The complement of a dodecahedral knot contains an essential closed surface

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Definition 1 ([3] p531)**.** A 2–string tangle diagram D_T is *strongly alternating* if both the denominator closure and the numerator closure of D_T are connected alternating diagrams with no nugatory crossings.

Definition 2 ([3] p532)**.** A *semi-alternating* link diagram is a non-alternating diagram as in Figure 1, where D_1 and D_2 are strongly alternating 2–string tangle diagrams.

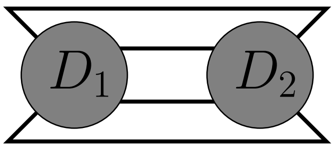

Figure 1

Theorem 3 ([3] Proposition 6)**.** *Any semi-alternating link L is not split.*

Theorem 4 ([1] Theorem 2.0.3 (see [2] proof of Theorem 1.2))**.** *Let K be a knot. Suppose that $\mathbb{S}^3 \setminus \mathcal{N}(K)$ contains an incompressible, planar, meridional surface. Then $\mathbb{S}^3 \setminus \mathcal{N}(K)$ contains an essential closed surface.*

Theorem 5. *Let K be either of the dodecahedral knots (which are shown in Figure 2). Then $\mathbb{S}^3 \setminus \mathcal{N}(K)$ contains an essential closed surface.*

Proof. Let S be the planar, meridional surface in $\mathbb{S}^3 \setminus \mathcal{N}(K)$ given by the line in Figure 2. We will show that S is incompressible. Theorem 4 then gives the result.

The surface S divides $(\mathbb{S}^3, K)$ into two 3–string tangles. Each tangle is one of the three given by the (equal) tangle diagrams shown in Figure 3. If S is not incompressible, there is a compressing disc for S in one tangle. This disc is disjoint from K, and so separates the three strings. The string in bold in Figure 3 is separated from at least one of the other two. We will show that no such disc can exist. By the rotational symmetry of the tangle, we only need to consider the case where we delete the string on the left of the tangle diagram in the top right of Figure 3. This leaves the tangle (B^3, T) given by the diagram D_T in Figure 4. Note that D_T is strongly alternating (see Figure 5).

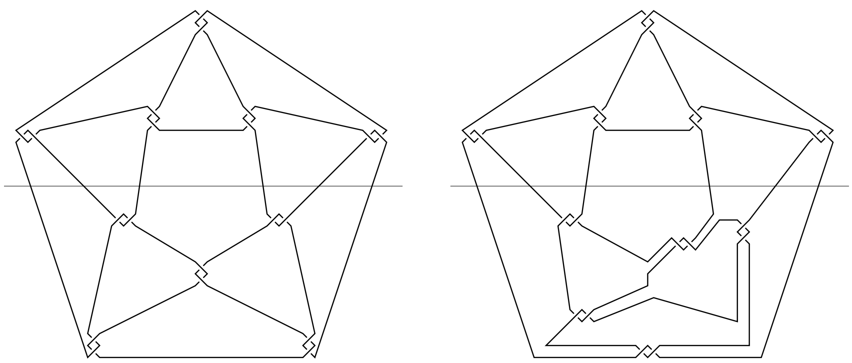

Figure 2

Suppose there exists a compression disc E for ∂B^3 in $B^3 \setminus T$. Then there is a sphere in the double $D(B^3)$ separating the link components. Thus the semi-alternating link formed from the tangle T and its mirror image (see Figure 6) is split. But this is not the case, by Theorem 3. Hence no such disc E exists, and S is incompressible, as required. □

References

[1] Marc Culler, C. McA. Gordon, J. Luecke, and Peter B. Shalen. Dehn surgery on knots. *Ann. of Math. (2)*, 125(2):237–300, 1987.

[2] Elizabeth Finkelstein and Yoav Moriah. Closed incompressible surfaces in knot complements. *Trans. Amer. Math. Soc.*, 352(2):655–677, 2000.

[3] W. B. R. Lickorish and M. B. Thistlethwaite. Some links with nontrivial polynomials and their crossing-numbers. *Comment. Math. Helv.*, 63(4):527–539, 1988.

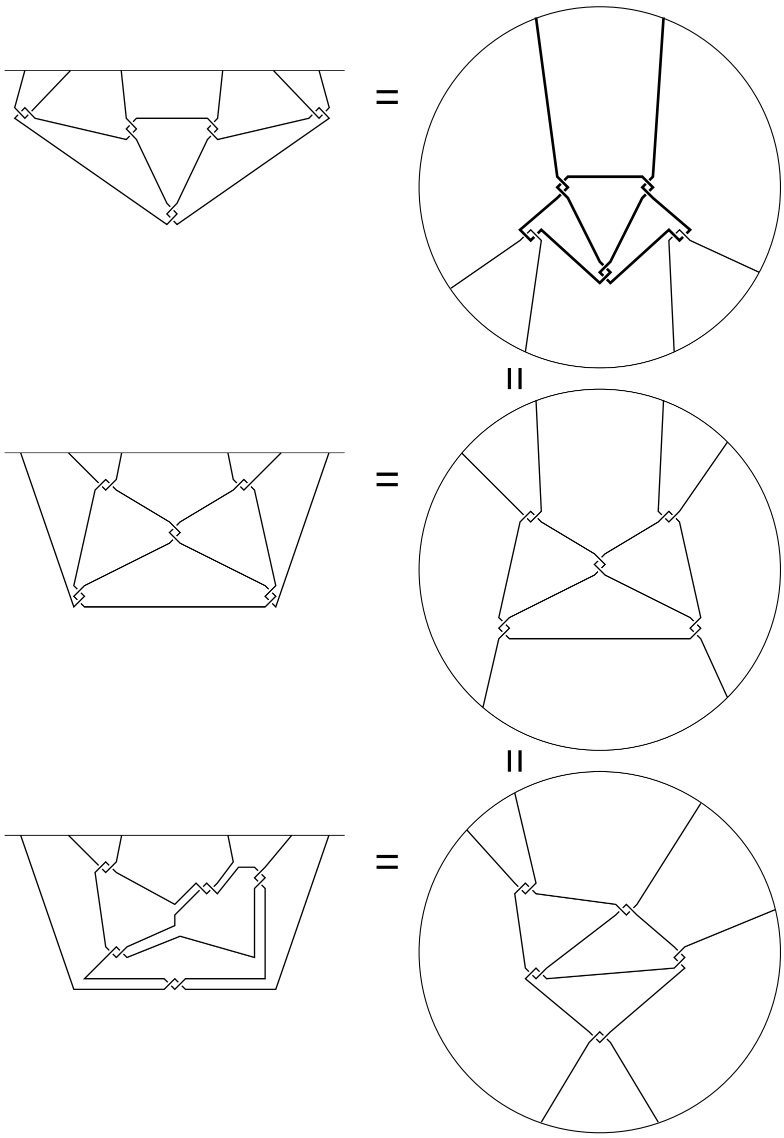

Figure 3

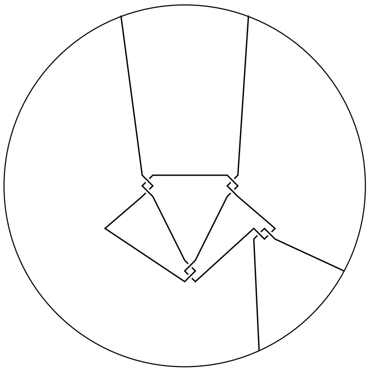

Figure 4

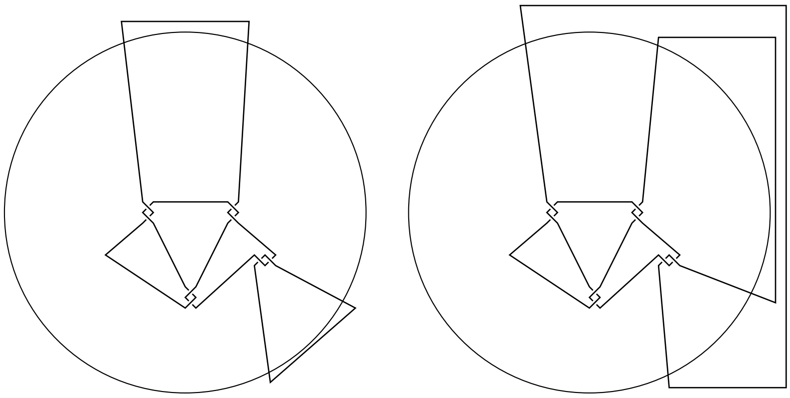

Figure 5

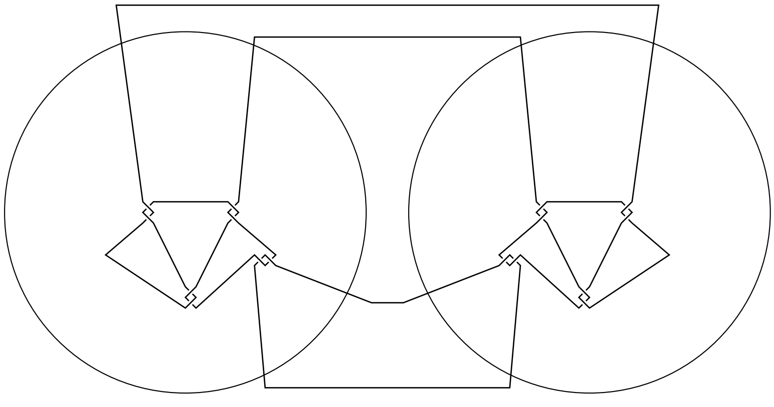

Figure 6