Colimit preservation from weaker large cardinals

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Background

Theorem (Rosický, Trnková & Adámek, 1990)

Assuming Vopěnka's Principle, for each full embedding $F : \mathcal{A} \to \mathcal{K}$ with $\mathcal{K}$ an accessible category, there is a regular cardinal λ such that F preserves λ-directed colimits.

Recall that a poset is λ-*directed* if every subset of cardinality less than λ has an upper bound. A λ-directed diagram is one whose index category is a λ-directed poset.

Vopěnka's Principle

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For any signature $\mathbf{\Sigma}$, and any proper class $\mathcal{C}$ of $\mathbf{\Sigma}$-structures, there are distinct structures A and B in $\mathcal{C}$ such that there exists a homomorphism from A to B.

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Question:

Can the colimit preservation theorem from the previous slide be stratified in this way?

Answer

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Yes!

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Theorem (Bagaria & B-T)

Suppose that $\mathcal{K}$ is a full subcategory of $\mathbf{Str}\,\boldsymbol{\Sigma}$ *for some signature* $\boldsymbol{\Sigma}$*. Let* $F : \mathcal{A} \to \mathcal{K}$ *be any* Σ_n*-definable full embedding with* Σ_n*-definable domain category* $\mathcal{A}$*, for some* $n > 0$*. If there exists a* $C^{(n)}$*-extendible cardinal greater than*

- *the rank of* $\boldsymbol{\Sigma}$*,*
- *the arity of each function or relation symbol in* $\boldsymbol{\Sigma}$*, and*
- *the ranks of the parameters used in some* Σ_n *definitions of* F *and* $\mathcal{A}$ *and in some definition of* $\mathcal{K}$*,*

then there exists a regular cardinal λ *such that* F *preserves* λ*-directed colimits.*

The set-theoretic framework

The von Neumann hierarchy

$$V_0 = \emptyset$$
$$V_{\alpha+1} = \mathcal{P}(V_\alpha)$$
$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha \quad \text{for limit ordinals } \lambda$$
$$V = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha, \quad \text{the full set-theoretic universe.}$$

The *rank* of a set x is the least α such that $x \subseteq V_\alpha$.

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Classes are collections of sets given by formulae (possibly with parameters): $\mathcal{C} = \{x \mid \varphi(x, p)\}$ for some formula φ and set p.

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Categories and functors are taken to be classes.

Formula complexity

Levy hierarchy

In the language of set theory, $\mathbf{\Sigma} = \{\in\}$, a formula is

- Σ_0 and Π_0 if all of its quantifiers are bounded (i.e., of the form $\forall x \in X$ or $\exists x \in X$).
- Σ_{n+1} if it is of the form $\exists x(\varphi(x))$ for some Π_n formula φ.
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A class (or category, or functor) is Σ_n if there is a Σ_n formula defining it.

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For a structure $\mathcal{M}$, we write $\mathcal{M} \models \varphi(m)$ for "$\mathcal{M}$ satisfies formula φ with parameter m".

Example

$$\langle \mathbb{Z}, + \rangle \models \forall x \exists y (x + y = 3)$$

$C^{(n)}$ cardinals

We denote by $C^{(n)}$ the class of cardinals κ such that $V_\kappa \prec_{\Sigma_n} V$, that is, for every Σ_n formula $\varphi(x)$ and set $x_0 \in V_\kappa$,

$$\langle V_\kappa, \in \rangle \models \varphi(x_0) \quad \text{if and only if} \quad \langle V, \in \rangle \models \varphi(x_0).$$

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For every n, $C^{(n)}$ is unbounded: given any cardinal γ, one can find a cardinal κ greater than γ in $C^{(n)}$.

Proof sketch

By induction on n. Iteratively take larger and larger κ in $C^{(n-1)}$ so that V_κ contains sets x witnessing statements of the form $\exists x(\varphi(x))$ with φ a Π_{n-1} formula. This process "closes off" at a limit point κ in $C^{(n)}$. □

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Note however that trying this for all formulae (i.e., all n) at once raises Gödelian, definability of definability problems.

$C^{(n)}$-extendible cardinals

Recall that an *elementary embedding* is a function preserving *all* formulae.

Definition

A cardinal κ *is* $C^{(n)}$*-extendible if for every* $\lambda > \kappa$ *there is a cardinal* $\mu > \lambda$ *and an elementary embedding* $j : V_\lambda \to V_\mu$ *such that*

1. $\kappa = \text{crit}(j)$, *i.e.,* κ *is the least ordinal such that* $j(\kappa) \neq \kappa$,
2. $j(\kappa) > \lambda$, *and*
3. $j(\kappa) \in C^{(n)}$.

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κ *is* $C^{(n)+}$*-extendible if moreover, for every* $\lambda > \kappa$ *in* $C^{(n)}$*, there is a* $\mu > \lambda$ *in* $C^{(n)}$ *and an elementary embedding* $j : V_\lambda \to V_\mu$ *such that (1), (2) and (3) hold.*

Relationships of large cardinals

Theorem (Bagaria & B-T)

For all α,

$$\exists\kappa > \alpha(\kappa \textit{ is } C^{(n)}\textit{-extendible})$$

is equivalent to

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Theorem (Bagaria, Casacuberta, Mathias & Rosický)

Vopěnka's Principle is equivalent to the existence of a proper class of $C^{(n)+}$*-extendible cardinals for every* n*. Morever, the existence of a* $C^{(n)+}$*-extendible* κ *corresponds exactly to Vopěnka's Principle for classes that are* Σ_{n+2}*-definable with parameters from* V_κ*.*

The main theorem again

Theorem (Bagaria & B-T)

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ii. *the arity of each function or relation symbol in* $\mathbf{\Sigma}$*, and*
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then there exists a regular cardinal λ *such that* F *preserves* λ*-directed colimits.*

Proof

Want to show $F : \mathcal{A} \to \mathcal{K}$ preserves λ-directed colimits.

Sufficient:

$$i \circ F : \mathcal{A} \to \mathbf{Str}\,\mathbf{\Sigma} \text{ preserves } \lambda\text{-directed colimits,}$$

where $i : \mathcal{K} \to \mathbf{Str}\,\mathbf{\Sigma}$ is the inclusion functor (and this notional inclusion doesn't change the quantifier complexity). So WLOG assume $F : \mathcal{A} \to \mathbf{Str}\,\mathbf{\Sigma}$.

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Note: $\mathbf{Str}\,\mathbf{\Sigma}$ has all λ-directed colimits, for λ greater than the arities of the symbols in $\mathbf{\Sigma}$ (i.e. cardinals as per (ii)).

Let β be sufficiently large as per (i), (ii) and (iii).

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Consider the following category $\mathcal{C}$:

Objects: $\mathbf{Str}\,\boldsymbol{\Sigma}$ morphisms $a : \bar{A} \to F(A)$ such that for some $\lambda > \beta$ and some λ-directed diagram $\mathcal{D}$ in $\mathcal{A}$,

- A is the colimit of $\mathcal{D}$ in $\mathcal{A}$,
- $\bar{A}$ is the colimit of $F\mathcal{D}$ in $\mathbf{Str}\,\boldsymbol{\Sigma}$, and
- a is the morphism induced by the image under F of the $\mathcal{A}$-colimit cocone from $\mathcal{D}$ to A.

Morphisms: From a to b: pairs $\langle g, h \rangle$ of $\mathbf{Str}\,\boldsymbol{\Sigma}$ morphisms such that

$$\begin{array}{ccc} \bar{A} & \xrightarrow{\;a\;} & F(A) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \bar{B} & \xrightarrow[\;b\;]{} & F(B). \end{array}$$

commutes.

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If the theorem fails, then $\mathcal{C}^*$ is not essentially small.

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The universal property of colimits makes the middle line Π_{n+1}.

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Let κ be a $C^{(n)+}$-extendible cardinal greater than β.

Let a be an object of $\mathcal{C}^*$ of rank $> \kappa$, arising from a λ_a-directed diagram $\mathcal{D}_a$ for some $\lambda_a > \kappa$.

Let $\lambda \in C^{(n)}$ be greater than the ranks of $a, \mathcal{D}_a, F\mathcal{D}_a$, and the corresponding cocones $\langle \bar{A}, \bar{\eta} \rangle_a$ and $\langle A, \eta \rangle_a$.

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$$V_\mu \vDash \lambda_a, \mathcal{D}_a, \langle \bar{A}, \bar{\eta} \rangle_a \text{ and } \langle A, \eta \rangle_a \text{ witness that } a \in \mathrm{Obj}(\mathcal{C}^*).$$

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Note that because $\kappa > \beta$, the definition of F is unaffected by j, so j commutes with F.

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Since j is elementary, we have a morphism in ${\mathcal{C}^*}^{V_\mu}$

$$\begin{array}{ccc} \bar{A} & \xrightarrow{\quad a \quad} & F(A) \\ {\scriptstyle j\restriction\bar{A}}\downarrow & & \downarrow{\scriptstyle j\restriction F(A)} \\ j(\bar{A}) & \xrightarrow[\quad j(a) \quad]{} & j(F(A)). \end{array}$$

Proof

Want to show $F : \mathcal{A} \to \mathbf{Str}\,\boldsymbol{\Sigma}$ preserves λ-directed colimits.
$\mathcal{C}$: cat. of λ-directed colimit morphisms $a : \bar{A} \to F(A)$
$\mathcal{C}^*$: full subcat. of non-isos; $\mathcal{C}^*$ is Σ_{n+2}-definable
Want to show $\mathcal{C}^*$ is essentially small
If not take κ a $C^{(n)+}$-extendible, $a \in \mathcal{C}^*$ "bigger", $\lambda \in C^{(n)}$ yet bigger,
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Now, $\mathcal{D}_a$ is λ_a-directed, so $j(\mathcal{D}_a)$ is $j(\lambda_a)$-directed, and $j(F\mathcal{D}_a) = Fj(\mathcal{D}_a)$ is $j(\lambda_a)$-directed.

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Now, $\mathcal{D}_a$ is λ_a-directed, so $j(\mathcal{D}_a)$ is $j(\lambda_a)$-directed, and $j(F\mathcal{D}_a) = Fj(\mathcal{D}_a)$ is $j(\lambda_a)$-directed. Since $j(\lambda_a) > j(\kappa) > \lambda > |\mathcal{D}_a|$, $j``F\mathcal{D}_a$ has an upper bound $F(d_0)$ in $Fj(\mathcal{D}_a)$.

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