
Peer reviewed version
License (if available):
CC BY-NC-ND
Link to published version (if available):
10.1016/j.ifacol.2016.09.070

Link to publication record in Explore Bristol Research
PDF-document

This is the author accepted manuscript (AAM). The final published version (version of record) is available online via Elsevier at http://www.sciencedirect.com/science/article/pii/S2405896316315439. Please refer to any applicable terms of use of the publisher.

University of Bristol - Explore Bristol Research
General rights
This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available: http://www.bristol.ac.uk/red/research-policy/pure/user-guides/ebr-terms/
Comparison of Aeroelastic Modeling and Robust Flutter Analysis of a Typical Section *

A. Iannelli * A. Marcos * M. Lowenberg *

* University of Bristol, BS8 1TR, United Kingdom (e-mail: andrea.iannelli/andres.marcos/m.lowenberg@bristol.ac.uk).

Abstract: A comparison of robust flutter analyses within the µ framework is presented. The chosen test bed is the typical section with unsteady aerodynamic loads, which enables basic modeling features to be captured and so extend the gained knowledge to practical problems treated with modern techniques. The two main approaches to pose the LFT problem are investigated and features such as numerical accuracy and physical uncertainty description are assessed. A criterion is proposed to correlate the families of plants originated by the uncertainty description of the aerodynamic operator when different approximation algorithms are employed.

Keywords: Robust stability, Flexible structures, Flutter analysis, Structured Singular Value

1. INTRODUCTION

The interaction among inertial and elastic forces in a mechanical system is the subject of structural dynamics. For systems such as lifting surfaces, blades or sails, the external loading is represented by aerodynamic forces. The study of these systems is then addressed by aeroelasticity, which investigates the coupled problem of a deformable structure surrounded by a fluid flow generating a pressure dependent on its geometry.

Flutter is a self-excited instability in which aerodynamic forces on a flexible body couple with its natural vibration modes producing oscillatory motion. The level of vibration may result in sufficiently large amplitudes to provoke failure and often this phenomenon dictates the design of the aerodynamic body. Thus, flutter analysis has been widely investigated and there are several techniques representing the state-of-practice (Edwards and Wieseman (2008)). The major methods, for example $k$ and $p-k$ method, are based on the frequency-domain as this is the framework in which the aerodynamic loads are more often expressed.

Despite the large amount of effort spent in understanding flutter, it is acknowledged that predictions based only on computational analyses are not totally reliable. Currently this is compensated by the addition of conservative safety margins to the analysis results and expensive flutter test campaigns. One of the main criticalities arises from the sensitivity of aeroelastic instability to small variations in parameter and modeling assumptions. In addressing this issue, in the last ten years researchers looked at robust modeling and analysis techniques from the robust control community, specifically linear fractional transformation (LFT) models and µ analysis (Packard and Doyle (1993); Balas et al. (1998)). The so-called flutter robust analysis aims to quantify the gap between the prediction of the nominal stability analysis (model without uncertainties) and the worst-case scenario when the whole set of uncertainty is contemplated. This is believed to be a powerful tool when used as a complement to the classical techniques in that it could highlight weak points of the model requiring more refinement and conversely identify parameters that can be coarsely estimated as they do not have a strong influence on the results. The most well-known robust flutter approaches are those from Lind and Brenner (2012), Borglund (2004), Idan et al. (1999), with the first even including on-line analysis during flight tests.

Each of the aforementioned robust flutter approaches used the same underlying µ analysis tools but a different LFT model development path, in addition to relying on different aerodynamic approximations (e.g. Roger or Minimum State). The goal of each of those robust flutter studies was to provide an end-to-end process, from robust modeling to robust analysis, and demonstrate the validity of the approach. Since this was their focus, no detailed study or comparison was performed on the effect the modeling choices have on the analysis –although it is well-known in the robust control community that this is a fundamental issue (Marcos and Balas (2004); Magni (2004); Marcos et al. (2015)). Thus, the goal of this article is to present a comparison of the modeling options and a better understanding of their effects on the flutter analysis.

The layout of the article is as follows. Section 2 presents the test bed adopted. This section also includes a cursory description of the algorithms employed for the aerodynamics rational approximations. Nominal flutter results are presented in Section 3 followed by a very brief description of LFT and µ analysis in Section 4. The comparison and discussion of results obtained with the different approaches is presented in Section 5, ending with the conclusions in Section 6.

* This work has received funding from the European Union’s Horizon 2020 research and innovation programme under grant agreement No 636307, project FLEXOP.
2. AEROELASTIC MODELING

2.1 Typical section

The typical section model was introduced in the early stages of aeroelasticity to investigate dynamic phenomena such as flutter (Bisplinghoff and Ashley (1962)). Despite its simplicity, it captures essential effects in a simple model representation, see Fig. 1.

![Fig. 1. Typical section sketch](image)

From the structural side, it basically consists of a rigid airfoil with lumped springs simulating the 3 degrees of freedom of the section: plunge, pitch and trailing edge flap. The positions of the elastic axis (EA), center of gravity (CG) and the aerodynamic center (AC) are also marked. The main parameters in the model, see Fig.1, are: \( K_h \), \( K_o \) and \( K_\beta \) - respectively the bending, torsional and control surface stiffness; half chord \( b \) and dimensionless distances \( a \), \( c \) from the mid-chord to respectively the flexural axis and the hinge location; \( x_a \) and \( x_\beta \) dimensionless distances from flexural axis and airfoil center of gravity and from hinge location and control surface center of gravity.

For the aerodynamic loads model, the unsteady formulation proposed by Theodorsen (1935) is employed. This approach is based on the assumption of a thin airfoil moving with small harmonic oscillations in a potential and incompressible flow. Despite its simplicity, such an aerodynamic assumption is pertinent to flutter analysis since the latter attempts to model the memory effect of the flow, which results in phase shift and magnitude change of the loads with respect to the former one. This is commonly referred to as time lag effect. A general two-part approximation model can then be obtained based on quasi-steady (QS) and lag contributions.

\[
\begin{bmatrix} A(s) \end{bmatrix} \approx \Gamma_{QS} + \Gamma_{lag} \tag{4}
\]

In this paper two among the most established algorithms are presented: Roger method and Minimum State method. They propose a formally identical expression for \( \Gamma_{QS} \):

\[
\Gamma_{QS} = [A_2] s^2 + [A_1] s + [A_0] \tag{5}
\]

Where \([A_2]\), \([A_1]\) and \([A_0]\) are real coefficient matrices. Roger proposed (Roger (1977)) that \( \Gamma_{lag} \) could be approximated as:

\[
\Gamma_{lag-Roger} = \sum_{L=3}^{N} \frac{\bar{s}}{s + \gamma L^{-2}} [A_L] \tag{6}
\]

The partial fractions inside the summation are the so-called lag terms and they basically represent high-pass filters with the aerodynamic roots \( \gamma \) as cross-over frequencies. The real coefficient matrices \([A_i]\) with \( i = 0 \ldots 2 \) are found using a linear least-square technique for a term-by-term fitting of the aerodynamic operator. The resulting state-space equation includes augmented states representing the aerodynamic lags, which are equal to the number of roots multiplied by the number of degrees of freedom.

The MS method (Karpel (1981)) tries to improve the efficiency of Roger’s in terms of number of augmented states per given accuracy of the approximation. There is no clear quantification of this advantage, but it has been stated (Idan et al. (1999)) that the number of aerodynamic states may typically be 6-8 times smaller for the same level of model accuracy. The \( \Gamma_{lag} \) expression is:

\[
\Gamma_{lag-MS} = [D'] \begin{bmatrix} 1 & \cdots & 0 \\ s + \gamma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s + \gamma N^{-2} \end{bmatrix} [E'] s \tag{7}
\]

The coefficients of \([D']\) and \([E']\) are iteratively determined through a nonlinear least square since (7) is bilinear in

Influence Coefficient (AIC) matrix, and is composed of generic terms \( A(s)_ij \) representing the transfer function from each degree of freedom \( j \) in \( X(s) \) to each aerodynamic load component \( i \) in \( L_a(s) \).

It is remarked that Theodorsen’s aerodynamic theory assumes harmonic motion, which means that the relation in (2) is pertinent only if \( \Gamma(s) \) is evaluated at \( \bar{s} = j \omega \), with \( \omega \) the reduced frequency. The final aeroelastic equilibrium is given by:

\[
\begin{bmatrix} M_a \end{bmatrix} s^2 + [C_a] s + [K_a] X = q \begin{bmatrix} A(s) \end{bmatrix} X \tag{3}
\]

2.2 Rational Approximations

The AIC matrix does not have a rational dependence on the Laplace variable \( s \) and this forces approximations of the aerodynamic operator to be pursued in order to provide an expression for (3) in state space, which is essential to deal with aeroelastic problems. The difference between a quasi-steady and an unsteady formulation of the aerodynamic loads is that the latter attempts to model the memory effect of the flow, which results in phase shift and magnitude change of the loads with respect to the former one. This is commonly referred to as time lag effect. A general two-part approximation model can then be obtained based on quasi-steady (QS) and lag contributions.
these two unknowns, while the matrices defining $\Gamma_{QS}$ (5) are now obtained imposing the constraint to match the aerodynamic operator at $\eta=0$ and at another selected reduced frequency $k$. The number of augmented states is now equal to the number of roots. The impact that the differences in the expression of $\Gamma_{lag}$ have on robust flutter analysis when lag terms are uncertain will be investigated in Section 5.

Both methods lead to the same short-hand state matrix:

$$
\begin{bmatrix}
\dot{X}_s \\
\dot{X}_a
\end{bmatrix} =
\begin{bmatrix}
\chi_{ss} & \chi_{sa} \\
\chi_{as} & \chi_{aa}
\end{bmatrix}
\begin{bmatrix}
X_s \\
X_a
\end{bmatrix}
$$

(8)

Where $X_s$ and $X_a$ are respectively the vector of structural and aerodynamic states.

3. NOMINAL FLUTTER ANALYSIS

Nominal flutter analysis studies the conditions at which the dynamic aeroelastic system loses its stability. As the air stream speed $V$ varies the system's behavior in terms of response and stability changes. The result is the prediction of the so-called flutter speed $V_f$, below which the system is guaranteed to be stable.

In principle it is possible to solve the problem studying either (3) or (8). In the later case, the stability of the system is related to the spectrum of the state-matrix. We will use the Roger and MS approximations for nominal comparison. In the former case (the most reliable and currently adopted approach), the objective is to find the flutter determinant roots $s$ such that nonzero solutions for $X$ exist. The complexity arises since $A(s)$ does not have a polynomial dependence on $s$ and thus iterative solutions have to be sought. We will use the $p-k$ method to baseline the comparison.

The parameter values for these analyses are taken from (Karpel (1981)). In Fig. 2 the eigenvalues corresponding to the structural modes are depicted as speed increases. The system exhibits a plunge-torsion flutter, featured by the merging of the frequencies just before instability occurrence (binary flutter). Table 1 summarizes the results. For the Roger method, 4 aerodynamic roots equally spaced between -0.1 and -0.6 were selected (state-matrix size equals 11). No substantial mismatches are found in the flutter speeds.

Table 1. Nominal flutter analyses results

<table>
<thead>
<tr>
<th>State Space - Roger</th>
<th>Flutter speed [m/s]</th>
<th>Flutter frequency [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>302.7</td>
<td>11.25</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>State Space - MS</th>
<th>Flutter speed [m/s]</th>
<th>Flutter frequency [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>302.5</td>
<td>11.2</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Frequency-domain p-k</th>
<th>Flutter speed [m/s]</th>
<th>Flutter frequency [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>301.8</td>
<td>11.2</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ref. Karpel (1981)</th>
<th>Flutter speed [m/s]</th>
<th>Flutter frequency [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>303.3</td>
<td>11.15</td>
<td></td>
</tr>
</tbody>
</table>

4. LFT AND $\mu$ ANALYSIS

In this section a very cursory presentation of the mathematical concepts behind linear fractional transformation (LFT) modeling and robust $\mu$ analysis is given. The interested reader is referred to Packard and Doyle (1993); Balas et al. (1998).

Fig. 2. Nominal system: pole locations in terms of speed

If the coefficient matrix $M$ is defined as a proper transfer matrix, $F_u$, namely the upper LFT, is the closed-loop transfer matrix from input $u$ to output $y$ when the nominal plant $M_{22}$ is subject to a perturbation matrix $\Delta$ (Fig. 3). $M_{11}, M_{12}$ and $M_{21}$ reflect a priori knowledge of how the perturbation affects the nominal map. Once all varying or uncertain parameters are pulled out of the nominal plant, the problem appears as a nominal system subject to an artificial feedback. The algebraic expression for $F_u$ is given by:

$$
F_u(M, \Delta_u) = M_{22} + M_{21} \Delta_u (I - M_{11} \Delta_u)^{-1} M_{12}
$$

(9)

This LFT is well posed if and only if the inverse of $(I - M_{11} \Delta_u)$ exists.

The structured singular value (s.s.v.) is a matrix function denoted by $\mu(\Delta)$ and defined as:

$$
\mu(\Delta) = \frac{1}{\min(\sigma(\Delta) : \det(I - M \Delta)) = 0)}
$$

(10)

where $\sigma(\Delta)$ is the maximum singular value of $\Delta$. Note that this definition can be specialized to determine whether the LFT $F_u(M, \Delta)$ is well posed once the generic matrix $M$ in the above definition is replaced by $M_{11}$ and $\Delta$ belongs to the corresponding uncertainty set $\Delta$. For ease of calculation and interpretation, $\Delta$ is typically norm-bounded $\|\Delta\|_\infty < 1$ (without loss of generality by scaling of $M$). In this manner, if $\mu(\Delta) \leq 1$ then the result guarantees that the analyzed system represented by the LFT is robust to the considered uncertainty level. The structured singular value is a robust stability (RS) test but can be used also for robust performance (RP). It is known that $\mu(\Delta)$ is non-polynomial (NP) hard with either pure real or mixed real-complex uncertainties, thus the algorithms implement upper and lower bound calculations (Balas et al. (1998)). The upper bound $\mu_{UB}$ provides the maximum size perturbation for which RS/RP is guaranteed, while the lower bound $\mu_{LB}$ guarantees the minimum size perturbation for which RS/RP is guaranteed to be violated. Along with this information, the lower
bound also provides the matrix $\Delta_{LB} = \Delta_C$ satisfying the determinant condition.

5. ROBUST FLUTTER ANALYSIS

Robust flutter analysis deals with flutter instability predictions when the aeroelastic model is subject to uncertainties. Examples of the latter are low confidence in the values of parameters and coefficients of the matrices, or neglected dynamics in the nominal model. Once a quantification of the uncertainty and its mathematical description are provided by means of LFT, $\mu$ analysis enables to predict at a given speed if the set of uncertainties is capable of leading to instability.

5.1 General preamble on flutter analysis with $\mu$

The standard $\mu$ framework applies to systems represented by rational transfer function matrices (or similarly described by ODE), which are easily recast in state-space. This is related to the existence of well established state-space algorithms employed in servo-control applications and marked the path for the development of the first robust $\mu$ analyses. But in fact, the $\mu$ technique and the intimately related concept of LFT are frequency-domain based. Recalling (9), the core problem is to study the well-posedness of the LFT describing a representative transfer function of the plant. What can be different is the LFT representing a proper LFT model development path applied. That is, either (3) or (8) function of the plant. What can be different is the LFT posingness of the LFT describing a representative transfer function.

Nonetheless robust predictions are expected to be identical different when the same range of variation is allowed. In the present work this is accomplished by writing the uncertain parameters in symbolic form and using the well consolidated LFR toolbox (Magni (2004)) leading straightforwardly to an LFT representation of the uncertain system which is a valid input for $\mu$ analysis.

In Fig. 4 the upper and lower bounds are reported for the case when only structural uncertainties are considered. The estimation given by $\mu_{LB}$ is that the system is flutter free for structural uncertainties up to approximately 75% ($\approx \frac{1}{5}$) of the assumed range, whereas no information is provided by $\mu_{LB}$ about the minimum size of the perturbation matrix which proves to cause plant instability.

Considering now errors in the aerodynamic model, generally uncertainties in the values of the lag roots are contemplated (Lind (2002)). Although these terms have the same physical meaning for Roger and MS methods, the way they enter the equations is different, as previously seen. An uncertainty of 5% in the value of the lag terms is considered in the analysis. Choice of number and nominal values for the roots reflects that of the nominal case. The uncertainty LFT block sizes are:

$$\Delta_{S}^{5,R} = \text{diag}(\delta_{M_{11}}, \delta_{M_{12}}, I_{2}, \delta_{M_{22}}, \delta_{K_{11}}, \delta_{K_{22}})$$

$$\Delta_{MS}^{5,R} = \text{diag}(\delta_{\gamma_{1}}, \delta_{\gamma_{2}}, \delta_{\gamma_{3}}, \delta_{\gamma_{4}}, \delta_{\gamma_{5}})$$

$$\Delta_{R}^{12,R} = \text{diag}(\delta_{\gamma_{1}}, I_{3}, \delta_{\gamma_{2}}, I_{3}, \delta_{\gamma_{3}}, I_{3}, \delta_{\gamma_{4}}, I_{3})$$

Results are shown in Fig.5. Remarkably different margins of stability are predicted by the Roger and MS algorithms. In other words, the capability of the two families of aerodynamic operators to perturb the stability is considerably different when the same range of variation is allowed. Nonetheless robust predictions are expected to be identical as long as the aerodynamics’ uncertainty descriptions are consistent. Consistency is here referred to a definition of the range of variation for the parameters such that the uncertainty operator $\Delta$ maps the two approximations in two similar families of plants. A rationale to perform this range definition is here sought.

When the uncertainties are inserted in approximate operators, each of them can be represented as an upper LFT $\mathcal{F}(\hat{A}, \Delta)$ (subscripts $R$ and $MS$ will be used to identify the two approximation options). It is known how the size of a transfer function (in terms of its $H_{\infty}$ norm, i.e. the maximum singular value over the frequency range considered) affected by uncertainties can be determined using a $\mu$ robust stability test (Zhou et al. (1996)). This
The frequency range under investigation: for each discrete parameter are embedded. This procedure implies a gridding with

\[ \| F \|_\infty = \| F \|_\infty \] (13)

Once defined a certain size \( \bar{\alpha} \), \( \bar{W} \) could represent a term of comparison among different uncertainty descriptions (and range definitions).

The proposed validation method is prompted by the observation of the two aerodynamic operators corresponding to the smallest perturbation leading to instability \( \Delta_{cr} \) in the uncertainty\( \bar{\alpha} \). The corresponding values for Roger’s \( \| A_R(\Delta_{cr}) \|_\infty \) and Minimum State \( \| A_{MS}(\Delta_{cr}) \|_\infty \) are respectively 13.3 and 13.7. These are just the values of \( \| F_R \|_\infty \) and \( \| F_{MS} \|_\infty \) when the corresponding sizes of uncertainty \( \bar{\alpha} \) (respectively \( \| \Delta_{cr} \|_\infty \) and \( \| \Delta_{cr} \|_\infty \) ) are defined.

Moreover, when the norm calculations are applied to the set adopted for the analyses reported in Fig. 5, fixing the size \( \bar{\alpha}=1 \), and considering a reduced frequency \( k_T = 0.28 \) (approximately the robust reduced flutter frequency), the values for \( \| F_R \|_\infty \) and \( \| F_{MS} \|_\infty \) are respectively 12.8 and 31. The discrepancies in the operator norms (despite identical nominal values) confirm that setting the same range of variation (5% in this case) for both of them lead to very different families of plant.

When the range of variation of the aerodynamic roots in \( F_{MS} \) is specified so as to lead to the same \( \bar{W} \) (it is found that this range has to be about 9 times smaller than the one used in \( F_R \)), the discrepancy in \( \mu \) prediction falls within 10% (Fig. 6). This mismatch is thought to be caused by the discrepancies in \( H_\infty \) norm shown before (13.3 and 13.7 for the two different cases).

### 5.3 Frequency-domain approach

Recalling Section 4, the matrix used by \( \mu \) analysis for well-posedness is \( M_{11} \). This is the transfer function seen by the perturbation block of the uncertain plant. Borglund (2004) proposed to perform the robust flutter analysis by starting from the plant formulated in the frequency-domain (3), with \( M_{11} \) manually assembled once the uncertain parameters are embedded. This procedure implies a gridding of the frequency range under investigation: for each discrete value \( \omega \) in the range, the terms involved in the definition

\[ \Delta_3^C = \text{diag}(\delta_{A_{12}}, \delta_{A_{21}}, \delta_{A_{22}}) \] (14)

The analysis for this plant is depicted in Fig.7. The most remarkable observation is that now lower and upper bounds coincide, leading to a precise estimation of the robust margin as opposed to that seen in Fig.4. The case of both structure and aerodynamic uncertainties,
Numeric quality of lower and upper bounds calculation for different sets of uncertainties and framework where the problem is posed are highlighted. Advantages in terms of uncertainty description and results accuracy of the frequency-domain approach are assessed; the state-space approach on the other hand offers a more straightforward way to pose the problem, due to well established toolbox and a general greater confidence in the formulation of the plants in state-space. Although this is commonly true for control problems, the role played by the aerodynamics in the aeroelastic plant and its inherent frequency-domain expression could alter this scenario when these kind of stability analyses are pursued.

REFERENCES


6. CONCLUSION

This work attempts to compare techniques developed to study robust flutter analysis adopting the typical section with unsteady loads as a test bed. The investigations concern effect of the uncertainty description when two different aerodynamic approximations for the unsteady AIC matrix are employed. In the latter case, a criterion to select the range of variation for the parameters in order to define similar families of plants is proposed.

These results provide examples of the aforementioned advantages when modeling is entirely approached in the frequency-domain. It is worth remarking however that these comments are pertinent to robust flutter stability analysis. The scenario may change when other tasks are involved, for example robust control design for flutter suppression and/or on-line robust predictions during flight tests. Indeed, the latter has only been demonstrated using the state-space approach (Lind and Brenner (2012)) and well-consolidated algorithms could dictate the same way for the former task. These aspects deserve then a better insight, and could be addressed in the future.