SUPPLEMENTARY MATERIAL FOR: INDIVIDUAL CONFIDENCE-WEIGHTING AND GROUP DECISION-MAKING

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1. Optimal Decision-Making

In general to optimise a decision-maker's expected payoff from decisions, such as whether or not a predator is present while foraging in an exposed area, we need to take account of the prior probability of states of the world, the payoffs (or losses) from different decision outcomes, and finally the accuracy of the decision maker under different states of the world (*e.g.* [6, 4]). In decision theory terms for binary classification decisions these can be captured in a prior probability vector **P**, a cost matrix **W** and a confusion matrix **C**. These are defined as

$$\mathbf{P} = \begin{pmatrix} p & (1-p) \end{pmatrix} \tag{1}$$

where p is the probability that the state of the world is 'positive' (*e.g.* predator present),

$$\mathbf{C} = \begin{pmatrix} a_P & (1-a_N) \\ (1-a_P) & a_N \end{pmatrix} \tag{2}$$

where a_P is the accuracy (probability of true positive) when the state of the world is 'positive', and a_N is the accuracy (probability of true negative) when the state of the world is 'negative', and

$$\mathbf{W} = \begin{pmatrix} W_{TP} & W_{FP} \\ W_{FN} & W_{TN} \end{pmatrix} \tag{3}$$

where W_{TP} is the cost, or loss, from a true positive, W_{FP} is the loss from a false-positive, W_{FN} the loss from a false negative, and W_{TN} that from a true negative. Hence in matrices **C** and **W** columns correspond to the true state of the world, and rows correspond to the state the decision maker perceives ('positive' then 'negative' in both cases).

Given the matrix definitions (1), (2) and (3) the expected loss of a decision-maker classifying an instance x is defined as

$$\mathrm{E}(L(x)) \quad = \quad \mathbf{P}(\mathrm{Diag}[\mathbf{C}^{\mathrm{T}}\mathbf{W}]) \tag{4}$$

where Diag[**X**] is the leading diagonal of matrix **X**, which simplifies to give

$$\mathrm{E}(L(x)) = p(a_P W_{TP} + (1-a_P)W_{FN}) + (1-p)(a_N W_{TN} + (1-a_N)W_{FP}). \tag{5}$$

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The expected loss is an economic concept that is well known in behavioural biology, for example in the application of signal detection theory to animal behaviour (*e.g.* [6, 4]). An optimal decision-maker minimises this loss, everything else being equal.

2. Consensus Decision-Making in Groups

We define the consensus decision of a group, reached by combining the individual decisions of constituent group members, as

$$H(x) = \sum_i \alpha_i h_i(x) \tag{6}$$

where h_i is the decision of the i-th group member as to whether the state of the world is 'positive' or 'negative' (*i.e.* $h_i \in \{-1, +1\}$), and α_i is a weight that individual puts on their decision, calculated below. In other words, the group's decision is a weighted sum of the decisions of its constituent members. Note that 'positive' and 'negative' are simply arbitrary labels we assign to the two possible states of the world, such as predator present and predator absent.

3. Optimising Consensus Decisions

Since the loss function for the group depends on the discontinuous losses in matrix $\mathbf{W}$, and minimisation over discontinuous functions is inherently difficult, in the following analyses we adapt a standard exponential bound on the loss from decisions. This upper bound has been used in, for example, the machine learning literature (*e.g.* [1]), to approximate losses from decisions. We define the correct classification $c(x) \in \{-1, +1\}$ to be -1 if the state of the world is 'negative' and $+1$ otherwise, and a group's hypothesis, $H(x) < 0$ or $H(x) > 0$ to indicate their belief about the state of the world ('negative' or 'positive' respectively); thus the group's decision is given by the sign of equation 6. If we assume for now that the decision-maker incurs a penalty of 1 if they make an incorrect prediction (*i.e.* the sign of the decision maker's classification is opposite to the sign of the environment state), and no penalty if they make a correct prediction (*i.e.* the sign of the decision maker's classification is opposite to the true sign of the environment state), then the loss arising from the decision-maker's decision is bounded above by an exponential function, as described in appendix A and represented in figure 1.

As described at the beginning of the section, in realistic decisions gains or losses from decision outcomes need not be all-or-nothing, nor need they be the same under all states of the world. Rather these gains and losses reflect the costs and benefits of different types of error and correct classification under different states of the world, as captured in the loss matrix $\mathbf{W}$ defined above in equation 3. However the exponential bound presented in figure 1 can be adapted to capture this increased detail and derive an optimal decision-weighting rule for the fully general case presented above, as described in appendix A. The optimal α_i^* for this general case, defined in equation A.8 in appendix A, appears rather complicated but in heuristic terms is relatively simple when visualised (figure 3 in appendix A) and, as discussed below, could be replaced by simpler functions that still provide an advantage over simple unweighted voting. Simplifying the general analysis to the special case where $W_{FN} - W_{TP} = W_{FP} - W_{TN}$ (so costs for errors are the same

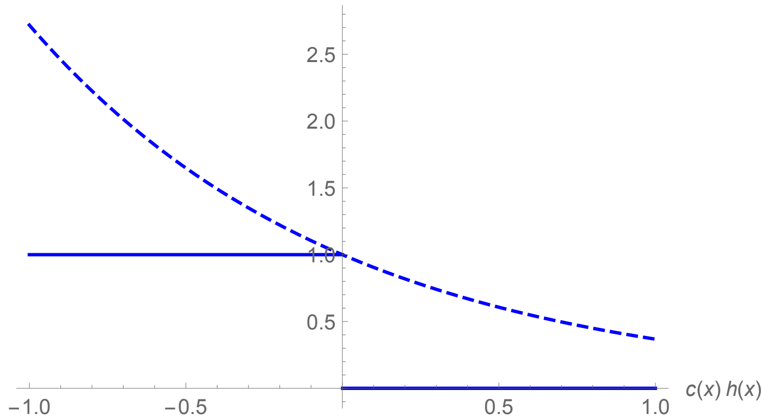

FIGURE 1. Realised zero-one loss from a decision (zero loss if correct, one loss if incorrect; solid lines) is upper-bounded by an exponential function (dashed line) as defined in equation A.1.

under positive and negative states of the world), and defining overall expected accuracy

$$a_i = pa_{P_i} + (1-p)a_{N_i}, \tag{7}$$

while noting that if all group members' weight their vote by the same constant factor then the consensus decision is unchanged, then as shown in appendix A this reduces to

$$\alpha_i^* \propto \ln\left(\frac{a_i}{1-a_i}\right), \tag{8}$$

which is the well-known optimal weighting rule from machine learning [1] and decision theory [2], presented in the main text. Note, however that the derivation presented in appendix A differs from previous derivations resting on assumed weighting of training examples for classifiers [1] or deriving as a corollary of Bayes' Theorem [5, 2, 1]. This optimal confidence-weighting is illustrated in Box 2 in the main text.

4. Optimal Confidence-Weighting from Optimal Individual Decision-Making

Kiani and Shadlen [3] show how a measure of subjective confidence, the log-odds ratio, can be calculated for a drift-diffusion decision maker (Box 3, main text) faced with uncertain information quality. Formally, Kiani and Shadlen note that the log odds-ratio of a correct decision

$$\log\frac{\mathrm{P}(\langle\mu\rangle>0|x(t))}{\mathrm{P}(\langle\mu\rangle<0|x(t))} = \log\frac{\sum_i \mathrm{P}(x(t)\big|\mu_i>0,|\mu_i|)\mathrm{P}(|\mu_i|\big|\mu_i>0)}{\sum_i \mathrm{P}(x(t)\big|\mu_i<0,|\mu_i|)\mathrm{P}(|\mu_i|\big|\mu_i<0)} + \log\frac{\mathrm{P}(\mu>0)}{\mathrm{P}(\mu<0)}. \tag{9}$$

Assuming that signal magnitude ($|\mu_i|$) is uncorrelated with signal direction ($\mathrm{sign}(\mu_i)$), and that both states of the world are equally likely, this simplifies to equation 5 in Box 3 in the main text.

Appendix A. Weighting Opinions by Individual Decision Accuracy Reduces Expected Group Decision Loss

As described in the main text we assume that the correct classification $c(x) \in \{-1, +1\}$ is -1 if the state of the world is negative and $+1$ otherwise, and a decision-maker's hypothesis $h(x) \in \{-1, +1\}$ indicates their belief about the state of the world (negative or positive). We also assume that the decision-maker incurs a penalty of 1 if they make an incorrect prediction, and no penalty if they make a correct prediction, then the loss arising from the decision-maker's decision is bounded above by the function

$$\overline{L} = e^{-c(x)h(x)}, \tag{A.1}$$

as illustrated in figure 1.

Since in general losses from decisions are not all or nothing (eq. 5), and need not be the same under each state of the world (so the cost for a false positive need not be the same as the cost for a false negative, for example), expression A.1 can be generalised to

$$\overline{L} = \begin{cases} W_{TN} + (W_{FP} - W_{TN})e^{h(x)} & \text{if } c(x) = -1 \\ W_{TP} + (W_{FN} - W_{TP})e^{-h(x)} & \text{if } c(x) = 1 \end{cases}, \tag{A.2}$$

as illustrated in figure 2.

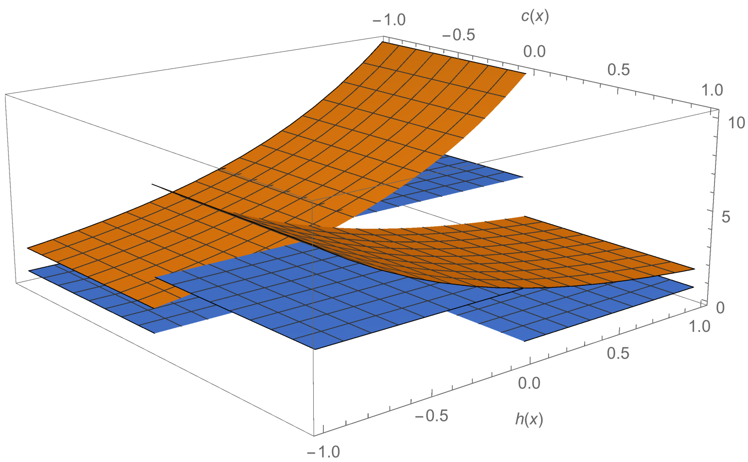

Figure 2. Realised loss from a decision, defined according to a loss matrix (blue planes) is upper-bounded by a piecewise exponential function (yellow surfaces) as defined in equation A.2.

We now consider how exponential bounds can be used to derive an individual confidence-weighting rule that minimises the group's expected loss. Since optimal decision rules should take account of the prior probability p of the state of the world (5), the upper bound on the expected loss of a group should also take this into account, so substituting equation 6 into A.2 and weighting by prior probability gives the expected loss for a group decision as

$$\text{(A.3)} \quad \mathrm{E}(\overline{L}) = p(W_{TP} + (W_{FN} - W_{TP})e^{-\sum_i \alpha_i h_i(x)}) + (1-p)(W_{TN} + (W_{FP} - W_{TN})e^{\sum_i \alpha_i h_i(x)})$$

$$\text{(A.4)} \quad = p(W_{TP} + (W_{FN} - W_{TP})\prod_i e^{-\alpha_i h_i(x)}) + (1-p)(W_{TN} + (W_{FP} - W_{TN})\prod_i e^{\alpha_i h_i(x)})$$

To add an individual to a group that minimises their contribution to that group's expected decision loss, we can see that they thus need to minimise

$$\text{(A.5)} \quad \mathrm{E}(\overline{L_i}) = p(W_{TP} + (W_{FN} - W_{TP})e^{-\alpha_i h_i(x)}) + (1-p)(W_{TN} + (W_{FP} - W_{TN})e^{\alpha_i h_i(x)})$$

Separating out into expected loss from correct and incorrect decisions, according to accuracies under positive and negative world states, this becomes

$$\text{(A.6)} \quad \mathrm{E}(\overline{L_i}) = p(a_{P_i}(W_{TP} + (W_{FN} - W_{TP})e^{-\alpha_i}) + (1-a_{P_i})(W_{TP} + (W_{FN} - W_{TP})e^{\alpha_i})) + (1-p)(a_{N_i}(W_{TN} + (W_{FP} - W_{TN})e^{-\alpha_i}) + (1-a_{N_i})(W_{TN} + (W_{FP} - W_{TN})e^{\alpha_i})),$$

where a_{P_i} and a_{N_i} refer to decision accuracies for individual i when faced with positive and negative states of the world respectively.

To minimise this contribution we differentiate with respect to individual 'confidence' α_i

$$\text{(A.7)} \quad \frac{\partial}{\partial \alpha_i}\mathrm{E}(\overline{L_i}) = (p(1-a_{P_i})(W_{FN} - W_{TP}) + (1-p)(1-a_{N_i})(W_{FP} - W_{TN}))e^{\alpha_i} - (pa_{P_i}((W_{FN} - W_{TP}) + (1-p)a_{N_i}(W_{FP} - W_{TN}))e^{-\alpha_i}.$$

then find the value of α_i at which (A.7) is zero, as this minimises $\mathrm{E}(\overline{L_i})$.

Rearranging and taking the natural logarithm we find that (A.7) is zero when

$$\text{(A.8)} \quad \alpha_i^* = \frac{1}{2}\ln\left(\frac{pa_p(W_{FN} - W_{TP}) + (1-p)a_N(W_{FP} - W_{TN})}{p(1-a_p)(W_{FN} - W_{TP}) + (1-p)(1-a_N)(W_{FP} - W_{TN})}\right).$$

Equation A.8 is visualised in figure 3

References

[1] Yoav Freund and Robert E Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. *Journal of Computer and System Sciences*, 55(1):119–139, 1997.
[2] Bernard Grofman, Guillermo Owen, and Scott L Feld. Thirteen theorems in search of the truth. *Theory and Decision*, 15:261–278, 1983.
[3] Roozbeh Kiani and Michael N Shadlen. Representation of confidence associated with a decision by neurons in the parietal cortex. *Science (New York, N.Y.)*, 324(5928):759–64, 2009.
[4] James A R Marshall, Pete C. Trimmer, Alasdair I. Houston, and John M. McNamara. On evolutionary explanations of cognitive biases. *Trends in Ecology and Evolution*, 28(8):469–473, 2013.
[5] Shmuel Nitzan and Jacob Paroush. Optimal decision rules in uncertain dichotomous choice situations. *International Economic Review*, pages 289–297, 1982.

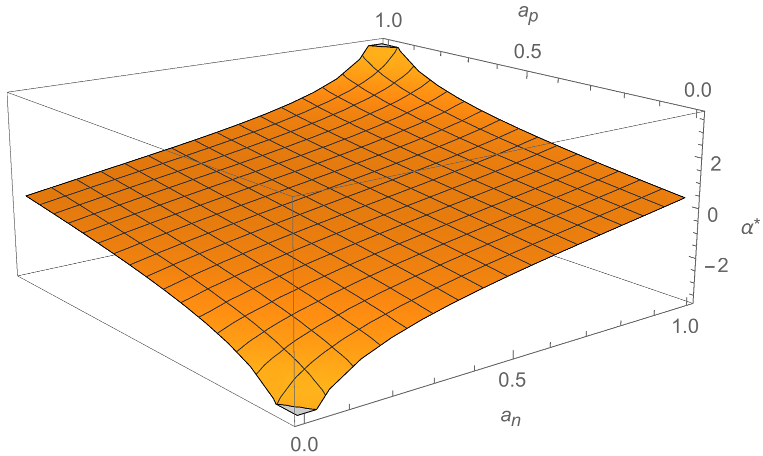

FIGURE 3. Optimal confidence-weighting parameter α^* as a function of individual decision-accuracies under positive (a_P) and negative (a_N) states of the world, calculated from equation A.8. Note that the individual's contribution to the group decision should be negatively-weighted if both its decision accuracies are less than chance (1/2). Note also that while standard ROC analysis predicts that the total accuracy of a decision-maker $a_P + a_N$ should exceed 1, combinations of accuracies satisfying this condition that result in a negative optimal weighting can be found.

[6] Pete C Trimmer, Alasdair I Houston, James AR Marshall, Rafal Bogacz, Elizabeth S Paul, Mike T Mendl, and John M McNamara. Mammalian choices: combining fast-but-inaccurate and slow-but-accurate decision-making systems. *Proceedings of the Royal Society of London B: Biological Sciences*, 275(1649):2353–2361, 2008.