_1 = 0$ or $x_2 = 0$ outside their intersection, as is easily shown from the switching layer systems on the different surfaces $x_1 = 0 \neq x_2$ and $x_2 = 0 \neq x_1$, (or performing standard Filippov analysis), showing that no sliding modes exist. Instead, the flow spirals around the intersection $x_1 = x_2 = 0$ by crossing through the switching planes, spiralling in towards the intersection for $x_3 < 0$ and away from it for $x_3 > 0$. We then consider the intersection point itself.

The canopy combination (6) applied to (49) simplifies to

$$\mathbf{f} = (\lambda_2 + \frac{1}{4}x_3(1 + \lambda_1)(1 + \lambda_2), -\lambda_1, \frac{1}{4}(1 + \lambda_1)(1 + \lambda_2)) , \quad (50)$$

and the switching layer system at the intersection, given by (11) with $r = 2$, is

$$\begin{aligned} \lambda_1' &= \lambda_2 + \frac{1}{4}x_3(1 + \lambda_1)(1 + \lambda_2) , \\ \lambda_2' &= -\lambda_1 , \\ \dot{x}_3 &= \frac{1}{4}(1 + \lambda_1)(1 + \lambda_2) . \end{aligned} \quad (51)$$

The dummy timescale (prime) system has equilibria at $(\lambda_1, \lambda_2) = (0, -\frac{x_3}{x_3+4})$, forming a sliding manifold $\mathcal{M}^S$ on which the sliding dynamics is given by $\dot{x}_3 = 1/(4 + x_3)$. The Jacobian derivative of the equilibrium in the (λ_1, λ_2) variables is $\begin{pmatrix} \frac{x_3}{x_3+4} & 1 + \frac{1}{4}x_3 \\ -1 & 0 \end{pmatrix}$, which for $x_3 > -1$ has complex eigenvalues. For $x_3 < 0$ the eigenvalues have negative real part, implying an attracting focus. For $x_3 > 0$ the eigenvalues have positive real part, implying a repelling focus. A drift along in the positive x_3 direction remains. So if a trajectory enters the intersection in $-1 < x_3 < 0$ it will spiral around in the (λ_1, λ_2) coordinates of the switching layer system, initially with decreasing radius around $(\lambda_1, \lambda_2) = (0, -\frac{x_3}{x_3+4})$ until it passes into $x_3 > 0$. It then begins spiralling outward until it reaches $|\lambda_1| = 1$ or $|\lambda_2| = 1$ and then exits.

Thus when a trajectory enters the intersection $x_1 = x_2 = 0$ for $x_3 < 0$, it does so with a unique value of (λ_1, λ_2) lying on the set

$$\mathcal{B} = \{(\lambda_1, \lambda_2) \in [-1, +1]^2 : (\lambda_1^2 - 1)(\lambda_2^2 - 1) = 0\}.$$

We can integrate (51) to find that λ_1 and λ_2 evolve through the region $(\lambda_1, \lambda_2) \in [-1, +1]$ until they again reach the bounding box $\mathcal{B}$, at which exit from the intersection occurs in $x_3 > 0$.

The dynamics *inside* the intersection is therefore well defined, but the entry and exit trajectories in $x_1 x_2 \neq 0$ may not be. It is therefore the dynamics outside the intersection that turns out to be the most interesting here.

Take a starting point $(x_1, x_2, x_3) = (0, \xi, \zeta)$ with $\xi > 0$ and $-1 < \zeta < 0$ at time $t = 0$ on one of the switching planes, and say its orbit crosses successive switching planes at times $t = t_1, t_2, t_3, t_4$. The map over time $t = 0$ to $t = t_4$ gives a return map on the half plane $x_1 = 0 < x_2$. In $0 < x_1, x_2$ we have $\dot{x}_2 = -1$ so to reach $x_2 = 0$ takes at time $t_1 = \xi$, arriving at $x_1(t_1) = \int_0^\xi (x_3 + 1) dt = \int_\zeta^{\xi+\zeta} (x_3 + 1) dx_3 = (\frac{1}{2}\xi + \zeta + 1)\xi$ and $x_3(t_1) = \xi + \zeta$. The next two sectors are a reflection so we arrive at $(-(\frac{1}{2}\xi + \zeta + 1)\xi, 0, \xi + \zeta)$ in time $t_3 - t_1 = 2(\frac{1}{2}\xi + \zeta + 1)\xi$. The last sector is a rotation to $(0, (\frac{1}{2}\xi + \zeta + 1)\xi, \xi + \zeta)$ in time $t_4 - t_3 = (\frac{1}{2}\xi + \zeta + 1)\xi$.

FIG. 16. Sketch of a Zeno trajectory (bold), taking infinitely many steps in finite time to spiral in towards the intersection in $x_3 < 0$ and out from the intersection in $x_3 > 0$. The inward and outward trajectories are connected by a codimension $r = 2$ sliding trajectory along the intersection. A simple trajectory which never reaches the intersection is also shown. The switching layer system, including a sliding manifold $\mathcal{M}^S$ with focal attraction, is shown inset (lower right).

Thus the overall rotation map on $\{x_1 = 0 < x_2\}$ is

$$\begin{aligned} \xi_n &= (1 + \zeta_{n-1} + \frac{1}{2}\xi_{n-1})\xi_{n-1}, \\ \zeta_n &= \zeta_{n-1} + \xi_{n-1}, \end{aligned} \quad (52)$$

which has an invariant

$$\xi_n - \frac{1}{2}\zeta_n^2 = \xi_{n-1} - \frac{1}{2}\zeta_{n-1}^2 = \dots = \xi_0 - \frac{1}{2}\zeta_0^2,$$

implying that the map $(\xi_{n-1}, \zeta_{n-1}) \mapsto (\xi_n, \zeta_n)$ on $x_1 = 0$ has trajectories lying on the parabolic contours of the function

$$\psi(\xi, \zeta) = \xi - \frac{1}{2}\zeta^2.$$

Therefore an orbit that reaches a point $\xi_n = 0$ does so with $\zeta_n = \sqrt{\zeta_0^2 - 2\xi_0}$, and can do so only if it starts on a curve such that $\zeta_0^2 - 2\xi_0 > 0$.

While we cannot solve the map, we can easily show that it exhibits the Zeno phenomenon.

Proposition 1. *An orbit starting at (ξ_0, ζ_0) such that $\sqrt{\zeta_0^2 - 2\xi_0} > 0$ and $-1 < \zeta_0 < 0$ converges to $\xi_n = 0$ as $n \rightarrow \infty$ in finite time $\sqrt{\zeta_0^2 - 2\xi_0} - \zeta_0$.*

Proof. An orbit starting at (ξ_0, ζ_0) such that $\sqrt{\zeta_0^2 - 2\xi_0} > 0$ and $-1 < \zeta_0 < 0$ will hit $\xi_n = 0$ when $\zeta_n = \sqrt{\zeta_0^2 - 2\xi_0}$. Since the speed of travel of the flow along the x_3 direction is unity, the time taken is $\Delta T_n = \zeta_n - \zeta_0 = \sqrt{\zeta_0^2 - 2\xi_0} - \zeta_0$, which is clearly finite. We must then show that this orbit takes infinitely many steps, i.e. $\xi_n = 0$ implies $n \rightarrow \infty$. Note that $\xi_n = 0$ is a fixed point of the map (52) for any ζ_n . Then by the ξ_n part of (52) we have $\frac{\xi_{n+1}}{\xi_n} = 1 + \zeta_n + \frac{1}{2}\xi_n$, and using the ζ_n part of (52) we can re-write this as $\frac{\xi_{n+1}}{\xi_n} = 1 + \zeta_n + \frac{1}{2}(\zeta_{n+1} - \zeta_n) = 1 + \frac{1}{2}(\zeta_n + \zeta_{n+1})$, which is negative since $\zeta_n, \zeta_{n+1} < 0$. This implies that ξ_n is strictly decreasing towards 0, and therefore cannot terminate at the fixed point 0 in finitely many steps, and thus ξ_n asymptotes towards 0 as $n \rightarrow \infty$. $\square$

Conversely, an orbit starting at the intersection in $\zeta_0 > 0$ takes infinitely many steps but finite time to exit from the intersection via the rotation map.

Because an orbit takes infinitely many rotations to reach the intersection, its entry point cannot be determined uniquely, and hence, even if the exit points from sliding can be determined from the switching layer system, the exit trajectory can not be determined uniquely, and the exit takes infinitely many steps in finite time.

We conclude with a few simulations of (49). In this case one finds, as predicted, that the exit point along the intersection is very sensitive to numerical imprecision. The simulations shown in figure 17 replace $\lambda_i = \text{sign}(x_i)$ with $\lambda_i \approx \phi(x_i/\varepsilon) = \tanh(x_i/\varepsilon)$ for $i = 1, 2$, with (i) $\varepsilon = 10^{-4}$, (i) $\varepsilon = 10^{-3}$, (i) $\varepsilon = 10^{-2}$. Here the value of the smoothing stiffness ε is more evident, determining how narrow (order ε) the funnelling along the intersection is. The results are consistent, however, as the exit points occur at similar coordinates $x_3 \approx 0.2$.

FIG. 17. Simulations using a tanh smoothing with ε values (i) 10^{-4} , (ii) 10^{-3} , (iii) 10^{-2} . For (i)-(ii) a magnification is shown of the funnel along the intersection.

The consistency of these results is further verified by using different smoothings of the sign function, taking the rational function $\phi(x_i/\varepsilon) = (x_i/\varepsilon)/\sqrt{1+(x_i/\varepsilon)^2}$ in (i-ii.a), or the ramp function $\phi(x_i/\varepsilon) = \text{sign}(x_i)$ for $|x_i| > \varepsilon$ and $\phi(x_i/\varepsilon) = x_i/\varepsilon$ for $|x_i| \leq \varepsilon$ in (i-ii.b). The results for these rational and ramp smoothings are qualitatively indistinguishable from the tanh smoothing, with some difference in the thickness of the funnel visible for $\varepsilon = 10^{-3}$, but with similar exit points around $x_3 \approx 0.2$.

V. EXAMPLE OF COUPLED OSCILLATORS

We have so far looked at single or double tangencies to a codimension $r = 2$ switching surface, as the basic mechanisms for exit from sliding. In systems of many dimensions with many switches, such as those suggested in (2)-(4), many such exits may occur at higher codimension intersections. A system with n dimensions and a switching surface comprised of r transverse manifolds may generically exhibit exit points consisting of up to $n - r$ tangencies on independent switching surfaces. This suggests that such events may tend to cluster and form exit cascades. The conditions for this to happen require more study, but cascades are observed to arise quite easily in models like (2)-(4), as we show here.

FIG. 18. The attractors in figure 17(i-ii) for the rational (a) and ramp (b) smoothings, with the same initial conditions, for $\varepsilon = 10^{-3}$.

Take an example of an oscillator system similar to those in the introduction, specifically

$$\left. \begin{aligned} \dot{z}_i &= y_i, \\ \dot{y}_i &= -M_{ij}^\rho y_j - M_{ij}^\kappa z_j - \mu(y_i - v), \end{aligned} \right\} \quad i = 1, \dots, n/2,$$

where M^ρ and M^κ are square matrices, and n is an even integer, and we sum over the repeated indices j . The matrix of damping coefficients is diagonal with components in the range $M_{ij}^\rho \in [\rho, 2\rho]$. The matrix of spring coefficients has an antisymmetric part and a diagonal part with components in the range $M_{ij}^\kappa \in [-\kappa, +\kappa]$.

The model represents a network of oscillators with displacements x_i and velocities y_i , connected via spring-damper couplings, with every oscillator also coupled to some parent object. The couplings are generated randomly, therefore M^ρ and M^κ are random matrices (up to symmetries). The parent slips at a constant speed v , and each oscillator experiences a dry (Coulomb) friction force with surface friction coefficient unity. The system is non-conservative due to the linear and frictional damping, and the energy input from the slipping surface.

Let $x_{2i-1} = z_i$, $x_{2i} = y_i$. We have $n/2$ switching surfaces $h_i = y_i - v = 0$. A trajectory crosses a switching surface transversally when an oscillator in the system alternates between left and right slipping motion. A trajectory slides on a codimension r switching surface intersection when r oscillators experience frictional sticking, so that their speeds are each fixed at $y_i = v$.

In simulations the oscillators typically either collapse to low codimension sliding (where most oscillators are slipping) or else exhibit complex transitions between higher and lower codimension sliding. One observes many oscillators sticking and releasing in complex patterns: when one oscillator slips it may trigger a cascade of many slip events across different oscillators. Each one

corresponds to a decrease in the sliding codimension r at an exit point as described above.

Visualizing the dynamics directly becomes difficult, of course. The simulations in figure 19 show three of the 400 dimensions, and give a fair representation of the dynamics. Two cases are shown (for different randomly generated matrices), one in which the system exhibits sustained complex oscillations which continually attach and exit from the switching manifolds (left), and one in which all sliding ceases and the system escapes, meaning that blocks change direction without sticking, and increasingly gain speed.

FIG. 19. Oscillator simulations for (iii) and (iv) for figure 20, respectively.

To view this usefully we can use the sliding codimension r . Drops in the value of r correspond to exit points. Figure 20 shows values of r calculated from simulations along a trajectory over large times, for: (i) 100 oscillators, $v = -0.45$, $\varepsilon = 0.03$, $\rho = 3$, $\kappa = 0.2$; (ii) 200 oscillators, $v = -0.2$, $\rho = 3.5$, $\kappa = 0.2$; (iii) 200 oscillators, $v = -0.2$, $\rho = 3.5$, $\kappa = 0.22$; (iv) 200 oscillators, $v = -0.3$, $\rho = 3.5$, $\kappa = 0.3$. We model each switch as $\text{sign}(h_i) = \tanh(h_i/\varepsilon)$ with $\varepsilon = 0.03$ (and simulations do not show qualitative dependence on ε for small enough values). In (i)-(iii), the system exhibits sustained and complex oscillatory dynamics, with erratic increases and decreases in r , all, however, tending to vary around an approximate value of $\sqrt{n/2}$. In (iv) the oscillations eventually die away and all blocks escape from sliding.

A continual decrease in r corresponds to a cascade of exit events (stick-to-slip events in mechanical terms). The frequency of cascades of size r is plotted in figure 21 for each of the simulations in figure 20, revealing a logarithmic distribution. This is seen whether the system remains in highly critical state or suffers complete eventual collapse. Ongoing work is examining the particular exit points and their roles in generating such cascades.

VI. CLOSING REMARKS

We are only at the beginning of the study of exit points. Rather than begin a classification that would be limited to low dimensions, we have focussed on dynamical phenomena such as exit points and determinacy-breaking, which form the basis for behaviour in higher dimensions and which might have distinct implications for applications. Particularly interesting is the collapse to higher

FIG. 20. Stick-slip events measured by sliding codimension r .

FIG. 21. Plot of frequency of cascade events by size. A cascade is a sequence of drops in sliding codimension r , the size being the overall decrease in r . The gradients are -0.86 , -0.70 , -0.82 , -0.89 (for fitting we disregard the last two data points).

order sliding that forms highly critical states, and the subsequent large scale reorganization through cascades of exit points which, at least in the example studied here, satisfy logarithmic size-frequency distributions.

We have analysed the basic mechanisms by which determinacy-breaking affects a piecewise smooth flow with one or two switches. The flow outside the switching surface suggests that determinacy fails at certain points where the flow is transversal to the switching thresholds. Whereas each initial condition has a unique forward time orbit almost everywhere in the flow, at the intersection the flow becomes set-valued. A switching layer analysis of the intersection is required to reveal what is happening in more detail, restoring determinacy to some extent, but revealing sensitivity to initial conditions and a dependence on parameters, which is not evident if we neglect the ‘hidden dynamics’ inside the switching surfaces.

Several cases studied here can be analyzed in more de-

tail. Our focus has been on bringing out their common features and the methods useful to study them. A systematic classification remains an open problem. From (48) in section IV A we showed that a double codimension $r = 1$ sliding tangency to a codimension $r = 2$ intersection has the same local form as the two-fold (a double codimension $r = 0$ tangency to a codimension $r = 1$ switching manifold). It is to be hoped that such results can be generalized. Particularly interesting for future work is to tangencies of multiple codimension $r - 1$

sliding flows to a codimension r intersection at a point, generalizing the $r = 1$ and $r = 2$ cases studied here.

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