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The quasi-isomorphism class of the Kakimizu complex

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Abstract

It has been shown that the Kakimizu complex of a knot is quasi-isomorphic to $\mathbb{Z}^n$ for some $n \geq 0$. We give a lower bound on n, matching the upper bound previously given.

A *Seifert surface* for a knot K in $\mathbb{S}^3$ is a compact, connected, orientable surface whose boundary is K. We consider Seifert surfaces up to ambient isotopy in the knot exterior $E = \mathbb{S}^3 \setminus \mathcal{N}(K)$. The *Kakimizu complex* $\mathrm{MS}(K)$ of K is a simplicial complex that records the structure of the set of minimal genus Seifert surfaces for K. The vertices are given by the isotopy classes of minimal genus Seifert surfaces for K, and distinct vertices span a simplex if the vertices can be realised disjointly in E. The Kakimizu complex of the unknot is a single vertex; we will assume in this paper that K is not the unknot (the result is immediate in this case).

It is known that, given K, there is an upper bound on the dimension of any simplex in $\mathrm{MS}(K)$. If K is either a torus knot or hyperbolic then $\mathrm{MS}(K)$ has only finitely many vertices, but if K is a satellite knot then $\mathrm{MS}(K)$ may be infinite ([6]) and even locally-infinite ([1]). In addition, Przytycki and Schultens have shown that $\mathrm{MS}(K)$ is contractible ([7]).

In [5], Johnson, Pelayo and Wilson proved that $\mathrm{MS}(K)$ is quasi-Euclidean. That is, there exists $n \in \mathbb{N} \cup \{0\}$ such that $\mathrm{MS}(K)$ is quasi-isomorphic to $\mathbb{Z}^n$. Here the metric on $\mathrm{MS}(K)^1$ is the graph metric where each edge has length 1. The authors give an upper bound on n, and suggest that this is also a lower bound. Our aim is to show that this is indeed a lower bound. To do so, we must recall the key elements of their proof.

Consider an incompressible torus T properly embedded in E. In $\mathbb{S}^3$, it must bound a solid torus on one side, and this solid torus necessarily contains K. We will describe the solid torus as being 'inside' T, and the knot-complement component of $E \setminus T$ as being 'outside' T.

We next need to consider the JSJ decomposition of E (see, for example, [3]). Choose a minimal collection $T_1, \ldots, T_N$ of incompressible tori, pairwise disjoint, such that the complement of $\bigcup T_i$ consists of Seifert fibered pieces and atoroidal pieces. Let $E_0, \ldots, E_N$ be the (closures of the) regions of $E \setminus \bigcup T_i$. We may arrange that E_0 meets $\mathcal{N}(K)$, and, for $1 \leq j \leq N$, that E_j has T_j as one of its boundary components with E_j lying outside T_j. Following the terminology of [5], we will refer to each E_j as a *block*. Set T_0 to be the torus $\partial E = \partial \mathcal{N}(K)$. Then T_0 is also incompressible in E and is a boundary component of E_0, with E_0 outside of T_0.

Johnson, Pelayo and Wilson define the *core* of E to be the union of the core blocks, where E_j is a *core block* if every (minimal genus) Seifert surface for K intersects E_j. Equivalently, E_j is a core block if K is homologically non-trivial in the solid torus V_j inside T_j. Note that the core of E is connected and contains E_0. The following result shows that each T_j contained in the interior of the core of K has a preferred slope. For each j, let K_j be the core of V_j, and view V_j as a neighbourhood of K_j.

Proposition 1 ([5] Proposition 2). *If E_j is a core block then there is a slope α_j on T_j such that, if R is any minimal genus Seifert surface for K, every curve of $R \cap T_j$ that is essential in T_j (of which there is at least one) is parallel to α_j. Moreover, α_j is the longitude of V_j (that is, α_j is the boundary of a Seifert surface for K_j, as an unoriented curve).*

Although the slope α_j is determined by the knot K_j in $\mathbb{S}^3$, the number and orientation of the curves $R \cap T_j$ are controlled by the position of the surface R. More precisely, $[R \cap T_j] = a_j[\alpha_j] = [K]$ in $H_1(V_j; \mathbb{Z})$ for some $a_j \in \mathbb{Z}$. Here $|a_j|$ is equal to the winding number of K in V_j.

A group action

Johnson, Pelayo and Wilson make use of an action of $\mathbb{Z}^N$ on $\mathrm{MS}(K)$. For a fixed k, choose a product neighbourhood W_k of T_k in E. Choose a product structure on W_k, expressing it as $\mathbb{S}^1 \times \mathbb{S}^1 \times I$, where the first $\mathbb{S}^1$ corresponds to the slope α_k on T_k. Define $\phi_{T_k} \colon E \to E$ by

$$\phi_{T_k}(x) = \begin{cases} x & x \notin W_k, \\ (z, e^{i(\theta + 2\pi t)}, t) & x = (z, e^{i\theta}) \in W_k. \end{cases}$$

Note that if the product neighbourhoods W_i are disjoint then these homeomorphisms of E act independently.

We then define the action $\Phi \colon \mathbb{Z}^N \times \mathrm{MS}(K) \to \mathrm{MS}(K)$ by

$$\Phi(r_1, \ldots, r_N, R) = \phi_{T_1}^{r_1} \circ \cdots \circ \phi_{T_N}^{r_N}(R).$$

The moral of [5] is that all infinite directions in $\mathrm{MS}(K)$ come from 'spinning around the tori' using this action. The upper bound on the quasi-dimension of $\mathrm{MS}(K)$ comes from counting the number of different ways of spinning around tori like this. To give a lower bound, we will fix a minimal genus Seifert surface R_K for K, and show that acting on R_K gives enough Seifert surfaces that are different (and distant in $\mathrm{MS}(K)$) from each other.

Basepoint

Choose a minimal genus Seifert surface R_K for K, for use as a reference point. We will now edit R_K to suit our purposes, but continue to denote it by R_K. For this we will use the following two results, which, although not explicitly stated, make up the proof of [4] Lemma 16.3.

Lemma 2. *Let K_S be a (satellite) knot, and let T_S be an essential torus in the complement of K_S. Let R_S be a minimal genus Seifert surface for K_S, in general position with respect to T_S. Then it is possible, by surgering along*

subdics and subannuli of T_S and discarding closed components, to change R_S to a minimal genus Seifert surface R'_S such that all components of $R'_S \cap T_S$ are parallel and oriented in the same direction.

Lemma 3. *Let K_S be a knot in $\mathbb{S}^3$, and let R_S be a connected, oriented surface properly embedded in the exterior of K_S, such that all boundary components of R_S are longitudes of K_S oriented in the same direction. Then R_S has at most one boundary component.*

Choose a JSJ torus T_k. We can edit R_K so that all components of $R_K \cap T_k$ are parallel and oriented in the same direction. If T_k does not lie in the interior of the core then R_K is disjoint from T_k. On the other hand, if T_k lies in the interior of the core then these curves are parallel to α_k and there are $|a_k|$ of them. Then $R_K \setminus V_k$ is formed of $|a_k|$ minimal genus Seifert surfaces for K_k. We may then replace these with $|a_k|$ parallel copies of a single one of those components. Note that these changes all take place in W_k or outside T_k, without affecting anything further inside.

By working in this way inductively outwards from K, we achieve the following result.

Lemma 4. *We may choose R_K such that, for each j, all curves of $R_K \cap T_j$ are parallel to α_j and oriented the same way, and all components of $R_K \cap E_j$ are parallel to each other with a single boundary component each on T_j.*

Some readers may find it helpful to picture the surface R_K we have just constructed in terms of branched surfaces. We will not explicitly use this viewpoint in this paper.

Fibred blocks

The upper bound on the dimension of $\mathrm{MS}(K)$ given in [5] depends on the number of core blocks that are fibred. For each core block E_j, we can ask whether a connected component R_j of $R_K \cap E_j$ is a fibre for E_j (that is, whether the complement of R_j in E_j is $R_j \times I$). Note that the answer to this question is determined only by the curves $R_K \cap \bigcup T_i$ (which depend only on K), and is not dependent on the specific choice of surface R_K. It is possible that a block might be fibred with a different 'boundary pattern', but we are not interested in such cases in this paper. The upper bound in [5] is one less that the number of core blocks that are not fibred. Denote this number by N'. Our aim is to construct a quasi-isometric embedding of $\mathbb{Z}^{N'}$ into $\mathrm{MS}(K)$.

The intuitive explanation for this value is that we can spin R_K around each torus it intersects, but spinning around T_0 can be reversed by isotopy, and if E_j is fibred then spinning around T_j has the same effect, up to isotopy, as spinning around each of the other boundary components of E_j.

For the purposes of our proof, we will need to forget about some of the tori $T_0, \ldots, T_N$, according to which ones we will use for spinning around. For convenience, we will re-label the objects we are considering. Starting with the list $T_0, \ldots, T_N$, remove each T_j such that E_j is not a core block. Also remove T_0. If E_j is fibred for $j \geq 1$ then remove T_j. Finally, if E_0 is fibred then remove one remaining torus that is now 'innermost', in the sense that it is not separated from K by any of the other remaining tori. Re-label the remaining list of tori as

$T'_1, \ldots, T'_{N'}$ and their neighbourhoods as $W'_1, \ldots, W'_{N'}$. For convenience, write $W' = \bigcup W'_i$. Also label the regions of $E \setminus \bigcup T'_i$ as $E'_0, \ldots, E'_{N'}$. As before we may arrange that T'_j is a boundary component of E'_j, with E'_j lying outside T'_j. The advantage of our new notation is that $T'_j \cap R_K \neq \emptyset$ for each j, and no $\dot{E}'_j$ is fibred.

We are now ready to define our quasi-isometric embedding using the group action Φ. Note that, in defining each Φ_j, we had some choice in the product structure on W_j. For notational convenience, we will assume that the product structure on each W'_j has been chosen such that, moving from the inside of W'_j to outside, $\phi_{T'_j}$ twists in the direction given by the orientation on the Seifert surface R_K. Figure 1 illustrates this convention for the torus T'_1.
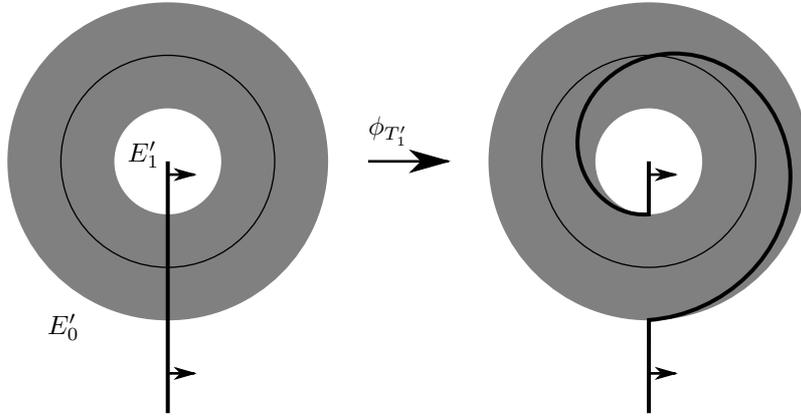

Figure 1

We define $\Theta \colon \mathbb{Z}^{N'} \to \mathrm{MS}(K)$ by

$$\Theta(r_1, \ldots, r_{N'}) = (\phi_{T'_1})^{5r_1} \circ \cdots \circ (\phi_{T'_{N'}})^{5r_{N'}}(R_K).$$

That is, up to re-labelling, Θ is the restriction of Φ^5 to the coordinates corresponding to the tori $T'_1, \ldots, T'_{N'}$ and the surface R_K. The use of the power 5 here is not significant; its purpose is to remove the need to consider 'small cases' later.

Distances

To show that Θ is a quasi-isometric embedding, we need to calculate distances in $\mathrm{MS}(K)$. The distance $\mathrm{d}_{\mathrm{MS}(K)}$ between two vertices in $\mathrm{MS}(K)$ is defined using the graph metric where each edge has length 1. In [6], Kakimizu gave a method for calculating the distance using the infinite cyclic cover of E corresponding to the kernel of the linking number $\mathrm{lk} \colon \pi_1(E) \to \mathbb{Z}$.

Choose a minimal genus Seifert surface R for K. We can build the infinite cyclic cover $\widetilde{E}$ of E as follows. Let E_R be E cut along the surface R. Then the boundary of E is divided into three parts: two copies of R, which can be distinguished using the orientation of R, and an annulus that is the torus ∂E cut along the simple closed curve ∂R. To form $\widetilde{E}$, stack countably many copies

of E_R by gluing the positive side of R in the nth copy of ∂E_R to the negative side of R in the $(n+1)$th copy. The quotient map is given by mapping each copy of E_R to E_R by the identity, then taking the quotient map from E_R to E. The covering transformation is given by translating along the line of copies of E_R.

Now choose a second minimal genus Seifert surface R' for K that is not isotopic to R. We can calculate the distance between vertices R and R' in $\mathrm{MS}(K)$ as follows. Choose a lift $\widetilde{R'}$ of R' to $\widetilde{E}$. Isotope $\widetilde{R'}$ within $\widetilde{E}$ to minimise the number, d, of copies of E_R that it intersects. Then $\mathrm{d}_{\mathrm{MS}(K)}(R, R') = d$. Note that $d = 1$ if and only if $\widetilde{R'}$ can be isotoped to be disjoint from all lifts of R in $\widetilde{E}$, which is as we would expect given the definition of adjacency in $\mathrm{MS}(K)$.

The difficult part of using this criterion is establishing when $\widetilde{R'}$ has been suitably positioned. The following result, which has its roots in work of Waldhausen, enables us to verify this by only considering the position of the surface R' relative to R within E. This version is restricted to the case of knots in $\mathbb{S}^3$ (the original was for use in more general manifolds).

Definition 5. Let S be a compact, connected, orientable surface, and let ρ be a finite (possibly empty, possibly disconnected) submanifold of ∂S. Let M_S be the manifold given by taking $S \times I$ and identifying $\{x\} \times I$ to a point for each $x \in \rho$. We call any manifold of this form a *product region*.

We say that the surfaces R and R' *bound a product region* if there exists a product region M_S of this form properly embedded in (the closure of) $E \backslash (R \cup R')$ such that $M_S \cap R = S \times \{0\}$ and $M_S \cap R' = S \times \{1\}$.

This definition should be viewed as the three-dimensional analogue of when two arcs or curves in a surface 'bound a bigon'.

Proposition 6 ([7] Proposition 3.2)**.** *If R and R' intersect transversely and do not bound a product region then R and R' realise $\mathrm{d}_{\mathrm{MS}(K)}(R, R')$.*

In other words, if we can arrange that R and R' are transverse and $E \backslash (R \cup R')$ does not include any product regions, then we can count the distance between R and R' without needing to consider any further isotopy of R' (or equivalently of $\widetilde{R'}$). This is the technique we will use to verify that the images of points under Θ are suitably far apart in $\mathrm{MS}(K)$.

Note that if M_S is a product region between R and R', the intersection $M_S \cap R$ is a connected, orientable surface. Thus if a component of $E \backslash (R \cup R')$ meets R on both the positive and the negative sides then this component is not a product region between R and R'.

Proof

Proposition 7. *The map Θ is a quasi-isometric embedding of $\mathbb{Z}^{N'}$ into $\mathrm{MS}(K)$.*

Proof. Let $(r_1, \ldots, r_{N'}), (s_1, \ldots, s_{N'}) \in \mathbb{Z}^{N'}$. Using the action Φ, we may assume without loss of generality that $(r_1, \ldots, r_{N'}) = (0, \ldots, 0)$. With this assumption, $\Theta(r_1, \ldots, r_{N'}) = R_K$. We may also assume that $(s_1, \ldots, s_{N'}) \neq (0, \ldots, 0)$, which implies that $\max(|s_1|, \ldots, |s_{N'}|) > 0$.

Denote by S a copy of $\Theta(s_1, \ldots, s_{N'})$. We will position S carefully with respect to R_K, show that there are no product regions bounded by R_K and S,

and read off a lower bound on $\mathrm{d}_{\mathrm{MS}(L)}(R_K, S)$. If there is a value of k such that $s_k = 0$ then the torus T'_k plays no part in this process. We should therefore forget about T'_k, as we have already forgotten about some of the other T_i. Rather than re-labelling the tori and complementary regions again, we will instead assume that $s_k \neq 0$ for each k. This does not impact on the method of proof; it is simply for notational convenience.

We can think of the surface S as being divided into different pieces. In $E'_j \setminus W'$, R_K and S coincide, and are made up of $|a_j|$ parallel copies of the same connected surface. Meanwhile, each component of $S \cap W'_j$ is an annulus that winds $|s_j|$ times around T_j relative to R_K. We will re-position S by considering these pieces separately.

First consider W'_k for some k. Each of $R_K \cap W'_k$ and $S \cap W'_k$ consists of parallel annuli properly embedded in W'_k. Picture the case where $R_K \cap W'_k$ is a single annulus A_K and $S \cap W'_k$ is a single annulus A_S. Note that initially $\partial A_K = \partial A_S$. Because A_S winds around W'_k at least once relative to A_K, there is a well-defined choice of direction to isotope each boundary component of A_S within a neighbourhood of ∂A_K to make ∂A_S and ∂A_K disjoint without otherwise affecting $A_K \cap A_S$ (see Figure 2a). When there are instead multiple components
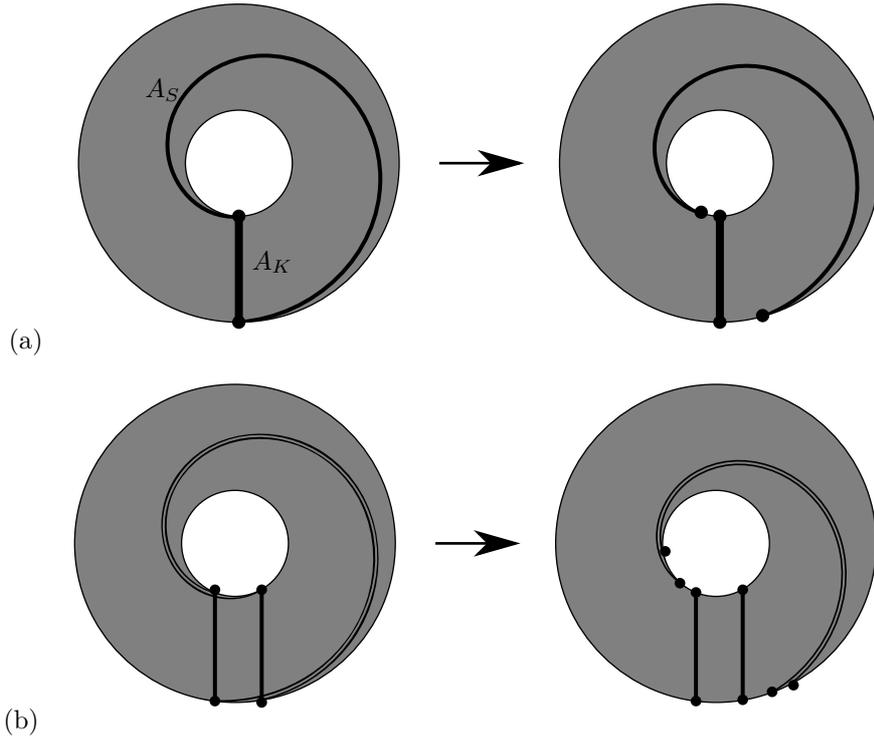

(a)

(b)

Figure 2

to consider, we treat all the parallel copies as a product neighbourhood of a single copy, and isotope this neighbourhood as we just described for one annulus (see Figure 2b).

Similarly, in each $E'_j \setminus W'$, we will treat all parallel copies of a connected

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component as a product neighbourhood of one component. Again, therefore, picture the case where $R_K \cap (E'_k \setminus W')$ has a single component (the same will then be true of $S \cap (E'_k \setminus W')$). Denote the component of R_K by B_K and the component of S by B_S.

Initially B_K and B_S coincide. We have already defined our isotopy on $\partial(E'_k \setminus W')$; we wish to extend this isotopy to $E'_k \setminus W'$ so that B_S becomes transverse to B_K. Each boundary component has a pre-defined direction that it needs to be moved. Since B_S is connected, a suitable isotopy of the surface B_S exists after which $|B_K \cap B_S| \leq 1$, with B_K and B_S disjoint if possible.

We can see more explicitly how the isotopy is chosen as follows. Each boundary component of B_S needs to be moved either in the direction given by the orientation on B_K or in the opposite direction. Mark a boundary component with a $+$ if the direction it needs to move agrees with the orientation of B_K, and with a $-$ otherwise. These signs can also be determined using the coefficients s_i. Recall that B_S lies outside T'_k. If $s_k > 0$ then the boundary component of B_S on $\partial W'_k$ has a $-$ sign, and if $s_k < 0$ then it has a $+$ sign. If B_S has a boundary component on $\partial W'_j$ for some $j \neq k$ then B_S lies inside T'_j. If $s_j > 0$ then this boundary component has a $+$, and if $s_j < 0$ then it has a $-$. If all boundary components of B_S have the same sign, we may isotope B_S to be disjoint from B_K in a way that behaves as required on the boundary. Otherwise, choose a single simple closed curve on B_S that separates all boundary components with a $+$ from all boundary components with a $-$. In this case we can choose a suitable isotopy that leaves this curve as the intersection between B_K and B_S. The section of B_S on the $+$ side of the curve is isotoped to the positive side of B_K, while the section on the $-$ side is isotoped to the negative side of B_K.

There is one case not included in this description. If $k = 0$ then B_S has one boundary component on T_0. There is no pre-determined position we must isotope this boundary component to. To avoid creating product regions, if all other boundary components have the same sign then we must also assign that sign to this boundary component. Otherwise, we may freely assign it either a $+$ or a $-$.

We have now made R_K and S transverse by an isotopy of S in E. To apply Proposition 6, we must verify that our choice of isotopy was a good one, that there are now no product regions bounded by R_K and S. We must therefore check each of the complementary regions of $E \setminus (R_K \cup S)$ to see if it is a product region.

The first thing to note is that any complementary region that lies between two parallel copies of a section of R_K will meet R_K on both its positive and negative sides, since we have chosen R_K such that all such sections of surface are oriented in the same direction. Therefore, these complementary regions cannot be product regions. The same holds for parallel sections of S. See Figure 3a; the shaded regions pick out one complementary region between parallel sections of R_K, one between parallel sections of S, and one coming from the intersection of parallel regions that therefore lies both between parallel sections of R_K and between parallel sections of S.

Hence once more we can imagine that each part of each of the surfaces R_K and S has a single component rather than multiple parallel copies of a component. Observe that, under this assumption, our choice of sign for the boundary component of S on T_0 ensures that every complementary region meets T'_j for some $j \in \{1, \ldots, N'\}$.
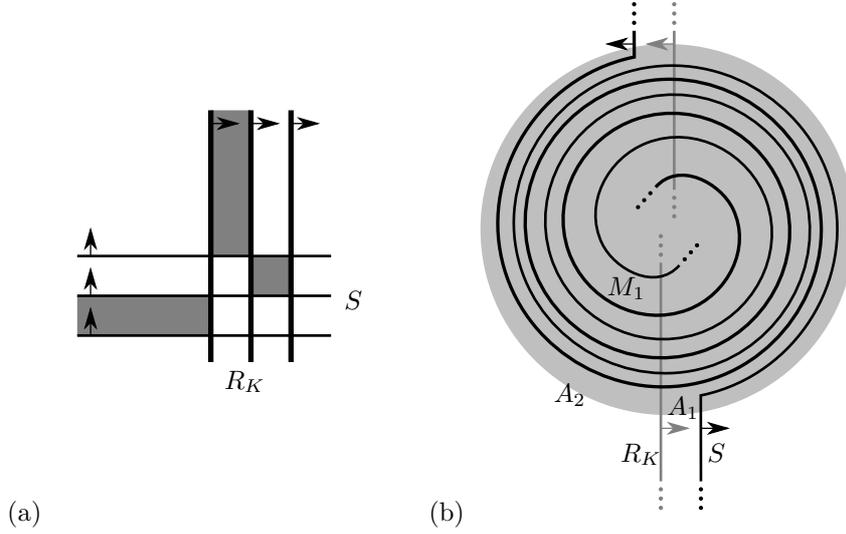
Figure 3

Next we turn our attention to the complementary regions that are contained entirely within W'. This is depicted in Figure 3b. Again we find that each such complementary region (such as that marked M_1 in Figure 3b) meets S on both the positive and negative sides. This is also true of any complementary region that intersects $E_j' \cap \partial W'$ in a 'small' sub-annulus of $E_j' \cap \partial W'$ across which S was isotoped (such as that marked A_1 in Figure 3b).

The final case to consider is a complementary region that intersects $E_j' \cap \partial W'$ in a 'larger' sub-annulus coming from a component of $(E_j' \cap \partial W') \setminus R_K$ (such as that marked A_2 in Figure 3b). There are two possibilities. If R_K and S intersect in $E_j' \setminus W'$ (that is, if there are boundary components of $S \cap E_j'$ that were marked with different signs) then once again the complementary region meets both R_K and S each on the positive side and on the negative side. The other possibility is that $S \cap E_j'$ and $R_K \cap E_j'$ are disjoint and parallel in $E_j' \setminus W'$. This time we cannot necessarily use the boundary pattern to rule out the possibility that the complementary region of interest is a product region. However, the fact that $R_K \cap E_j'$ is not a fibre for E_j' tells us this instead.

As R_K and S do not bound a product region, Proposition 6 allows us to use S to calculate the distance between R_K and S in $\mathrm{MS}(K)$ without any further isotopy. Choose $k \in \{1, \ldots, N'\}$ such that $s_k = \max(|s_1|, \ldots, |s_{N'}|)$. An annulus of $S \cap W_k'$ contains at least $5|s_k| - 1$ curves of intersection with R_K, each a core curve of the annulus and all oriented in the same direction. Therefore, in the cover $\widetilde{E}$ of E constructed using R_K, a lift of S intersects at least $5|s_k|$ lifts of $E \setminus R_K$. Hence $d_{\mathrm{MS}(K)}(R_K, S) \geq 5|s_k| = 5\max(|s_1|, \ldots, |s_{N'}|) = 5\max(|s_1 - r_1|, \ldots, |s_{N'} - r_{N'}|)$.

We can also find an upper bound on $d_{\mathrm{MS}(K)}(R_K, S)$, since this is at most $|R_K \cap S| + 1$. Set $M = \max(|R_K \cap T_1'|, \ldots, |R_K \cap T_{N'}'|)$. Then

$$|R_K \cap S| \leq M^2(5\max(|s_1|, \ldots, |s_{N'}|) - 1) + (N' + 1),$$

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so

$$\mathrm{d}_{\mathrm{MS}(K)}(R_K, S) \leq 5M^2 \max(|s_1 - r_1|, \ldots, |s_{N'} - r_{N'}|) + (N' + 2).$$

These two inequalities together show that Θ is a quasi-isometric embedding. $\square$

Corollary 8. *The upper bound on dimension given in [5] is also a lower bound. That is, for a knot K in $\mathbb{S}^3$, the Kakimizu complex $\mathrm{MS}(K)$ of K is quasi-isometric to $\mathbb{Z}^M$, where M is equal to one less than the number of core JSJ blocks that are not fibred.*

A word on links

The results in this paper, like those in [5], are specifically stated for knots, rather than links in general. The definition of the Kakimizu complex and the metric on it can be extended to links. However, the definitions should be stated in a different form before generalising. For more details on this see [7] and [2].

The reason for the restriction to knots comes in Theorem 7 of [5], which shows that there are only finitely many subsurfaces in each block that are relevant for the main proof. This is proved using the classification of Seifert fibred submanifolds of $\mathbb{S}^3$ given by Budney in [3]. Although Budney's result applies equally well for multi-component link complements as for knot complements, the same is not true of [5] Theorem 7.

As a counter-example, consider the $(9, 6)$ torus link (that is, three parallel copies of a trefoil), with all components oriented in parallel. The complement of this link is Seifert fibred over a punctured sphere with two exceptional fibres. Since there is therefore only one block in the link complement, we would want to conclude that the Kakimizu complex is quasi-isomorphic to a point. On the other hand, there is an essential torus in the link complement separating two of the link components from the third, which can be used for spinning around. In calculating the dimension of the Kakimizu complex of a link complement, it is thus important to allow for the presence of toroidal Seifert fibred pieces in the JSJ decomposition.

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