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Spectral Bounds for the Torsion Function

M. van den Berg

Abstract. Let Ω be an open set in Euclidean space \mathbb{R}^m , $m = 2, 3, \dots$, and let v_Ω denote the torsion function for Ω . It is known that v_Ω is bounded if and only if the bottom of the spectrum of the Dirichlet Laplacian acting in $\mathcal{L}^2(\Omega)$, denoted by $\lambda(\Omega)$, is bounded away from 0. It is shown that the previously obtained bound $\|v_\Omega\|_{\mathcal{L}^\infty(\Omega)}\lambda(\Omega) \geq 1$ is sharp: for $m \in \{2, 3, \dots\}$, and any $\epsilon > 0$ we construct an open, bounded and connected set $\Omega_\epsilon \subset \mathbb{R}^m$ such that $\|v_{\Omega_\epsilon}\|_{\mathcal{L}^\infty(\Omega_\epsilon)}\lambda(\Omega_\epsilon) < 1 + \epsilon$. An upper bound for v_Ω is obtained for planar, convex sets in Euclidean space \mathbb{R}^2 , which is sharp in the limit of elongation. For a complete, non-compact, m -dimensional Riemannian manifold M with non-negative Ricci curvature, and without boundary it is shown that v_Ω is bounded if and only if the bottom of the spectrum of the Dirichlet–Laplace–Beltrami operator acting in $\mathcal{L}^2(\Omega)$ is bounded away from 0.

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1. Introduction

Let Ω be an open set in \mathbb{R}^m , and let Δ be the Laplace operator acting in $L^2(\mathbb{R}^m)$. Let $(B(s), s \geq 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be Brownian motion on \mathbb{R}^m with generator Δ . For $x \in \Omega$ we denote the first exit time, and expected lifetime of Brownian motion by

$$T_\Omega = \inf \{s \geq 0 : B(s) \notin \Omega\},$$

and

$$v_\Omega(x) = \mathbb{E}_x[T_\Omega], \quad x \in \Omega, \quad (1)$$

respectively, where \mathbb{E}_x denotes the expectation associated with \mathbb{P}_x . Then v_Ω is the torsion function for Ω , i.e. the unique solution of

$$-\Delta v = 1, \quad v \in H_0^1(\Omega). \quad (2)$$

The bottom of the spectrum of the Dirichlet Laplacian acting in $\mathcal{L}^2(\Omega)$ is denoted by

$$\lambda(\Omega) = \inf_{\varphi \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |D\varphi|^2}{\int_{\Omega} \varphi^2}. \quad (3)$$

It was shown in [1, 2] that $\|v_{\Omega}\|_{\mathcal{L}^{\infty}(\Omega)}$ is finite if and only if $\lambda(\Omega) > 0$. Moreover, if $\lambda(\Omega) > 0$, then

$$\lambda(\Omega)^{-1} \leq \|v_{\Omega}\|_{\mathcal{L}^{\infty}(\Omega)} \leq (4 + 3m \log 2) \lambda(\Omega)^{-1}. \quad (4)$$

The upper bound in (4) was subsequently improved (see [3]) to

$$\|v_{\Omega}\|_{\mathcal{L}^{\infty}(\Omega)} \leq \frac{1}{8}(m + cm^{1/2} + 8)\lambda(\Omega)^{-1},$$

where

$$c = (5(4 + \log 2))^{1/2}.$$

In Theorem 1 below we show that the coefficient 1 of $\lambda(\Omega)^{-1}$ in the left-hand side of (4) is sharp.

Theorem 1. *For $m \in \{2, 3, \dots\}$, and any $\epsilon > 0$ there exists an open, bounded, and connected set $\Omega_{\epsilon} \subset \mathbb{R}^m$ such that*

$$\|v_{\Omega_{\epsilon}}\|_{\mathcal{L}^{\infty}(\Omega_{\epsilon})} \lambda(\Omega_{\epsilon}) < 1 + \epsilon. \quad (5)$$

The set Ω_{ϵ} is constructed explicitly in the proof of Theorem 1.

It has been shown by L. E. Payne (see (3.12) in [4]) that for any convex, open $\Omega \subset \mathbb{R}^m$ for which $\lambda(\Omega) > 0$,

$$\|v_{\Omega}\|_{\mathcal{L}^{\infty}(\Omega)} \lambda(\Omega) \geq \frac{\pi^2}{8}, \quad (6)$$

with equality if Ω is a slab, i.e. the connected, open set, bounded by two parallel $(m - 1)$ -dimensional hyperplanes. Theorem 2 below shows that for any sufficiently elongated, convex, planar set (not just an elongated rectangle) $\|v_{\Omega}\|_{\mathcal{L}^{\infty}(\Omega)} \lambda(\Omega)$ is approximately equal to $\frac{\pi^2}{8}$. We denote the width and the diameter of a bounded open set Ω by $w(\Omega)$ (i.e. the minimal distance of two parallel lines supporting Ω), and $\text{diam}(\Omega) = \sup\{|x - y| : x \in \Omega, y \in \Omega\}$ respectively.

Theorem 2. *If Ω is a bounded, planar, open, convex set with width $w(\Omega)$, and diameter $\text{diam}(\Omega)$, then*

$$\|v_{\Omega}\|_{\mathcal{L}^{\infty}(\Omega)} \lambda(\Omega) \leq \frac{\pi^2}{8} \left(1 + 7 \cdot 3^{2/3} \left(\frac{w(\Omega)}{\text{diam}(\Omega)} \right)^{2/3} \right).$$

In the Riemannian manifold setting we denote the bottom of the spectrum of the Dirichlet–Laplace–Beltrami operator by (3). We have the following.

Theorem 3. *Let M be a complete, non-compact, m -dimensional Riemannian manifold, without boundary, and with non-negative Ricci curvature. There*

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exists $K < \infty$, depending on M only, such that if $\Omega \subset M$ is open, and $\lambda(\Omega) > 0$, then

$$\lambda(\Omega)^{-1} \leq \|v_\Omega\|_{\mathcal{L}^\infty(\Omega)} \leq 2^{(3m+8)/4} \cdot 3^{m/2} K^2 \lambda(\Omega)^{-1}, \quad (7)$$

where K is the constant in the Li-Yau inequality in (35) below.

The proofs of Theorems 1, 2, and 3 will be given in Sects. 2, 3 and 4 respectively.

Below we recall some basic facts on the connection between torsion function and heat kernel. It is well known (see [5–7]) that the heat equation

$$\Delta u(x; t) = \frac{\partial u(x; t)}{\partial t}, \quad x \in M, \quad t > 0,$$

has a unique, minimal, positive fundamental solution $p_M(x, y; t)$, where $x \in M$, $y \in M$, $t > 0$. This solution, the heat kernel for M , is symmetric in x, y , strictly positive, jointly smooth in $x, y \in M$ and $t > 0$, and it satisfies the semigroup property

$$p_M(x, y; s + t) = \int_M dz p_M(x, z; s) p_M(z, y; t),$$

for all $x, y \in M$ and $t, s > 0$, where dz is the Riemannian measure on M . See, for example, [8] for details. If Ω is an open subset of M , then we denote the unique, minimal, positive fundamental solution of the heat equation on Ω by $p_\Omega(x, y; t)$, where $x \in \Omega, y \in \Omega, t > 0$. This Dirichlet heat kernel satisfies,

$$p_\Omega(x, y; t) \leq p_M(x, y; t), \quad x \in \Omega, y \in \Omega, t > 0.$$

Define $u_\Omega : \Omega \times (0, \infty) \mapsto \mathbb{R}$ by

$$u_\Omega(x; t) = \int_\Omega dy p_\Omega(x, y; t).$$

Then,

$$u_\Omega(x; t) = \mathbb{P}_x[T_\Omega > t],$$

and by (1)

$$v_\Omega(x) = \int_0^\infty dt \mathbb{P}_x [T_\Omega > t] = \int_0^\infty dt \int_\Omega dy p_\Omega(x, y; t). \quad (8)$$

It is straightforward to verify that v_Ω as in (8) satisfies (2).

2. Proof of Theorem 1

We introduce the following notation. Let $C_L = (-\frac{L}{2}, \frac{L}{2})^{m/2}$ be the open cube with measure L^m , and delete from C_L , N^m closed balls with radii δ , where each ball $B(c_i; \delta)$ is positioned at the centre of an open cube Q_i with measure $(L/N)^m$. These open cubes are pairwise disjoint, and contained in C_L . Let $0 < \delta < \frac{L}{2N}$, and put (Fig. 1)

$$\Omega_{\delta, N, L} = C_L - \cup_{i=1}^{N^m} B(c_i; \delta).$$

The set $\Omega_{\delta, N, L}$ also features in [9], where the sharpness of an inequality due to Pólya has been established.

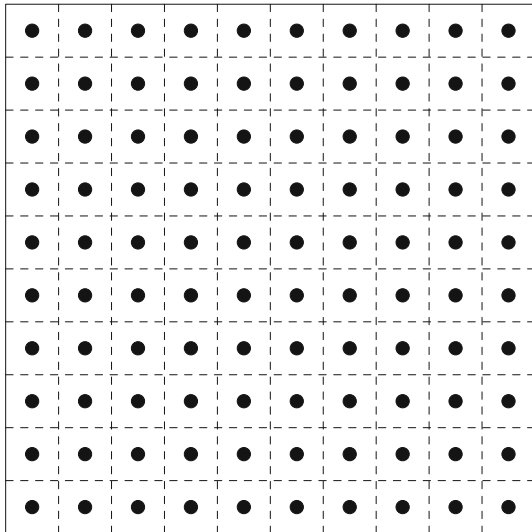


FIGURE 1. $\Omega_{\delta, N, L}$ with $m = 2, N = 10, \delta = \frac{L}{8N}$

Below we will show that for any $\epsilon > 0$ we can choose δ, N such that

$$\|v_{\Omega_{\delta, N, L}}\|_{\mathcal{L}^\infty(\Omega_{\delta, N, L})} \lambda(\Omega_{\delta, N, L}) < 1 + \epsilon.$$

In Lemma 4 below we show that $\lambda(\Omega_{\delta, N, L})$ is approximately equal to the first eigenvalue, $\mu_{1, B(0; \delta), L/N}$, of the Laplacian with Neumann boundary conditions on $\partial C_{L/N}$, and with Dirichlet boundary conditions on $\partial B(0; \delta)$. The requirement $\mu_{1, B(0; \delta), L/N}$ not being too small stems from the fact that the approximation of replacing the Neumann boundary conditions on C_L is a surface effect which should not dominate the leading term $\mu_{1, B(0; \delta), L/N}$.

Lemma 4. *If $\delta \leq \frac{L}{4N}$, $N \geq 10$, and $\frac{N}{L^2} \leq \mu_{1, B(0; \delta), L/N}$, then*

$$\lambda(\Omega_{\delta, N, L}) \leq \mu_{1, B(0; \delta), L/N} + 32m \left(\frac{5}{4}\right)^m \left(\frac{N}{L^2} + \frac{1}{N^{1/2}} \mu_{1, B(0; \delta), L/N}\right).$$

Proof. Let $\varphi_{1, B(0; \delta), L/N}$ be the first eigenfunction (positive) corresponding to $\mu_{1, B(0; \delta), L/N}$, and normalised in $\mathcal{L}^2(C_{L/N} - B(0; \delta))$. In order to prove the lemma we construct a test function by periodically extending $\varphi_{1, B(0; \delta), L/N}$ to all cubes Q_1, \dots, Q_{N^m} of $\Omega_{\delta, N, L}$. We denote this periodic extension by f . We define

$$C_{L, N} = C_{L(1 - \frac{2}{N})}.$$

So $C_{L, N}$ is the sub-cube of C_L with the outer layer of cubes of size L/N removed. Let

$$\tilde{f} = \left(1 - \frac{\text{dist}(x, C_{L, N})}{L/(4N)}\right)_+ f.$$

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Then $\tilde{f} \in H_0^1(\Omega_{\delta,N,L})$, and

$$\|\tilde{f}\|_{\mathcal{L}^2(\Omega_{\delta,N,L})}^2 \geq \int_{C_{L,N}} f^2 = (N-2)^m, \quad (9)$$

since f restricted to any of the cubes Q_i in $\Omega_{\delta,N,L}$ is normalised. Furthermore

$$\begin{aligned} |D\tilde{f}|^2 &\leq \left(1 - \frac{\text{dist}(x, C_{L,N})}{L/(4N)}\right)^2 |Df|^2 + 1_{C_L - C_{L,N}} \left(\left(\frac{4N}{L}\right)^2 f^2 + \frac{8N}{L} f |Df| \right) \\ &\leq |Df|^2 + \left(\frac{4N}{L}\right)^2 1_{C_L - C_{L,N}} f^2 + \frac{8N}{L} 1_{C_L - C_{L,N}} f |Df|. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\Omega_{\delta,N,L}} |D\tilde{f}|^2 &\leq \int_{\Omega_{\delta,N,L}} |Df|^2 + \left(\frac{4N}{L}\right)^2 \int_{C_L - C_{L,N}} f^2 \\ &\quad + \frac{8N}{L} \left(\int_{C_L - C_{L,N}} |Df|^2 \right)^{1/2} \left(\int_{C_L - C_{L,N}} f^2 \right)^{1/2} \\ &= N^m \mu_{1,B(0;\delta),L/N} + (N^m - (N-2)^m) \left(\left(\frac{4N}{L}\right)^2 + \frac{8N}{L} (\mu_{1,B(0;\delta),L/N})^{1/2} \right) \\ &\leq N^m \mu_{1,B(0;\delta),L/N} + (N^m - (N-2)^m) \left(\left(\frac{4N}{L}\right)^2 + 8N^{1/2} \mu_{1,B(0;\delta),L/N} \right), \end{aligned} \quad (10)$$

where we have used the last hypothesis in the lemma. By (9), (10), the Rayleigh-Ritz variational formula, and the hypothesis $N \geq 10$,

$$\begin{aligned} \lambda(\Omega_{\delta,N,L}) &\leq \mu_{1,B(0;\delta),L/N} \\ &\quad + \frac{N^m - (N-2)^m}{(N-2)^m} \left(\left(\frac{4N}{L}\right)^2 + (8N^{1/2} + 1) \mu_{1,B(0;\delta),L/N} \right) \\ &\leq \mu_{1,B(0;\delta),L/N} + 32m \left(\frac{5}{4}\right)^m \left(\frac{N}{L^2} + \frac{1}{N^{1/2}} \mu_{1,B(0;\delta),L/N} \right). \end{aligned} \quad (11)$$

□

To obtain an upper bound for $\|v_{\Omega_{\delta,N,L}}\|_{\mathcal{L}^\infty(\Omega_{\delta,N,L})}$, we change the Dirichlet boundary conditions on ∂C_L to Neumann boundary conditions. This increases the corresponding heat kernel, torsion function, and \mathcal{L}^∞ norm respectively. By periodicity, we have that

$$\|v_{\Omega_{\delta,N,L}}\|_{\mathcal{L}^\infty(\Omega_{\delta,N,L})} \leq \|\tilde{v}_{C_{L/N} - B(0;\delta)}\|_{\mathcal{L}^\infty(C_{L/N} - B(0;\delta))}, \quad (12)$$

where $\tilde{v}_{C_{L/N} - B(0;\delta)}$ is the torsion function with Neumann boundary conditions on $\partial C_{L/N}$, and Dirichlet boundary conditions on $\partial B(0;\delta)$. Denote the spectrum of the corresponding Laplacian by $\{\mu_j := \mu_{j,B(0;\delta),L/N}, j = 1, 2, \dots\}$, and let $\{\varphi_j := \varphi_{1,B(0;\delta),L/N}, j = 1, 2, \dots\}$ denote a corresponding

orthonormal basis of eigenfunctions. We denote by $\pi_{\delta,N/L}(x, y; t)$, $x \in C_{L/N} - B(0; \delta)$, $y \in C_{L/N} - B(0; \delta)$, $t > 0$ the corresponding heat kernel. Then

$$\pi_{\delta,N/L}(x, y; t) = \sum_{j=1}^{\infty} e^{-t\mu_j} \varphi_j(x) \varphi_j(y), \quad (13)$$

and

$$\begin{aligned} & \tilde{v}_{C_{L/N} - B(0; \delta)}(x) \\ &= \int_0^{\infty} dt \int_{C_{L/N} - B(0; \delta)} dy \pi_{\delta,N/L}(x, y; t) \left(\frac{\varphi_1(y)}{\|\varphi_1\|} + 1 - \frac{\varphi_1(y)}{\|\varphi_1\|} \right) \\ &= \frac{1}{\mu_1} \frac{\varphi_1(x)}{\|\varphi_1\|} + \int_0^{\infty} dt \int_{C_{L/N} - B(0; \delta)} dy \pi_{\delta,N/L}(x, y; t) \left(1 - \frac{\varphi_1(y)}{\|\varphi_1\|} \right) \\ &\leq \frac{1}{\mu_1} + \int_0^T dt \int_{C_{L/N} - B(0; \delta)} dy \pi_{\delta,N/L}(x, y; t) \\ &\quad + \int_T^{\infty} dt \int_{C_{L/N} - B(0; \delta)} dy \pi_{\delta,N/L}(x, y; t) \left(1 - \frac{\varphi_1(y)}{\|\varphi_1\|} \right) \\ &\leq \frac{1}{\mu_1} + T + \int_T^{\infty} dt \int_{C_{L/N} - B(0; \delta)} dy \pi_{\delta,N/L}(x, y; t) \left(1 - \frac{\varphi_1(y)}{\|\varphi_1\|} \right), \quad (14) \end{aligned}$$

where $\|\varphi_1\| = \|\varphi_1\|_{\mathcal{L}^{\infty}(C_{L/N} - B(0; \delta))}$. By (13), we have that the third term in the right-hand side of (14) equals

$$\sum_{j=1}^{\infty} \mu_j^{-1} e^{-T\mu_j} \varphi_j(x) \int_{C_{L/N} - B(0; \delta)} dy \varphi_j(y) \left(1 - \frac{\varphi_1(y)}{\|\varphi_1\|} \right). \quad (15)$$

The term with $j = 1$ in (15) is bounded from above by

$$\begin{aligned} & \mu_1^{-1} \|\varphi_1\| \int_{C_{L/N} - B(0; \delta)} \|\varphi_1\| \left(1 - \frac{\varphi_1}{\|\varphi_1\|} \right) \\ &= \mu_1^{-1} \|\varphi_1\| \int_{C_{L/N} - B(0; \delta)} (\|\varphi_1\| - \varphi_1) \\ &\leq \mu_1^{-1} \left(\|\varphi_1\|^2 \left(\frac{L}{N} \right)^m - 1 \right), \end{aligned}$$

where we used the fact that $1 = \int_{C_{L/N} - B(0; \delta)} \varphi_1^2 \leq \|\varphi_1\| \int_{C_{L/N} - B(0; \delta)} \varphi_1$. It was shown on p.586, lines -3,-4, in [9] (with appropriate adjustment in notation) that

$$\|\varphi_1\|^2 \leq \left(\frac{N}{L} \right)^m \left(1 - s\mu_1 - \frac{mL^2}{3esN^2} \right)^{-1}, \quad s \geq 0,$$

provided the last term in the round brackets is non-negative. The optimal choice for s gives that

$$\|\varphi_1\|^2 \leq \left(\frac{N}{L} \right)^m \left(1 - \frac{(4m\mu_1)^{1/2}L}{(3e)^{1/2}N} \right)^{-1}, \quad \mu_1 < \frac{3eN^2}{4mL^2}.$$

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By further restricting the range for μ_1 , we have that the first term with $j = 1$ in (15) is then bounded from above by

$$\mu_1^{-1} \frac{2L(m\mu_1/(3eN^2))^{1/2}}{1 - 2L(m\mu_1/(3eN^2))^{1/2}} \leq \frac{(2m)^{1/2}L}{\mu_1^{1/2}N}, \quad \mu_1 \leq \frac{3eN^2}{16mL^2}. \quad (16)$$

The terms with $j \geq 2$ in (15) give, by Cauchy–Schwarz for both the series in j , and the integral over $C_{L/N} - B(0; \delta)$, a contribution

$$\begin{aligned} & \left| \sum_{j=2}^{\infty} \mu_j^{-1} e^{-T\mu_j} \varphi_j(x) \int_{C_{L/N} - B(0; \delta)} \varphi_j \left(1 - \frac{\varphi_1}{\|\varphi_1\|} \right) \right| \\ & \leq \mu_2^{-1} \sum_{j=2}^{\infty} e^{-T\mu_j} |\varphi_j(x)| \int_{C_{L/N} - B(0; \delta)} |\varphi_j| \\ & \leq \mu_2^{-1} \left(\frac{L}{N} \right)^{m/2} \left(\sum_{j=2}^{\infty} e^{-T\mu_j} \right)^{1/2} \left(\sum_{j=2}^{\infty} e^{-T\mu_j} |\varphi_j(x)|^2 \right)^{1/2} \\ & \leq \mu_2^{-1} \left(\frac{L}{N} \right)^{m/2} \left(\sum_{j=2}^{\infty} e^{-T\mu_j} \right)^{1/2} (\pi_{\delta, N/L}(x, x; T))^{1/2}. \end{aligned} \quad (17)$$

To bound the first series in the right-hand side of (17), we note that the μ_j 's are bounded from below by the Neumann eigenvalues of the cube $C_{L/N}$. So choosing $T = (L/N)^2$ we get that

$$\left(\sum_{j=2}^{\infty} e^{-L^2\mu_j/N^2} \right)^{1/2} \leq \left(1 + \sum_{j=1}^{\infty} e^{-\pi^2 j^2} \right)^{m/2} \leq \left(\frac{4}{3} \right)^{m/2}.$$

Similarly to the proof of Lemma 3.1 in [9], we have that

$$\begin{aligned} (\pi_{\delta, N/L}(x, x; L^2/N^2))^{1/2} & \leq (\pi_{0, N/L}(x, x; L^2/N^2))^{1/2} \\ & \leq \left(\frac{N}{L} \right)^{m/2} \left(1 + 2 \sum_{j=1}^{\infty} e^{-\pi^2 j^2} \right)^{m/2} \\ & \leq \left(\frac{4}{3} \right)^{m/2} \left(\frac{N}{L} \right)^{m/2}. \end{aligned} \quad (18)$$

Finally, $\mu_2 \geq \frac{\pi^2 N^2}{L^2}$, together with (12), (14), (16), (17), (18), and the choice $T = (L/N)^2$ gives that

$$\|v_{\Omega_{\delta, N, L}}\|_{\mathcal{L}^\infty(\Omega_{\delta, N, L})} \leq \mu_1^{-1} + \frac{(2m)^{1/2}L}{\mu_1^{1/2}N} + \left(\frac{4}{3} \right)^m \frac{L^2}{N^2}, \quad \mu_1 \leq \frac{3eN^2}{16mL^2}. \quad (19)$$

Proof of Theorem 1. Let $1 < \alpha < 2$. By (11) and (19), we have that

$$\begin{aligned} \lambda(\Omega_{\delta,N,L}) \|v_{\Omega_{\delta,N,L}}\|_{\mathcal{L}^\infty(\Omega_{\delta,N,L})} &\leq \left(\mu_1 + 32m \left(\frac{5}{4}\right)^m \left(\frac{N}{L^2} + \frac{1}{N^{1/2}} \mu_1 \right) \right) \\ &\quad \times \left(\mu_1^{-1} + \frac{(2m)^{1/2} L}{\mu_1^{1/2} N} + \left(\frac{4}{3}\right)^m \frac{L^2}{N^2} \right), \end{aligned} \quad (20)$$

provided

$$\frac{N}{L^2} \leq \mu_1 \leq \frac{3eN^2}{16mL^2}.$$

First consider the planar case $m = 2$. Recall Lemma 3.1 in [9]: for $\delta < L/(6N)$,

$$\frac{N^2}{100L^2} \left(\log \frac{L}{2\delta N} \right)^{-1} \leq \mu_{1,B(0;\delta),L/N} \leq \frac{8\pi N^2}{(4-\pi)L^2} \left(\log \frac{L}{2\delta N} \right)^{-1}. \quad (21)$$

Let

$$\delta^* := \delta^*(\alpha, N, L) = \frac{L}{2N} e^{-N^{2-\alpha}}, \quad (22)$$

where $1 < \alpha < 2$. Let $N_1 \in \mathbb{N}$ be such that for all $N \geq N_1$, $\delta^* < L/(6N)$. We now use (21) to see that there exists $C > 1$ such that

$$C^{-1} \frac{N^\alpha}{L^2} \leq \mu_{1,B(0;\delta^*),L/N} \leq C \frac{N^\alpha}{L^2}. \quad (23)$$

(In fact $C = \max\{100, 8\pi/(4-\pi)\}$). We subsequently let $N_2 \in \mathbb{N}$ be such that for all $N \geq N_2$,

$$\frac{N}{L^2} \leq C^{-1} \frac{N^\alpha}{L^2} \leq C \frac{N^\alpha}{L^2} \leq \frac{3eN^2}{16mL^2}.$$

By (20), (23), and all $N \geq \max\{N_1, N_2\}$ we have that

$$\lambda(\Omega_{\delta^*,N,L}) \|v_{\Omega_{\delta^*,N,L}}\|_{\mathcal{L}^\infty(\Omega_{\delta^*,N,L})} \leq 1 + \mathcal{C}(N^{1-\alpha} + N^{(\alpha-2)/2}), \quad (24)$$

where \mathcal{C} depends on C and on m only. Finally, we let $N_3 \in \mathbb{N}$ be such that for all $N \geq N_3$,

$$\mathcal{C}(N^{1-\alpha} + N^{(\alpha-2)/2}) < \epsilon.$$

We conclude that (5) holds with $\Omega_\epsilon = \Omega_{\delta^*,N,L}$ with δ^* given by (22), and $N \geq \max\{N_1, N_2, N_3\}$.

Next consider the case $m = 3, 4, \dots$. We apply Lemma 3.2 in [9] to the case $K = B(0; \delta)$, and denote the Newtonian capacity of K by $\text{cap}(K)$. Then $\text{cap}(B(0; \delta)) = \kappa_m \delta^{m-2}$, where κ_m is the Newtonian capacity of the ball with radius 1 in \mathbb{R}^m . Then Lemma 3.2 gives that there exists $C \geq 1$ such that

$$C^{-1} \left(\frac{N}{L}\right)^m \delta^{m-2} \leq \mu_{1,B(0;\delta),L/N} \leq C \left(\frac{N}{L}\right)^m \delta^{m-2}, \quad (25)$$

provided

$$\kappa_m \delta^{m-2} \leq \frac{1}{16} (L/N)^{m-2}. \quad (26)$$

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We choose

$$\delta^* := \delta^*(\alpha, N, L) = LN^{(\alpha-m)/(m-2)}. \quad (27)$$

This gives inequality (23) by (25). The requirement (26) holds for all $N \geq N_1$, where N_1 is the smallest natural number such that $N_1^{2-\alpha} \geq 16\kappa_m$. The remainder of the proof follows the lines below (23) with the appropriate adjustment of constants, and the choice of δ^* as in (27). \square

We note that the choice $\alpha = \frac{4}{3}$ in either (22) or in (27) gives, by (24), the decay rate

$$\lambda(\Omega_{\delta^*, N, L}) \|v_{\Omega_{\delta^*, N, L}}\|_{\mathcal{L}^\infty(\Omega_{\delta^*, N, L})} \leq 1 + 2\mathcal{C}N^{-1/3}. \quad (28)$$

3. Proof of Theorem 2

In view of Payne's inequality (6) it suffices to obtain an upper bound for $\|v_\Omega\|_{\mathcal{L}^\infty(\Omega)}\lambda(\Omega)$. We first observe, that by domain monotonicity of the torsion function, v_Ω is bounded by the torsion function for the (connected) set bounded by the two parallel lines tangent to Ω at distance $w(\Omega)$. Hence

$$\|v_\Omega\|_{\mathcal{L}^\infty(\Omega)} \leq \frac{w(\Omega)^2}{8}. \quad (29)$$

In order to obtain an upper bound for $\lambda(\Omega)$, we introduce the following notation. For a planar, open, convex set, with finite measure, we let z_1, z_2 be two points on the boundary of Ω which realise the width. That is there are two parallel lines tangent to $\partial\Omega$, at z_1 and z_2 respectively, and at distance $w(\Omega)$. Let the x -axis be perpendicular to the vector z_1z_2 , containing the point $\frac{1}{2}(z_1 + z_2)$. We consider the family of line segments parallel to the x -axis, obtained by intersection with Ω , and let l_1, l_2 be two points on the boundary of Ω which realise the maximum length L of this family. The quadrilateral with vertices, z_1, z_2, l_1, l_2 is contained in Ω . This quadrilateral in turn contains a rectangle with side-lengths h , and $(1 - \frac{h}{w(\Omega)})L$ respectively, where $h \in [0, w(\Omega))$ is arbitrary. Hence, by domain monotonicity of the Dirichlet eigenvalues, we have that

$$\lambda(\Omega) \leq \pi^2 h^{-2} + \pi^2 \left(1 - \frac{h}{w(\Omega)}\right)^{-2} L^{-2}.$$

Minimising the right-hand side above with respect to h gives that

$$h = \frac{(w(\Omega)L^2)^{1/3}}{1 + \left(\frac{L}{w(\Omega)}\right)^{2/3}}.$$

It follows that

$$\lambda(\Omega) \leq \frac{\pi^2}{w(\Omega)^2} \left(1 + \left(\frac{w(\Omega)}{L}\right)^{2/3}\right)^3. \quad (30)$$

As $w(\Omega) \leq L$ we obtain by (30) that

$$\lambda(\Omega) \leq \frac{\pi^2}{w(\Omega)^2} \left(1 + 7 \left(\frac{w(\Omega)}{L} \right)^{2/3} \right). \quad (31)$$

In order to complete the proof we need the following.

Lemma 5. *If Ω is an open, bounded, convex set in \mathbb{R}^2 , and if L is the length of the longest line segment in the closure of Ω , perpendicular to $z_1 z_2$, then*

$$\text{diam}(\Omega) \leq 3L. \quad (32)$$

Proof. Let $d_1, d_2 \in \partial\Omega$ such that $|d_1 - d_2| = \text{diam}(\Omega)$. We denote the projections of d_1, d_2 onto the line through z_1, z_2 by e_1, e_2 respectively. Let z be the intersection of the lines through z_1, z_2 and d_1, d_2 respectively. Then, by the maximality of L , we have that $|d_1 - e_1| \leq L, |d_2 - e_2| \leq L$. Furthermore, by convexity, $|e_1 - z| + |e_2 - z| \leq w(\Omega)$. Hence,

$$|d_1 - d_2| \leq |d_1 - e_1| + |e_1 - z| + |d_2 - e_2| + |e_2 - z| \leq 2L + w(\Omega) \leq 3L. \quad \square$$

By (31), we have that

$$\lambda(\Omega) \leq \frac{\pi^2}{w(\Omega)^2} \left(1 + 7 \cdot 3^{2/3} \left(\frac{w(\Omega)}{\text{diam}(\Omega)} \right)^{2/3} \right).$$

This implies Theorem 2 by (29). □

4. Proof of Theorem 3

We denote by $d : M \times M \mapsto \mathbb{R}^+$ the geodesic distance associated to (M, g) . For $x \in M, R > 0, B(x; R) = \{y \in M : d(x, y) < R\}$. For a measurable set $A \subset M$ we denote by $|A|$ its Lebesgue measure. The Bishop–Gromov Theorem (see [10]) states that if M is a complete, non-compact, m -dimensional, Riemannian manifold with non-negative Ricci curvature, then for $p \in M$, the map $r \mapsto \frac{|B(p; r)|}{r^m}$ is monotone decreasing. In particular

$$\frac{|B(p; r_2)|}{|B(p; r_1)|} \leq \left(\frac{r_2}{r_1} \right)^m, \quad 0 < r_1 \leq r_2. \quad (33)$$

Corollary 3.1 and Theorem 4.1 in [11], imply that if M is complete with non-negative Ricci curvature, then for any $D_2 > 2$ and $0 < D_1 < 2$ there exist constants $0 < C_1 \leq C_2 < \infty$ such that for all $x \in M, y \in M, t > 0$,

$$\begin{aligned} C_1 \frac{e^{-d(x,y)^2/(2D_1 t)}}{(|B(x; t^{1/2})| |B(y; t^{1/2})|)^{1/2}} &\leq p_M(x, y; t) \\ &\leq C_2 \frac{e^{-d(x,y)^2/(2D_2 t)}}{(|B(x; t^{1/2})| |B(y; t^{1/2})|)^{1/2}}. \end{aligned} \quad (34)$$

Spectral Bounds for the Torsion Function

Finally, since by (33) the measure of any geodesic ball with radius r is bounded polynomially in r , the theorems of Grigor'yan in [6] imply stochastic completeness. That is, for all $x \in M$, and all $t > 0$,

$$\int_M dy p_M(x, y; t) = 1.$$

Proof of Theorem 3. We choose $D_1 = 1$, $D_2 = 3$ in (34), and define the corresponding number $K = \max\{C_2, C_1^{-1}\}$. Then

$$K^{-1} e^{-d(x,y)^2/(2t)} \leq \left(|B(x; t^{1/2})| |B(y; t^{1/2})| \right)^{1/2} p_M(x, y; t) \leq K e^{-d(x,y)^2/(6t)}. \quad (35)$$

Let $q \in M$ be arbitrary, and let $R > 0$ be such that $\Omega(q; R) := B(q; R) \cap \Omega \neq \emptyset$. The spectrum of the Dirichlet Laplacian acting in $L^2(\Omega(q; R))$ is discrete. Denote the bottom of this spectrum by $\lambda(\Omega(q; R))$. Then $\lambda(\Omega(q; R)) \geq \lambda(\Omega)$. By the spectral theorem, monotonicity of Dirichlet heat kernels, and the Li-Yau bound (35), we have that

$$\begin{aligned} p_{\Omega(q; R)}(x, x; t) &\leq e^{-t\lambda(\Omega(q; R))/2} p_{\Omega(q; R)}(x, x; t/2) \\ &\leq e^{-t\lambda(\Omega(q; R))/2} p_M(x, x; t/2) \\ &\leq K e^{-t\lambda(\Omega(q; R))/2} |B(x; (t/2)^{1/2})|^{-1}. \end{aligned} \quad (36)$$

By the semigroup property and the Cauchy–Schwarz inequality, for any open set $\Omega \subset M$, we have that

$$\begin{aligned} p_{\Omega}(x, y; t) &= \int_{\Omega} dz p_{\Omega}(x, z; t/2) p_{\Omega}(z, y; t/2) \\ &\leq \left(\int_{\Omega} dz p_{\Omega}^2(x, z; t/2) \right)^{1/2} \left(\int_{\Omega} dz p_{\Omega}^2(z, y; t/2) \right)^{1/2} \\ &= (p_{\Omega}(x, x; t) p_{\Omega}(y, y; t))^{1/2}. \end{aligned} \quad (37)$$

We obtain by (36), (37) (for $\Omega = \Omega(q; R)$), and $p_{\Omega(q; R)}(x, y; t) \leq p_M(x, y; t)$, that

$$\begin{aligned} p_{\Omega(q; R)}(x, y; t) &\leq (p_{\Omega(q; R)}(x, x; t) p_{\Omega(q; R)}(y, y; t))^{1/4} p_M(x, y; t)^{1/2} \\ &\leq K^{1/2} e^{-t\lambda(\Omega(q; R))/4} (|B(x; (t/2)^{1/2})| |B(y; (t/2)^{1/2})|)^{-1/4} p_M^{1/2}(x, y; t). \end{aligned} \quad (38)$$

By (38) and (35), we have that

$$\begin{aligned} p_{\Omega(q; R)}(x, y; t) &\leq K e^{-t\lambda(\Omega(q; R))/4} (|B(x; (t/2)^{1/2})| |B(y; (t/2)^{1/2})|)^{-1/4} \\ &\quad \times (|B(x; t^{1/2})| |B(y; t^{1/2})|)^{-1/4} e^{-d(x,y)^2/(12t)}. \end{aligned} \quad (39)$$

By the Li-Yau lower bound in (35), we can rewrite the right-hand side of (39) to yield,

$$p_{\Omega(q;R)}(x, y; t) \leq K^2 e^{-t\lambda(\Omega(q;R))/4} p_M(x, y; 6t) \times \frac{\left(|B(x; (6t)^{1/2})||B(y; (6t)^{1/2})|\right)^{1/2}}{\left(|B(x; (t/2)^{1/2})||B(y; (t/2)^{1/2})||B(x; t^{1/2})||B(y; t^{1/2})|\right)^{1/4}}. \quad (40)$$

By Bishop–Gromov (33), we have that the volume quotients in the right-hand side of (40) are bounded by $2^{3m/4} \cdot 3^{m/2}$ uniformly in x and y . Hence

$$p_{\Omega(q;R)}(x, y; t) \leq 2^{3m/4} \cdot 3^{m/2} K^2 e^{-t\lambda(\Omega(q;R))/4} p_M(x, y; 6t).$$

Since manifolds with non-negative Ricci curvature are stochastically complete, we have that

$$\int_{\Omega(q;R)} dy p_{\Omega(q;R)}(x, y; t) \leq 2^{3m/4} \cdot 3^{m/2} K^2 e^{-t\lambda(\Omega(q;R))/4} \int_M dy p_M(x, y; 6t) = 2^{3m/4} \cdot 3^{m/2} K^2 e^{-t\lambda(\Omega(q;R))/4}.$$

Integrating the inequality above with respect to t over $[0, \infty)$ yields,

$$v_{\Omega(q;R)}(x) \leq 2^{(3m+8)/4} \cdot 3^{m/2} K^2 \lambda(\Omega(q; R))^{-1} \leq 2^{(3m+8)/4} \cdot 3^{m/2} K^2 \lambda(\Omega)^{-1}.$$

Finally letting $R \rightarrow \infty$ in the left-hand side above yields the right-hand side of (7).

The proof of the left-hand side of (7) is similar to the one in Theorem 5.3 in [1] for Euclidean space. We have that

$$v_{\Omega(q;R)}(x) = \int_0^\infty dt \int_{\Omega(q;R)} dy p_{\Omega(q;R)}(x, y; t). \quad (41)$$

We first observe that $|\Omega(q; R)| < \infty$, and so the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega(q; R))$ is discrete and is denoted by $\{\lambda_j(\Omega(q; R)), j \in \mathbb{N}\}$, with a corresponding orthonormal basis of eigenfunctions $\{\varphi_{j, \Omega(q; R)}, j \in \mathbb{N}\}$. These eigenfunctions are in $\mathcal{L}^\infty(\Omega(q; R))$. Then, by (41) and the eigenfunction expansion of the Dirichlet heat kernel for $\Omega(q; R)$, we have that

$$\begin{aligned} v_{\Omega(q;R)}(x) &\geq \int_0^\infty dt \int_{\Omega(q;R)} dy p_{\Omega(q;R)}(x, y; t) \frac{\varphi_{1, \Omega(q;R)}(y)}{\|\varphi_{1, \Omega(q;R)}\|_{\mathcal{L}^\infty(\Omega(q;R))}} \\ &= \int_0^\infty dt e^{-t\lambda_1(\Omega(q;R))} \frac{\varphi_{1, \Omega(q;R)}(x)}{\|\varphi_{1, \Omega(q;R)}\|_{\mathcal{L}^\infty(\Omega(q;R))}} \\ &= \lambda_1(\Omega(q; R))^{-1} \frac{\varphi_{1, \Omega(q;R)}(x)}{\|\varphi_{1, \Omega(q;R)}\|_{\mathcal{L}^\infty(\Omega(q;R))}}. \end{aligned} \quad (42)$$

First taking the supremum over all $x \in \Omega(q; R)$ in the left-hand side of (42), and subsequently taking the supremum over all such x in the right-hand side of (42) gives

$$\|v_{\Omega(q;R)}\|_{\mathcal{L}^\infty(\Omega(q;R))} \geq \lambda(\Omega(q;R))^{-1}. \quad (43)$$

Observe that the torsion function is monotone increasing in R . Taking the limit $R \rightarrow \infty$ in the left-hand side of (43), and subsequently in the right-hand side of (43) completes the proof. \square

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M. van den Berg

M. van den Berg (✉)
School of Mathematics
University of Bristol
University Walk
Bristol BS8 1TW
UK
e-mail: mamvdb@bristol.ac.uk

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