Wooley, T. D. (2017). A Superpowered Euclidean Prime Generator. *American Mathematical Monthly*, 124(4), 351-352.
<https://doi.org/10.4169/amer.math.monthly.124.4.351>

Peer reviewed version

License (if available):
CC BY-NC

Link to published version (if available):
[10.4169/amer.math.monthly.124.4.351](https://doi.org/10.4169/amer.math.monthly.124.4.351)

[Link to publication record in Explore Bristol Research](#)
PDF-document

This is the author accepted manuscript (AAM). The final published version (version of record) is available online via MAA at <http://www.maa.org/press/periodicals/american-mathematical-monthly/american-mathematical-monthly-april-2017>. Please refer to any applicable terms of use of the publisher.

University of Bristol - Explore Bristol Research

General rights

This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available:
<http://www.bristol.ac.uk/red/research-policy/pure/user-guides/ebr-terms/>

A Superpowered Euclidean Prime Generator

Trevor D. Wooley

Abstract. A variant of Euclid's prime generator is discussed with some of its brethren.

When $\{\pi_1, \dots, \pi_k\}$ is a finite set of primes, the least divisor exceeding 1 of $\pi_1 \cdots \pi_k + 1$ is a prime distinct from $\pi_1, \dots, \pi_k$. In this way, as every schoolchild knows, one sees that there are infinitely many primes: the assumption that there are just finitely many leads to a contradiction. This is essentially the proof attributed to Euclid, who observed that all primes dividing $\pi_1 \cdots \pi_k + 1$ are distinct from $\pi_1, \dots, \pi_k$. But is *every* prime delivered by iterating this algorithm? To be precise, if we put $\pi_1 = 2$ and then define

$$\pi_{k+1} = \min \{d > 1 : d \text{ divides } \pi_1 \cdots \pi_k + 1\} \quad (k \geq 1),$$

is it the case that the sequence $(\pi_k)_{k=1}^\infty$ contains all the primes? The widely held conjecture that the answer is in the affirmative remains open more than half a century after Mullin [3] posed this question. There are, however, variants of Euclid's construction that do yield every prime. Given a set of primes $\{\pi_1, \dots, \pi_k\}$, Pomerance [2, §1.1.3] defines π_{k+1} to be the least prime distinct from $\pi_1, \dots, \pi_k$ that divides a number of the form $d + 1$ for some divisor d of $n = \pi_1 \cdots \pi_k$. He shows that starting with $\pi_1 = 2$, every prime is delivered by this iterative process, and moreover (by extensive computations) that π_k is the k -th smallest prime for $k \geq 5$. Booker [1] instead considers the prime divisors p of the integers $d + n/d$, and shows that at each stage in the iteration, choices for d and p may be made so that, taking $\pi_{k+1} = p$, every prime is delivered.

The iterative processes of Booker [1] and Pomerance [2] involve some kind of ambiguity, in the latter case involving a choice of the divisor d of $n = \pi_1 \cdots \pi_k$, and in the former case a choice of both d and the prime divisor of $d + n/d$. In this note we present a variant of Euclid's prime generator in which the sequence of primes is determined in order by a single choice of divisor.

Theorem 1. *Let $\pi_1 = 2$, and when $k \geq 1$, define π_{k+1} to be the least divisor exceeding 1 of $n^{\pi_k} - 1$, where $n = \pi_1 \cdots \pi_k$. Then for each k , the integer π_k is the k -th smallest prime.*

This “superpowered” variant of Euclid's prime generator has computational value that can only be described as rather less than nanoscopic. However, it has the merit of succinctly delivering the $(k + 1)$ -st smallest prime in terms of the k smallest primes. The proof is immediate from the following lemma, the proof of which is reminiscent of the argument underlying the Pollard $p - 1$ factorization method (see [2, §5.4], [4]).

Lemma. When n is a positive integer, the least prime divisor of $n^{\pi_n} - 1$ is the smallest prime not dividing n .

Proof. We may plainly suppose that $n \geq 2$, for when $n = 1$ the desired conclusion is immediate. Let p be the smallest prime not dividing n , and let the primes dividing n be $\pi_1, \dots, \pi_k$. Then $p \leq \pi_1 \cdots \pi_k + 1 \leq n + 1$, as Euclid could have told us. Moreover, all prime divisors of $p - 1$ lie in $\{\pi_1, \dots, \pi_k\}$, and since $\pi_i^n \geq 2^n \geq n + 1 \geq p$ for

each i , we find that $p - 1$ divides $(\pi_1 \cdots \pi_k)^n$, and hence also n^n . But then, defining the integer λ by writing $n^n = \lambda(p - 1)$, and noting that p does not divide n , we find from Fermat's Little Theorem that $n^{n^n} = (n^\lambda)^{p-1} \equiv 1 \pmod{p}$, which is to say that p divides $n^{n^n} - 1$. ■

The argument just described makes it apparent that less profligate exponents are viable. The conclusion of Theorem 1 remains valid, for example, when $n^{n^n} - 1$ is replaced by $n^{n^m} - 1$, in which $m = \lceil (\log n)/(\log 2) \rceil$. In this context, we note also that if p_k denotes the k -th smallest prime for each k , and $n = p_1 \cdots p_k$, then the argument of the proof of the lemma shows that *all* primes p with $p_{k+1} \leq p < 2p_{k+1}$ divide $n^{n^n} - 1$.

A Euclidean disciple even more orthodox than enthusiasts of Theorem 1 might demand a means of obtaining the next prime without knowing a single one of the previous (smaller) primes. Even zero-knowledge demands such as this can be met by a direct consequence of the lemma.

Theorem 2. *When N is a positive integer, the smallest prime exceeding N is the least divisor exceeding 1 of $N!^{N!} - 1$.*

For a proof, simply apply the lemma with $n = N!$. We encourage readers to entertain themselves by establishing that for each natural number N , the smallest prime exceeding N is the least divisor exceeding 1 of $N!^{N!} - 1$ (the author is grateful to Andrew Booker and Andrew Granville for pointing out this refinement).

ACKNOWLEDGMENT. The author is grateful to the referees for their valuable suggestions and comments.

REFERENCES

1. A. R. Booker, A variant of the Euclid-Mullin sequence containing every prime, *J. Integer Seq.* **19** (2016) Article 16.6.4, 5 pp.
2. R. Crandall and C. Pomerance, *Prime numbers: a computational perspective*, Second edition, Springer, New York, 2005.
3. A. A. Mullin, Research Problem 8. Recursive function theory, *Bull. Amer. Math. Soc.* **69** (1963) 737.
4. J. M. Pollard, Theorems on factorization and primality testing, *Math. Proc. Cambridge Philos. Soc.* **76** (1974) 521–528.

School of Mathematics, University of Bristol, University Walk, Clifton, Bristol BS8 1TW, United Kingdom
matdw@bristol.ac.uk