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The solution of the optimal control algorithm in each time interval \([t_k, t_{k+1}]\) can be transformed into a boundary value problem by applying Pontryagin’s minimum principle [1]. We start by constructing the Hamiltonian as follows

\[
H(X, u, \lambda) = \frac{1}{2} \theta_{\sigma}(\dot{x} - r_{\sigma})^2 + \frac{1}{2} \eta_m u^2 + \lambda^T \begin{pmatrix}
y \\
-(\alpha x^2 + \beta y^2 - \gamma)y - \omega^2 x + u
\end{pmatrix}
\]

where \(X = [x, \dot{x}]^T = [x, y]^T\) and \(\lambda = [\lambda_1, \lambda_2]^T\). Using the minimum principle gives optimal open loop control

\[
u^* = \arg\min_{u \in \mathbb{R}} H(X^*, u, \lambda) = -\eta_m^{-1} \lambda^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\eta_m^{-1} \lambda_2
\]

and optimal state equation

\[
\dot{X}^* = \nabla_\lambda H = \begin{pmatrix} y^* \\ -(\alpha x^* + \beta y^* - \gamma)y^* - \omega^2 x^* - \eta_m^{-1} \lambda_2
\end{pmatrix}
\]

with initial condition \(X(t_k) = [x(t_k), \dot{x}(t_k)]^T\) and optimal costate equation

\[
\dot{\lambda} = -\nabla_X H = \begin{pmatrix}
\lambda_1(2\alpha x^* y^* + \omega^2) \\
\lambda_2(\alpha x^* + 3\beta y^* - \gamma) - \lambda_1 - \theta_{\sigma}(y^* - r_{\sigma})
\end{pmatrix}
\]

with the terminal condition

\[
\lambda(t_{k+1}) = \begin{pmatrix}
\theta_p(x^*(t_{k+1}) - \hat{r}_p(t_{k+1})) \\
0
\end{pmatrix}
\]

Let \(\hat{x}\) denote the approximation of the optimal solution \(x^*\), then it is feasible to estimate the position error between the VP and the HP based on the collocation method as.

\[
|x^* - \hat{r}_p| = |x^* - \hat{x} + \hat{x} - \hat{r}_p| \leq |x^* - \hat{x}| + |\hat{x} - \hat{r}_p|
\]

Notice that \(|x^* - \hat{x}|\) is negligible due to the high approximation accuracy of numerical methods [2]. In particular, considering that normally the optimal solution \(x^*\) is not available, the approximate
solution $\tilde{x}$ exactly corresponds to the position of the VP in the simulation. Thus, we mainly focus on the estimation of $|\tilde{x} - \hat{r}_p|$. For simplicity, we define $\tilde{x}(t) = a_0 + a_1(t - t_k) + a_2(t - t_k)^2$, $\lambda_1(t) = b_0 + b_1(t - t_k) + b_2(t - t_k)^2$ and $\lambda_2(t) = c_0 + c_1(t - t_k) + c_2(t - t_k)^2$, where $a_i, b_i$ and $c_i, i \in \{0, 1, 2\}$ are unknown constants and $t \in [t_k, t_{k+1}]$. Substituting $\tilde{x}(t), \lambda_1(t)$ and $\lambda_2(t)$ into the above optimal state equation and costate equation at the boundary points yields the linear matrix equation

$$A_k X_k = B_k$$

where

$$A_k = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\theta_p & \theta_p T & \theta_p T^2 & -1 & -T & -T^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & T \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & \eta_m^{-1} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \eta_m & (2\alpha x(t_k)y(t_k) + \omega^2) & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -2\alpha x(t_k)y(t_k) - \omega^2 & 0 & 0 \\
\theta_p & T\theta_p + \theta_\sigma & T(T\theta_p + 2\theta_\sigma) & 0 & 0 & 0 & 0 & 1 & 2T \\
0 & 0 & 0 & 0 & 0 & 1 & 2T & 0 & 0 \\
\end{pmatrix}$$

and

$$B_k = \begin{pmatrix}
x(t_k) \\
y(t_k) \\
\theta_p \hat{r}_p \\
0 \\
-(\alpha x(t_k)^2 + \beta y(t_k)^2 - \gamma)y(t_k) - \omega^2 x(t_k) \\
0 \\
-\theta_\sigma(y(t_k) - r_\sigma(t_k)) \\
\theta_p \hat{r}_p + \theta_\sigma r_\sigma(t_{k+1}) \\
0 \\
\end{pmatrix}, \quad X_k = \begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
b_0 \\
b_1 \\
b_2 \\
c_0 \\
c_1 \\
c_2 \\
\end{pmatrix}$$

Solving equation (1) determines the vector of unknown constants

$$X_k = A_k^{-1} B_k$$

Thus, we obtain the approximate solution

$$\tilde{x}(t) = x(t_k) + y(t_k)(t - t_k) + \frac{N}{D}(t - t_k)^2, \quad t \in [t_k, t_{k+1}]$$
where

\[ N = 2T \left[ \frac{r_σ(t_k) + r_σ(t_{k+1})}{2} - y(t_k) \right] \theta_σ + (\hat{r}_p - x(t_k) - Ty(t_k)) \theta_p \]
\[ - \eta_m \frac{T^2 \omega^2}{2} + \alpha T^2 x(t_k) y(t_k) + \alpha T x(t_k)^2 + 3 \beta T y(t_k)^2 - \gamma T + 2 \]
\[ \left[ (\alpha x(t_k)^2 + \beta y(t_k)^2 - \gamma) y(t_k) + \omega^2 x(t_k) \right] \]

and

\[ D = 2T^2 (\theta_p T + \theta_σ) + 2 \eta_m \frac{T^2 \omega^2}{2} + \alpha T^2 x(t_k) y(t_k) + \alpha T x(t_k)^2 + 3 \beta T y(t_k)^2 - \gamma T + 2. \]

Then we can compute

\[ |\tilde{x}(t_{k+1}) - \hat{r}_p(t_{k+1})| = \lim_{t \to t_{k+1}} |\tilde{x}(t) - \hat{r}_p(t_{k+1})| \]
\[ = |x(t_k) + Ty(t_k) + \frac{N}{D} T^2 - \hat{r}_p(t_{k+1})| \]
\[ \leq T^2 (1 - \theta_p) \left| \frac{2(x(t_k) - \hat{r}_p(t_{k+1})) + T(r_σ(t_k) + r_σ(t_{k+1}))}{|D|} \right| + \eta_m \frac{|L \cdot M|}{|D|} \]

where

\[ L = \frac{T^2 \omega^2}{2} + \alpha T^2 x(t_k) y(t_k) + \alpha T x(t_k)^2 + 3 \beta T y(t_k)^2 - \gamma T + 2 \]

and

\[ M = 2(x(t_k) + Ty(t_k) - \hat{r}_p(t_{k+1})) - T^2 (y(t_k)(\alpha x(t_k)^2 + \beta y(t_k)^2 - \gamma) + \omega^2 x(t_k)) \]

Since \( \hat{r}_p, r_σ, D, L \) and \( M \) are all bounded, it follows from inequality (2) that the bound on the tracking error \( |\tilde{x}(t_{k+1}) - \hat{r}_p(t_{k+1})| \) converges to 0 as \( \theta_p \to 1 \) and \( \eta_m \to 0 \). Similarly, we can estimate the velocity error between the VP and the reference signal encoding the desired signature as follows

\[ |\dot{x}(t_{k+1}) - r_σ(t_{k+1})| = \lim_{t \to t_{k+1}} |\dot{x}(t) - r_σ(t)| \]
\[ = |y(t_k) + \frac{2N}{D} T - r_σ(t_{k+1})| \]
\[ \leq (1 - \theta_σ) \frac{2T^2 |T(y(t_k) - r_σ(t_{k+1})) + 2(\hat{r}_p(t_{k+1}) - x(t_k) - Ty(t_k))|}{|D|} \]
\[ + \theta_σ \frac{2T^2 |r_σ(t_k) - y(t_k)|}{|D|} + 2 \eta_m \frac{|L \cdot P|}{|D|} \]

where

\[ P = y(t_k) - r_σ(t_{k+1}) - T[(\alpha x(t_k)^2 + \beta y(t_k)^2 - \gamma) y(t_k) + \omega^2 x(t_k)] \]

According to inequality (3), the bound of the velocity error goes to 0 if \( \theta_σ \to 1 \), \( \eta_m \to 0 \) and \( r_σ(t_k) = y(t_k) \).
References
