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A New Proof of Reichert's Theorem

Sara Y. Zhang, Jason Z. Jiang and Malcolm C. Smith

Abstract—Reichert's theorem (1969), a fundamental theorem of network synthesis, completely characterises minimum reactive synthesis of positive-real biquadratic impedances. The crucial part of the original approach depends on a complicated topological argument. This paper provides an alternative proof using the recently introduced concept of regular positive-real functions.

I. INTRODUCTION

Reichert's theorem [1] asserts that any impedance of a one-port electrical network which can be realised with two reactive elements and an arbitrary number of resistors can be realised with two reactive elements and three resistors. In Reichert's original German language publication [1], use is made of a characterisation of realisable biquadratic impedances published by Auth [2]. The crucial part of the proof in [1] is a topological argument that a certain system of polynomial equations has no solution in some region of its variable parameters. Even though this part has been further clarified in [3], a more transparent proof is preferable.

The approach of the present paper builds on [4], [5] and [6]. By using Cederbaum's necessary condition for the realisation of a purely resistive n -port network [7], five six-element networks with four resistors have been identified in [5], which can cover all biquadratic impedances realisable with two reactive elements. In [4], the concept of a regular positive real function was introduced. It has been shown that five-element networks with two reactive elements were sufficient to realise any regular biquadratics and only two bridge networks in this class can realise non-regular biquadratics. Based on the concept and properties of regularity [4], it was published out in [6] that three of the five networks in [5] can only realise regular biquadratics. Hence, to show the correctness of Reichert's theorem, the only step left is to prove that the remaining two networks in [5] cannot realise any non-regular biquadratics that cannot be realised by the two non-regular bridge networks in [4]. This step turns out to be not straightforward at all. The present paper fills in the gap and provides a complete alternative proof of Reichert's theorem. For reasons of space, full proofs of some lemmas are presented in the supplementary material [8].

II. PRELIMINARIES

In this section, we review the concept of network quartet, regular positive-real functions and some key conclusions in

Fig. 1: Transformations relating members of a network quartet.

[4] and [5]. The classification of networks is facilitated by the following transformations on the impedance $Z(s)$:

- 1) Frequency inversion: $s \rightarrow s^{-1}$,
- 2) Impedance inversion: $Z \rightarrow Z^{-1}$.

The first transformation corresponds to replacing inductors with capacitors of reciprocal value (and vice versa), and the second to taking the network dual. These transformations allow networks to be arranged into groups of four, which we call *network quartets* [9], [4] shown in Fig. 1. It should be noted that a network quartet may sometimes reduce to two or even one distinct network(s). We now recall the concept of a *regular* positive-real function.

Definition([6]): A positive-real function $Z(s)$ is defined to be *regular* if the smallest value of $\text{Re}(Z(j\omega))$ or $\text{Re}(Z^{-1}(j\omega))$ occurs at $\omega = 0$ or $\omega = \infty$.

For a biquadratic function

$$Z(s) = \frac{As^2 + Bs + C}{Ds^2 + Es + F}, \quad (1)$$

where A, B, C, D, E , and $F \geq 0$, it is well known [10], [11], [12] that $Z(s)$ is positive real if and only if $\sigma = BE - (\sqrt{AF} - \sqrt{CD})^2 \geq 0$.

Lemma 1 ([4]). *A positive-real biquadratic impedance (1) is regular if and only if the conditions of at least one of the following four cases are satisfied:*

- Case 1. $AF - CD \geq 0$ and $\lambda_1 = E(BF - CE) - F(AF - CD) \geq 0$,
- Case 2. $AF - CD \geq 0$ and $\lambda_2 = B(AE - BD) - A(AF - CD) \geq 0$,
- Case 3. $AF - CD \leq 0$ and $\lambda_3 = D(AF - CD) - E(AE - BD) \geq 0$,
- Case 4. $AF - CD \leq 0$ and $\lambda_4 = C(AF - CD) - B(BF - CE) \geq 0$.

Lemma 2 ([4]). *Let $Z(s)$ be a regular positive-real function. Then $\alpha Z(s)$, $Z(\beta s)$, $Z(s^{-1})$, $Z^{-1}(s)$ are all regular, where $\alpha, \beta > 0$.*

Sara Y. Zhang and Jason Z. Jiang are with the Department of Mechanical Engineering, University of Bristol, Bristol BS8 1TR, U.K. yz13229@bristol.ac.uk, z.jiang@bristol.ac.uk.

Malcolm C. Smith is with the Department of Engineering, University of Cambridge, Cambridge CB2 1PZ, U.K. mcs@eng.cam.ac.uk.

Fig. 2: The five networks with four resistors shown in [5]

Fig. 3: The five-element bridge network quartet which can realise non-regular biquadratics

Lemma 3 ([4]). *Let $Z(s)$ be a regular positive-real function. Then $Z(s) + R$ and $Z(s) + R^{-1}$ are both regular, where R is non-negative.*

Lemma 4 ([4]). *A positive-real biquadratic impedance (1) with $Z(0) = Z(\infty) \neq 0, \infty$ (which implies $AF - CD = 0$) is regular.*

The authors of [5] have studied all the networks with two reactive elements and an arbitrary number of resistors. From the results, the following conclusion for a biquadratic positive-real function $Z(s)$ can be drawn.

Theorem 1 ([5]). *A biquadratic impedance (1) can be realised by two reactive elements and an arbitrary number of resistors if and only if it can be realised by at least one of the five networks shown in Fig. 2.*

A complete analysis of all five element networks with two reactive elements was carried out in [4]. The following has been obtained.

Theorem 2 ([4]). *A biquadratic impedance (1) can be realised by five-element series-parallel networks with two reactive elements if and only if it is regular.*

Theorem 3 ([4]). *Bridge networks with two reactive and three resistive elements can only realise regular immittances except for the network quartet of Fig. 3.*

III. PROOF OF REICHERT'S THEOREM

Using Theorem 2 and Lemma 3, we can immediately see that networks of Figs. 2(a) and (b) can only realise regular impedances. Based on Theorem 3 and Lemma 3, Fig. 2(c) cannot realise any non-regular impedances, either. Hence, an improved version of Theorem 1 can be obtained.

Theorem 4 ([6]). *A biquadratic impedance (1) can be realised with at most two reactive elements if and only if it satisfies one of the following conditions:*

1. $Z(s)$ is regular;
2. $Z(s)$ is realisable by the network shown in Figs. 2(d) or (e).

From the analysis above, we can see that an alternative proof of Reichert's theorem would follow if it can be shown that there are no non-regular biquadratic impedances (1) that can be realised by the networks of Figs. 2(d) and (e) that cannot be realised by the networks of Fig. 3. It can be seen that Fig. 2(e) is the $s \leftrightarrow s^{-1}$ transformation of Fig. 2(d). Hence, we just need to compare the non-regular realisable region of Fig. 2(d) with that of Fig. 3(a) and Fig. 3(b). Without loss of generality, we assume all element values of Fig. 2(d) to be positive and finite.

The rest part of this section focuses on the analysis of the non-regular biquadratics that are realisable by Fig. 2(d). Lemmas 5, 6, 7, 9 and 10 show that when $AF - CD \geq 0$, Fig. 2(d) can realise non-regular biquadratics which can also be realised by Fig. 3(a). Lemma 11 demonstrates that with the condition $AF - CD < 0$, Fig. 2(d) can only realise regular biquadratics.

A. The Network of Fig. 2(d) with $AF - CD \geq 0$

For the network shown in Fig. 2(d), the parameters $A, B, \dots, F$ of the biquadratic function (1) can be expressed as:

$$A = \frac{L_1(R_1 R_2 R_3 U_4 + R_1 R_2 + R_1 R_3 + R_2 R_3)}{k}, \quad (2)$$

$$B = \frac{L_1 S_1 (R_1 + R_3)(R_2 U_4 + 1) + R_1 R_2 R_3}{k}, \quad (3)$$

$$C = \frac{R_2 (R_1 + R_3) S_1}{k}, \quad (4)$$

$$D = \frac{L_1 (R_2 + R_3)(R_1 U_4 + 1)}{k}, \quad (5)$$

$$E = \frac{n_1}{k}, \quad (6)$$

$$F = \frac{(R_1 + R_2 + R_3) S_1}{k}, \quad (7)$$

where

$$U_4 = \frac{1}{R_4}, S_1 = \frac{1}{C_1}. \quad (8)$$

and $n_1 = L_1 S_1 (U_4 (R_1 + R_2 + R_3) + 1) + R_1 (R_2 + R_3)$, It can be calculated that $AF - CD = -n_2/k^2$, where

$$n_2 = L_1 R_1 S_1 (R_1 R_2^2 U_4 - R_1 R_2 - R_1 R_3 - 2R_2 R_3 - R_3^2). \quad (9)$$

Hence $AF - CD \geq 0$ is equivalent to $U_4 \leq x_1$, where

$$x_1 = \frac{R_1 R_2 + R_1 R_3 + 2R_2 R_3 + R_3^2}{R_1 R_2^2}. \quad (10)$$

Lemma 5. *When $U_4 = x_1$, the network of Fig. 2(d) can only realise regular biquadratics.*

Proof. When $U_4 = x_1$, $AF - CD = 0$. The result follows immediately from Lemma 4. $\square$

Fig. 4: The curves of λ_1 (solid) and λ_2 (dashed) with $R_1 = 0.7$, $R_2 = 0.5$, $R_3 = 0.8$ and (a) $U_4 = 10$ ($y_{1+} < y_{2+}$), (b) $U_4 = 0.25$ ($y_{1+} > y_{2+}$).

By substituting (2)-(7) into λ_1 , λ_2 in Lemma 1, the following expressions can be obtained:

$$\lambda_1 = \frac{(R_1 + R_3)^2((R_1 + R_2 + R_3)U_4 + 1)S_1y^2}{k^3} - \frac{2R_1R_2(R_2 + R_3)(R_1 + R_3)S_1y}{k^3} - \frac{R_1^3R_2^2(R_2 + R_3)S_1}{k^3}, \quad (11)$$

$$\lambda_2 = -\frac{R_1R_2(R_2U_4 + 1)(R_1 + R_3)(U_4 - x_3)^2L_1y^2}{k^3} + \frac{2R_1^2R_2^2(R_2 + R_3)(R_1 + R_3)(U_4 - x_3)L_1y}{k^3} + \frac{R_1^3R_2R_3(R_2 + R_3)^2L_1}{k^3}. \quad (12)$$

where

$$y = L_1S_1, x_3 = \frac{R_3}{R_1R_2}. \quad (13)$$

It can be seen that (11) and (12) are quadratic functions of y . The figures of λ_1, λ_2 versus y are shown in Fig. 4 with two sets of R_1, R_2, R_3 and U_4 values. To find out the regularity of the impedance with $AF - CD > 0$, one needs to check whether λ_1 and λ_2 can both be negative for a given positive y . Since x_3 is always smaller than x_1 , the following conclusion can be immediately obtained:

Lemma 6. *When $U_4 = x_3$, the network of Fig. 2(d) can only realise regular biquadratics.*

Proof. From (12), it can be seen that when $U_4 = x_3$, $\lambda_2 = R_1^3R_2R_3(R_2 + R_3)^2L_1/k^3$, which is always positive. $\square$

Fig. 5: The curves of y_{1+} and y_{21+} with $U_4 \in (0, x_3)$, $R_1 = 10$, $R_2 = 1$ and $R_3 = 2$ for the case $x_2 \geq 0$.

For $U_4 \neq x_3$, we need to compare the positive roots for $\lambda_1 = 0$ and $\lambda_2 = 0$.

It can be calculated that the positive root for $\lambda_1 = 0$ is y_{1+} , where

$$y_{1+} = \frac{(R_2 + R_3 + \sqrt{\Delta_{\lambda_1}})R_2R_1}{(R_1 + R_3)((R_1 + R_2 + R_3)U_4 + 1)}, \quad (14)$$

and $\Delta_{\lambda_1} = (R_2 + R_3)(R_1 + R_2 + R_3)(R_1U_4 + 1)$. The positive root for $\lambda_2 = 0$ can be expressed as y_{21+} and y_{22+} , for the cases $0 < U_4 < x_3$ and $x_3 < U_4 < x_1$ respectively, where

$$y_{21+} = \frac{(R_2 + R_3)(R_1R_2 + R_2R_3 + \sqrt{\Delta_{\lambda_2}})}{R_2(R_1 + R_3)(R_2U_4 + 1)(U_4 - x_3)} \quad (15)$$

$$y_{22+} = \frac{(R_2 + R_3)(R_1R_2 + R_2R_3 - \sqrt{\Delta_{\lambda_2}})}{R_2(R_1 + R_3)(R_2U_4 + 1)(U_4 - x_3)} \quad (16)$$

and $\Delta_{\lambda_2} = R_2(R_1 + R_3)(R_1R_2R_3U_4 + R_1R_2 + R_2R_3 + R_1R_3)$. The expressions of y_{1+} , y_{21+} and y_{22+} can be regarded as functions of U_4 , with R_1, R_2 and R_3 as parameters. Since the positive root for $\lambda_2 = 0$ has different expressions with $U_4 \in (0, x_3)$ and (x_3, x_1) , we consider these two cases separately.

(i) $U_4 \in (0, x_3)$, for the extreme case, where $U_4 = 0$ (this happens when the network of Fig. 2(d) collapses down to Fig. 3(a)), it can then be calculated that the condition $y_{21+} \geq y_{1+}$ is equivalent to $x_2 \geq 0$, where x_2 can be found in [8]. The curves of y_{1+} and y_{21+} versus $U_4 \in (0, x_3)$ are plotted in Figs. 5 and 6 to illustrate the cases $x_2 \geq 0$ and $x_2 < 0$ respectively.

Lemma 7 ([8], Lemma 3). *When $U_4 \in (0, x_3)$ and $x_2 \geq 0$, the network of Fig. 2(d) can only realise regular biquadratics.*

From Fig. 6, we obtain that y_{1+} can be greater than y_{21+} with $x_2 < 0$, which means the network of Fig. 2(d) can realise non-regular biquadratics. In order to compare the non-regular realisable region of Fig. 2(d) with Fig. 3(a), the following lemma is derived, based on the necessary and sufficient realisability conditions for network of Fig. 3(a) and regularity conditions for biquadratics in [4]:

Lemma 8. *A non-regular biquadratic impedance can be realised by the network of Fig. 3(a) with R_1, R_2, R_3, L_1 and C_1 positive and finite if and only if:*

$$AF - CD > 0, \quad (17)$$

Fig. 6: The curves of y_{1+} and y_{21+} with $U_4 \in (0, x_3)$, $R_1 = 0.7$, $R_2 = 0.5$ and $R_3 = 0.8$ for the case $x_2 < 0$.

$$\eta_1 \geq 0, \quad (18)$$

$$\eta_2 > 0, \quad (19)$$

where

$$\eta_1 = (AF - CD)(AF - 9CD) + 4BCDE - (AE - BD)(BF - CE), \quad (20)$$

$$\eta_2 = (AE - BD)(BF - CE) - (AF - CD)(AF - 3CD). \quad (21)$$

By analysing whether the non-regular biquadratics realisable by Fig. 2(d) can satisfy the conditions in Lemma 8, we can obtain the following lemma for the case $x_2 < 0$:

Lemma 9 ([8], Lemma 4). *When $U_4 \in (0, x_3)$ and $x_2 < 0$, a non-regular biquadratic (1) can be realised by Fig. 2(d) only if it can be realised by Fig. 3(a).*

Proof Sketch. Since $U_4 \in (0, x_3)$, (17) has been satisfied, then we need to prove (18) and (19) can also be satisfied in this case.

By substituting (2)-(7) into η_1 and η_2 in (20) and (21), the following equations can be obtained:

$$\eta_1 = (R_1 + R_3)^2 f_1 y^2 + 2R_1 R_2 (R_2 + R_3) \times (R_1 + R_3) f_2 y + R_1^2 R_2^2 (R_2 + R_3)^2 f_3, \quad (22)$$

$$\eta_2 = -(R_1 + R_3)^2 (R_1 R_2 U_4 - R_3)^2 y^2 - 2R_1 R_2 (R_2 + R_3) (R_1 + R_3) f_4 y - R_1^4 R_2^2 (R_2 + R_3)^2, \quad (23)$$

where f_1, f_2, f_3 , and f_4 can be found in [8].

It can be seen from (22) that η_1 is a convex quadratic function of y . By calculating the discrimination of η_1 , we obtain

$$\Delta(\eta_1) = \frac{16R_1^3 R_2^4 U_4}{k^8} (R_2 + R_3)^2 (R_1 + R_3)^2 \times (R_1 U_4 + 1)(x_1 - U_4) f_5, \quad (24)$$

where f_5 can be found in [8]. It can be obtained that $\eta_1 > 0$ since $x_2 < 0$ and $\Delta(\eta_1) < 0$.

Then we need to prove $\eta_2 > 0$ if $y_{21+} < y_{1+}$ holds. It can be seen that η_2 is a concave quadratic function of y and it can be proved that $\eta_2 = 0$ has two positive roots which have been noted as y_{31+} and y_{32+} respectively, with $y_{31+} < y_{32+}$.

Fig. 7: The curves of y_{1+} and y_{22+} with $U_4 \in (x_3, x_1)$, $R_1 = 0.7$, $R_2 = 0.5$ and $R_3 = 0.8$.

Then, it can be shown that $y_{21+} > y_{31+}$ and $y_{32+} > y_{1+}$ with $U_4 \in (0, x_3)$ and $x_2 < 0$. So, when $y_{21+} < y_{1+}$, $y_{31+} < y_{21+} < y_{1+} < y_{32+}$ always holds. Based on the property of concave quadratic function, $\eta_2 > 0$ holds when $y \in (y_{21+}, y_{1+})$. The details of the proof can be found in [8]. $\square$

(ii) $U_4 \in (x_3, x_1)$, the curves of y_{1+} and y_{22+} are drawn in Fig. 7 as functions of U_4 with $U_4 \in (x_3, x_1)$. By comparing y_{22+} with y_{1+} , the following lemma can be obtained with $U_4 \in (x_3, x_1)$.

Lemma 10 ([8], Lemma 5). *When $U_4 \in (x_3, x_1)$, the network of Fig. 2(d) can only realise regular biquadratics.*

Proof Sketch. Subtracting y_{22+} and y_{1+} , we obtain:

$$y_{22+} - y_{1+} = \frac{f_6 + f_7 + f_8}{R_1 R_2 ((R_1 + R_2 + R_3) U_4 + 1)} \times \frac{1}{(R_1 + R_3)(R_2 U_4 + 1)(U_4 - x_3)} \quad (25)$$

The expressions of f_6, f_7, f_8 can be found in [8]. First, it can be proved that f_8 is positive. Then it can be further proved that $f_6 + f_7 + f_8 > 0$ when $U_4 \in (x_3, x_1)$. Hence, $y_{22+} - y_{1+} > 0$ always holds with $U_4 \in (x_3, x_1)$, which means at least one of $\lambda_1 > 0$ or $\lambda_2 > 0$ holds. The detailed proof can be found in [8]. $\square$

B. The Network of Fig. 2(d) with $AF - CD < 0$

It has been shown above that Fig. 2(e) is the $s \leftrightarrow s^{-1}$ transformation of Fig. 2(d). From the regularity conditions shown in Lemma 1, it can be seen that frequency inversion ($s \leftrightarrow s^{-1}$) will not change the regularity of the biquadratics $Z(s)$. Hence, we can consider the network of Fig. 2(e) with $AF - CD > 0$ alternatively. For the network shown in Fig. 2(e), the parameters $A, B, \dots, F$ of the biquadratic function (1) can be expressed as:

$$A = \frac{R_2(R_1 + R_3)S_1}{k}, \quad (26)$$

$$B = \frac{R_1 R_2 R_3 + L_1(R_1 + R_3)S_1(R_2 U_4 + 1)}{k}, \quad (27)$$

$$C = \frac{L_1(R_1 R_2 R_3 U_4 + R_1 R_2 + R_1 R_3 + R_2 R_3)}{k}, \quad (28)$$

$$D = \frac{(R_1 + R_2 + R_3)S_1}{k}, \quad (29)$$

$$E = \frac{R_1(R_2 + R_3) + L_1 S_1(1 + U_4(R_1 + R_2 + R_3))}{k}, \quad (30)$$

$$F = \frac{L_1(R_2 + R_3)(R_1 U_4 + 1)}{k}. \quad (31)$$

where U_4 and S_1 are defined in (8). It can be calculated that $AF - CD = n_2/k^2$, with n_2 defined in (9). Hence $AF - CD > 0$ is equivalent to $U_4 > x_1$, with x_1 defined in (10). Then λ_1 , λ_2 can be expressed as quadratic function of y , where

$$\lambda_1 = \frac{((R_1 + R_2 + R_3)U_4 + 1)(R_1 R_2 U_4 - R_3)^2 L_1 y^2}{k^3} - \frac{2R_1(R_2 + R_3)^2(R_1 R_2 U_4 - R_3)L_1 y}{k^3} - \frac{R_1^3(R_2 + R_3)^3 L_1}{k^3} \quad (32)$$

$$\lambda_2 = -\frac{(R_1 + R_3)^3(R_2 U_4 + 1)S_1 y^2}{k^3} + \frac{2R_1 R_2^2(R_1 + R_3)^2 S_1 y}{k^3} + \frac{R_2^3 S_1 R_1^3 R_3}{k^3} \quad (33)$$

with y shown in (13).

By showing the network of Fig. 2(e) can only realise regular biquadratics with the condition $AF - CD > 0$, and based on Lemma 2, we can obtain the following lemma:

Lemma 11 ([8], Lemma 6). *When $U_4 > x_1$, the network in Fig. 2(d) can only realise regular biquadratics.*

Proof Sketch. From previous analysis, similar to the network of Fig. 2(d), to find out the regularity of the impedance of Fig. 2(e) with $AF - CD > 0$, one needs to check whether λ_1 and λ_2 shown in (32), (33) can both be negative for a given positive y . Hence, we need to compare the positive roots for $\lambda_1 = 0$ and $\lambda_2 = 0$. It can be calculated that the positive real roots for $\lambda_1 = 0$ and $\lambda_2 = 0$ are y'_{1+} and y'_{2+} , respectively, where

$$y'_{1+} = \frac{(R_2 + R_3 + \sqrt{\Delta_{\lambda_1}})(R_2 + R_3)R_1}{((R_1 + R_2 + R_3)U_4 + 1)(R_1 R_2 U_4 - R_3)}, \quad (34)$$

$$y'_{2+} = \frac{R_1 R_2 (R_1 R_2 + R_2 R_3 + \sqrt{\Delta_{\lambda_2}})}{(R_2 U_4 + 1)(R_1 + R_3)^2} \quad (35)$$

Subtracting y'_{2+} and y'_{1+} , we can obtain

$$y'_{2+} - y'_{1+} = \frac{R_1}{f_{12}}(f_9 + f_{10} + f_{11}) \quad (36)$$

The expressions of f_9 , f_{10} , f_{11} and f_{12} have been derived in [8]. It can be checked that f_{10} is negative, f_9 is positive and f_{11} is always positive when $U_4 > x_1$ holds. By separating the positive and negative parts in (36), we then define

$$f_{13} = (f_9 + f_{11})^2 - f_{10}^2 \\ = (R_1 + R_3)((R_1 + R_2 + R_3)U_4 + 1)(f_{14} + f_{15})$$

where f_{14} , f_{15} can be found in [8]. Then it can be proved that f_{14} and f_{15} are both monotonically increasing regarding to U_4 when $U_4 > x_1$. By calculating $(f_{14} + f_{15})|_{U_4=x_1} = 0$, f_{13}

is always positive, which means y'_{2+} is always greater than y'_{1+} . So the network of Fig. 2(e) can only realise regular biquadratics with $AF - CD > 0$. $\square$

IV. CONCLUSION

Theorem 5 (Reichert's theorem 1969, [1]). *Any impedance function of a one-port electrical network which can be realised with two reactive elements and an arbitrary number of resistors can be realised with two reactive elements and three resistors.*

Proof. If $Z(s)$ is realisable by two reactive elements and an arbitrary number of resistors, based on Theorem 4, it can either be regular or non-regular but realisable by the network of either Fig. 2(d) or (e). For the case that $Z(s)$ is regular, based on Theorem 2, it can be realised by series-parallel networks with two reactive elements and three resistors. For the case that $Z(s)$ is non-regular but realisable by the network of Fig. 2(d), based on Lemmas 5, 6, 7, 9, 10 and 11, it can also be realised by Fig. 3(a), which has three resistors. The case that $Z(s)$ is non-regular but realisable by the network of Fig. 2(e) can be proved similarly observing the fact that Fig. 2(e) and Fig. 3(b) are the $s \leftrightarrow s^{-1}$ transformations of Fig. 2(d) and Fig. 3(a), respectively. $\square$

REFERENCES

- [1] M. Reichert, "Die kanonisch und übertragerfrei realisierbaren Zweipolfunktionen zweiten Grades (transformerless and canonic realisation of biquadratic immittance functions)," *Arch. Elek. Übertragung*, vol. 23, pp. 201–208, Apr. 1969.
- [2] L. V. Auth, "RLC biquadratic driving-point synthesis using the resistive three-port," *IEEE Trans. on Circuit Theory*, pp. 82–88, 1964.
- [3] J. Z. Jiang and M. C. Smith, "On the theorem of Reichert," *Systems and Control Letters*, vol. 61, pp. 1124–1131, 2012.
- [4] —, "Regular positive-real functions and five-element network synthesis for electrical and mechanical networks," *IEEE Trans. Automatic Control*, vol. 56, no. 6, pp. 1275–1290, June 2011.
- [5] M. Z. Q. Chen and M. C. Smith, "Electrical and mechanical passive network synthesis," in *Recent Advances in Learning and Control*. Springer-Verlag, 2008.
- [6] J. Z. Jiang and M. C. Smith, "Regular positive-real functions and passive networks comprising two reactive elements," *European Control Conference (ECC)*, August 2009, pp. 219–224.
- [7] I. Cederbaum, "Conditions for the impedance and admittance matrices of n-ports without ideal transformers," vol. 105. *Proc. IEE*, 1958, pp. 245–251.
- [8] S. Y. Zhang, J. Z. Jiang, and M. C. Smith, "Supplementary material to: A new proof of Reichert's theorem," Department of Mechanical Engineering, University of Bristol, Tech. Rep. 2016/1, 2016.
- [9] G. Dittmer, "Zur realisierung von RLC-Brückenzweipolen mit zwei Reaktanzen und mehr als drei Widerständen (on the realisation of RLC two-terminal bridge networks with two reactive and more than three resistive elements)," *Nachrichtentechnische Zeitschrift*, vol. 23, pp. 225–230, 1970.
- [10] R. M. Foster and E. L. Ladenheim, "A class of biquadratic impedances," *IEEE Trans. on Circuit Theory*, pp. 262–265, 1963.
- [11] M. E. V. Valkenburg, *Introduction to Modern Network Synthesis*. John Wiley & Sons, 1960.
- [12] M. Z. Q. Chen and M. C. Smith, "A note on tests for positive-real functions," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 390–393, 2009.