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Detailed Analysis for Remark 5.1 in the Manuscript

The solution of the optimal control algorithm in each time interval \([t_k, t_{k+1}]\) can be transformed into a boundary value problem by applying Pontryagin’s minimum principle [1]. We start by constructing the Hamiltonian as follows

\[
H(X, u, \lambda) = \frac{1}{2} \theta \sigma (\dot{x} - r_\sigma)^2 + \frac{1}{2} \eta_m u^2 + \lambda^T \begin{pmatrix}
y \\
-(\alpha x^2 + \beta y^2 - \gamma)y - \omega^2 x + u
\end{pmatrix}
\]

where \(X = [x, \dot{x}]^T = [x, y]^T\) and \(\lambda = [\lambda_1, \lambda_2]^T\). Using the minimum principle gives optimal open loop control

\[
u^* = \arg \min_{u \in \mathbb{R}} H(X^*, u, \lambda) = -\eta_m^{-1} \lambda^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\eta_m^{-1} \lambda_2
\]

and optimal state equation

\[
\dot{X}^* = \nabla_\lambda H = \begin{pmatrix}
\dot{y}^* \\
-(\alpha x^* y^2 + \beta y^* - \gamma) y^* - \omega^2 x^* - \eta_m^{-1} \lambda_2
\end{pmatrix}
\]

with initial condition \(X(t_k) = [x(t_k), \dot{x}(t_k)]^T\) and optimal costate equation

\[
\dot{\lambda} = -\nabla_X H = \begin{pmatrix}
\lambda_2 (2\alpha x^* y^* + \omega^2) \\
\lambda_2 (\alpha x^2 + 3\beta y^* - \gamma) - \lambda_1 - \theta \sigma (y^* - r_\sigma)
\end{pmatrix}
\]

with the terminal condition

\[
\lambda(t_{k+1}) = \begin{pmatrix}
\theta_p (x^* (t_{k+1}) - \hat{r}_p(t_{k+1})) \\
0
\end{pmatrix}
\]

Let \(\hat{x}\) denote the approximation of the optimal solution \(x^*\), then it is feasible to estimate the position error between the VP and the HP based on the collocation method as.

\[
|x^* - \hat{r}_p| = |x^* - \hat{x} + \hat{x} - \hat{r}_p| \leq |x^* - \hat{x}| + |\hat{x} - \hat{r}_p|
\]

Notice that \(|x^* - \hat{x}|\) is negligible due to the high approximation accuracy of numerical methods [2]. In particular, considering that normally the optimal solution \(x^*\) is not available, the approximate
solution \( \tilde{x} \) exactly corresponds to the position of the VP in the simulation. Thus, we mainly focus on the estimation of \(|\tilde{x} - \hat{r}_p|\). For simplicity, we define \( \tilde{x}(t) = a_0 + a_1(t - t_k) + a_2(t - t_k)^2 \), \( \lambda_1(t) = b_0 + b_1(t - t_k) + b_2(t - t_k)^2 \) and \( \lambda_2(t) = c_0 + c_1(t - t_k) + c_2(t - t_k)^2 \), where \( a_i, b_i \) and \( c_i, i \in \{0, 1, 2\} \) are unknown constants and \( t \in [t_k, t_{k+1}] \). Substituting \( \tilde{x}(t), \lambda_1(t) \) and \( \lambda_2(t) \) into the above optimal state equation and costate equation at the boundary points yields the linear matrix equation

\[
A_k X_k = B_k
\]  

where

\[
A_k = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\theta_p & \theta_p T & \theta_p T^2 & -1 & -T & -T^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & T & T^2 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & \eta_m^{-1} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -(2\alpha x(t_k)y(t_k) + \omega^2) & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -(\alpha x(t_k)^2 + 3\beta y(t_k)^2 - \gamma) & 1 & 0 \\
\theta_p & T \theta_p + \theta_\sigma & T (T \theta_p + 2 \theta_\sigma) & 0 & 0 & 0 & 0 & 1 & 2T \\
0 & 0 & 0 & 0 & 1 & 2T & 0 & 0 & 0 \\
\end{pmatrix}
\]

and

\[
B_k = \begin{pmatrix}
x(t_k) \\
y(t_k) \\
\theta_p \hat{r}_p \\
0 \\
-(\alpha x(t_k)^2 + \beta y(t_k)^2 - \gamma) y(t_k) - \omega^2 x(t_k) \\
0 \\
-\theta_\sigma (y(t_k) - r_\sigma(t_k)) \\
\theta_p \hat{r}_p + \theta_\sigma r_\sigma(t_{k+1}) \\
0 \\
\end{pmatrix}, \quad X_k = \begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
b_0 \\
b_1 \\
b_2 \\
c_0 \\
c_1 \\
c_2 \\
\end{pmatrix}
\]

Solving equation (1) determines the vector of unknown constants

\[
X_k = A_k^{-1} B_k
\]

Thus, we obtain the approximate solution

\[
\tilde{x}(t) = x(t_k) + y(t_k)(t - t_k) + \frac{N}{D}(t - t_k)^2, \quad t \in [t_k, t_{k+1}]
\]
where
\[
\mathcal{N} = 2T \left[ \frac{r_\sigma(t_k) + r_\sigma(t_{k+1})}{2} - y(t_k) \theta_\sigma + (\dot{r}_p - x(t_k) - Ty(t_k)) \theta_p \right] 
- \eta_m \left( \frac{\omega^2}{2} + \alpha T^2 x(t_k) y(t_k) + \alpha T x(t_k)^2 + 3 \beta T y(t_k)^2 - \gamma T + 2 \right) [\alpha x(t_k)^2 + \beta y(t_k)^2 - \gamma y(t_k) + \omega^2 x(t_k)]
\]
and
\[
\mathcal{D} = 2T^2 (\theta_p T + \theta_\sigma) + 2 \eta_m \left( \frac{\omega^2}{2} + \alpha T^2 x(t_k) y(t_k) + \alpha T x(t_k)^2 + 3 \beta T y(t_k)^2 - \gamma T + 2 \right).
\]

Then we can compute
\[
|\tilde{x}(t_{k+1}) - \dot{r}_p(t_{k+1})| = \lim_{t \to t_{k+1}} |\tilde{x}(t) - \dot{r}_p(t_{k+1})|
= |x(t_k) + Ty(t_k) + \frac{\mathcal{N}}{\mathcal{D}} T^2 - \dot{r}_p(t_{k+1})|
\leq T^2 (1 - \theta_p) \left[ \frac{2(x(t_k) - \dot{r}_p(t_{k+1})) + T(r_\sigma(t_k) + r_\sigma(t_{k+1}))}{|\mathcal{D}|} \right] + \eta_m \frac{\mathcal{L} \cdot \mathcal{M}}{|\mathcal{D}|}
\]
where
\[
\mathcal{L} = \frac{T^2 \omega^2}{2} + \alpha T^2 x(t_k) y(t_k) + \alpha T x(t_k)^2 + 3 \beta T y(t_k)^2 - \gamma T + 2
\]
and
\[
\mathcal{M} = 2(x(t_k) + Ty(t_k) - \dot{r}_p(t_{k+1})) - T^2 (y(t_k)(\alpha x(t_k)^2 + \beta y(t_k)^2 - \gamma) + \omega^2 x(t_k))
\]

Since \( \hat{r}_p, r_\sigma, \mathcal{D}, \mathcal{L} \) and \( \mathcal{M} \) are all bounded, it follows from inequality (2) that the bound on the tracking error \(|\tilde{x}(t_{k+1}) - \dot{r}_p(t_{k+1})|\) converges to 0 as \( \theta_p \to 1 \) and \( \eta_m \to 0 \). Similarly, we can estimate the velocity error between the VP and the reference signal encoding the desired signature as follows
\[
|\dot{x}(t_{k+1}) - r_\sigma(t_{k+1})| = \lim_{t \to t_{k+1}} |\dot{x}(t) - r_\sigma(t)|
= |y(t_k) + \frac{2\mathcal{N}}{\mathcal{D}} T - r_\sigma(t_{k+1})|
\leq (1 - \theta_\sigma) \left[ \frac{2T^2 |T(y(t_k) - r_\sigma(t_{k+1})) + 2(\dot{r}_p(t_{k+1}) - x(t_k) - Ty(t_k))|}{|\mathcal{D}|} \right] + \theta_\sigma \frac{2T^2 |r_\sigma(t_k) - y(t_k)|}{|\mathcal{D}|} + 2 \eta_m \frac{\mathcal{L} \cdot \mathcal{P}}{|\mathcal{D}|}
\]
where
\[
\mathcal{P} = y(t_k) - r_\sigma(t_{k+1}) - T[(\alpha x(t_k)^2 + \beta y(t_k)^2 - \gamma)y(t_k) + \omega^2 x(t_k)]
\]

According to inequality (3), the bound of the velocity error goes to 0 if \( \theta_\sigma \to 1 \), \( \eta_m \to 0 \) and \( r_\sigma(t_k) = y(t_k) \).
References
