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A note on balanced independent sets in the cube

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Abstract

Ramras conjectured that the maximum size of an independent set in the discrete cube $Q_n$ containing equal numbers of sets of even and odd size is $2^{n-1} - \binom{n-1}{(n-1)/2}$ when $n$ is odd. We prove this conjecture, and find the analogous bound when $n$ is even. The result follows from an isoperimetric inequality in the cube.

The discrete hypercube $Q_n$ is the graph with vertices the subsets of $[n] = \{1, \ldots, n\}$ and edges between sets whose symmetric difference contains a single element. The cube $Q_n$ is bipartite, with classes $X_0$ and $X_1$ consisting of the sets of even and odd size respectively. The maximum-sized independent sets in $Q_n$ are precisely $X_0$ and $X_1$. Ramras [3] asked: how large an independent set can we find with half its elements in $X_0$ and half in $X_1$? Call such an independent set balanced. The following result verifies the conjecture made by Ramras for the case where $n$ is odd.

Theorem 1. The largest balanced independent set in $Q_n$ has size

\[
2^{n-1} - 2 \binom{n-2}{(n-2)/2} \quad \text{if } n \text{ is even,}
\]

\[
2^{n-1} - \binom{n-1}{(n-1)/2} \quad \text{if } n \text{ is odd.}
\]

For a set $A$ of vertices of $Q_n$, write $N(A)$ for the set of vertices adjacent to an element of $A$. The maximal independent sets in $Q_n$ all have the form $A \cup (X_1 \setminus N(A))$ for some $A \subseteq X_0$. So for a maximum-sized balanced independent set we seek the largest $A \subseteq X_0$ for which

\[
|A| \leq |X_1 \setminus N(A)|.
\]
We use the following isoperimetric theorem for even-sized sets, due independently to Bezrukov [1] and Körner and Wei [2] (see also Tiersma [4]). Recall that \( x < y \) in the simplicial order on \( Q_n \) if either \( |x| < |y| \), or \( |x| = |y| \) and \( x < y \) lexicographically.

**Theorem 2** ([1], [2]). Let \( A \subseteq X_0 \), and let \( B \) be the initial segment of the simplicial order restricted to \( X_0 \) with \( |B| = |A| \). Then \( |N(B)| \leq |N(A)| \), and \( X_1 \setminus B \) is a terminal segment of the simplicial order restricted to \( X_1 \).

**Proof of Theorem 1.** We will exhibit an initial segment \( A \) of the simplicial order restricted to \( X_0 \), and a terminal segment \( B \) of the simplicial order restricted to \( X_1 \), with \( N(A) \cap B = \emptyset \) and \( |A| = |B| \) as large as possible. It follows from Theorem 2 that \( A \cup B \) will be a maximum-sized balanced independent set.

The form of \( A \) and \( B \) depends on the residue of \( n \) mod 4. For \( n = 4k \) we take

\[
A = [n]^{(0)} \cup [n]^{(2)} \cup \ldots \cup [n]^{(2k-2)} \cup (12 + [3, n]^{(2k-2)})
\]

\[
B = (1 + [3, n]^{(2k)}) \cup [2, n]^{(2k+1)} \cup [n]^{(2k+3)} \cup \ldots \cup [n]^{(n-3)} \cup [n]^{(n-1)},
\]

where, for instance,

\[
12 + [3, n]^{(2k-2)} = \{1, 2\} \cup x : x \subseteq \{3, 4, \ldots, n\}, |x| = 2k - 2.
\]

For \( n = 4k + 1 \) we take

\[
A = [n]^{(0)} \cup [n]^{(2)} \cup \ldots \cup [n]^{(2k-2)} \cup (1 + [2, n]^{(2k-1)})
\]

\[
B = [2, n]^{(2k+1)} \cup [n]^{(2k+3)} \cup \ldots \cup [n]^{(n-2)} \cup [n]^{(n)}.
\]

For \( n = 4k + 2 \) we take

\[
A = [n]^{(0)} \cup [n]^{(2)} \cup \ldots \cup [n]^{(2k-2)} \cup (1 + [2, n]^{(2k-1)}) \cup (2 + [3, n]^{(2k-1)})
\]

\[
B = [3, n]^{(2k+1)} \cup [n]^{(2k+3)} \cup \ldots \cup [n]^{(n-3)} \cup [n]^{(n-1)}.
\]

Finally, for \( n = 4k + 3 \) we take

\[
A = [n]^{(0)} \cup [n]^{(2)} \cup \ldots \cup [n]^{(2k)}
\]

\[
B = [n]^{(2k+3)} \cup \ldots \cup [n]^{(n-2)} \cup [n]^{(n)}.
\]

Verifying that these sets have the claimed sizes, and that \( |A| = |B| \) in each case, is a simple application of the identities \( \binom{m}{r} = \binom{m-1}{r} + \binom{m-1}{r-1} \), \( \binom{m}{r} = \binom{m}{m-r} \) and \( \sum_{r=0}^{m} \binom{m}{r} = 2^m \). \( \square \)

The maximum-sized balanced independent sets constructed above are also maximal independent sets. For example, if \( n = 4k + 3 \), then any set not in the family is adjacent to a complete layer; the other cases are similar, with slight complications in the middle layers of the cube.
References


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