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QUOTIENTS OF HYPERELLIPTIC CURVES AND ÉTALE COHOMOLOGY

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ABSTRACT. We study hyperelliptic curves C with an action of an affine group of automorphisms G . We establish a closed form expression for the quotient curve C/G and for the first étale cohomology group of C as a representation of G . The motivation comes from the arithmetic of hyperelliptic curves over local fields, specifically their local Galois representations and the associated invariants.

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1. INTRODUCTION

This paper studies hyperelliptic curves C/k with a group of affine automorphisms G . We show that both the equation of the quotient curve C/G and the G -representation $H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)$ admit simple closed form descriptions in terms of the defining equation of C . The application we have in mind is to Galois representations of hyperelliptic curves over local fields [5].

Throughout the paper C will be a non-singular projective hyperelliptic curve over a field k with $\text{char } k \neq 2$. It will be given by an (affine) equation

$$C : Y^2 = f(X)$$

with $f(X) \in k[X]$ a squarefree polynomial of degree ≥ 1 that factors over \bar{k} as

$$f(X) = c \prod_{r \in R} (X - r), \quad R \subset \bar{k}.$$

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Let $G \subset \text{Aut}_k C$ be an affine group of automorphisms. Thus $g \in G$ acts as

$$g(X) = \alpha(g)X + \beta(g), \quad g(Y) = \gamma(g)Y$$

for some $\alpha(g), \beta(g), \gamma(g) \in k$. In particular, G acts naturally on the set of roots R through the X -coordinate. The punchline is that both the quotient curve C/G and the étale cohomology group $H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)$ with G -action can be very simply expressed in terms of γ and the G -action on R . Explicitly, writing R/G for the set of orbits of G on R , and $\mathbb{C}[R]$ for the permutation representation, we prove:

Theorem 1.1 (Quotient curve).

- (1) If G contains the hyperelliptic involution, then $C/G \simeq \mathbb{P}_k^1$.
- (2) If G acts trivially on the Y -coordinate, then

$$C/G : y^2 = c(-1)^{|R|-|R/G|} \prod_{O \in R/G} (x - \prod_{r \in O} r).$$

- (3) Otherwise,

$$C/G : y^2 = c(-1)^{(m-1)(|R/G|-1)} (x - \mu) \prod_{\substack{O \in R/G \\ O \neq U}} (x - \prod_{r \in O} r),$$

where U is the set of $a \in k$ that are fixed points of some non-trivial $g \in G$ acting on k through the X -coordinate, $m = |G|/|U|$ and $\mu = \prod_{a \in U} a^m$.

It is easy to see that in case (3), $|G| = m$, $|U| = 1$ when $\text{char } k = 0$, and $|G| = mp^j$, $|U| = p^j$, $p \nmid m$ when $\text{char } k = p > 0$ (Lemma 2.1).

Theorem 1.2 (Étale cohomology). For every prime $l \neq \text{char } k$, and every embedding $\mathbb{Q}_l \hookrightarrow \mathbb{C}$,

$$H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l) \otimes \mathbb{C} \simeq V \oplus \epsilon$$

as a complex representation of G , where

$$V = \tilde{\gamma} \otimes (\mathbb{C}[R] \oplus \mathbf{1}), \quad \epsilon = \begin{cases} 0 & \text{if } |R| \text{ is odd,} \\ \det V & \text{if } |R| \text{ is even,} \end{cases}$$

and $\tilde{\gamma} : G \rightarrow \mathbb{C}^\times$ is any one-dimensional representation with $\ker \tilde{\gamma} = \ker \gamma$. The representation $\mathbf{1} \oplus \epsilon$ is the permutation action of G on the points at infinity of C .

The assumptions $\text{char } k \neq 2$ and $l \neq \text{char } k$ are certainly necessary. When $\text{char } k = 2$, the curve C can be given $y^2 + y = f(x)$, but neither the set of roots of f nor any other G -set will compute $H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)$ as in the theorem: for example if $C/\bar{\mathbb{F}}_2$ is an elliptic curve with $G = Q_8$ or $G = \text{SL}_2(\mathbb{F}_3)$, then $H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)$ is not a 1-dimensional twist of a representation realisable over \mathbb{Q} . Similarly, $H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)$ behaves differently when $l = \text{char } k$: its dimension drops, and it is not necessarily a rationally traced representation (e.g. for an ordinary elliptic curve over \mathbb{F}_p with $G = C_4$).

Our motivation comes from the arithmetic of hyperelliptic curves over local fields, specifically their $H_{\text{ét}}^1$ as a Galois representation and the associated

invariants: the conductor, local polynomial, root number etc. Suppose for simplicity that C/K is such a curve, and that it acquires good reduction over a finite Galois extension F/K . The inertia group $G = I_{F/K}$ and Frobenius element $\phi \in \text{Gal}(\bar{K}/K)$ act naturally on the reduced curve \bar{C} over the residue field k of K , and G acts as a group of affine automorphisms. It turns out that $H_{\text{ét}}^1(C_{\bar{K}}, \mathbb{Q}_l)$ as a $\text{Gal}(\bar{K}/K)$ -module is the same as $H_{\text{ét}}^1(\bar{C}_{\bar{k}}, \mathbb{Q}_l)$ with its G - and ϕ -action, and Theorem 1.2 describes the G -action explicitly. To complement this, we need to describe the action of Frobenius-like maps on the quotient curve C/G :

Theorem 1.3. *Suppose $\text{char } k = p > 2$, G does not contain the hyperelliptic involution, and $\Phi : C \rightarrow C$ is a morphism of the form*

$$\Phi X = aX^q + b, \quad \Phi Y = dY^q \quad (q \text{ power of } p)$$

that normalises¹ G . Then Φ descends to a morphism $\Psi : C/G \rightarrow C/G$ given by

$$\begin{aligned} \Psi x &= a^{|G|} x^q + \prod_{g \in G} (\alpha(g)b + \beta(g)), \\ \Psi y &= \begin{cases} d y^q & \text{if } \gamma = \mathbf{1}, \\ a^{\lfloor m/2 \rfloor |G|/m} d y^q & \text{if } \gamma \neq \mathbf{1}, \end{cases} \end{aligned}$$

where m is the prime-to- p part of $|G|$, and the model for C/G is that of Theorem 1.1.

See §6 for an example of how this theorem can be used over local fields to determine the local Galois representation of a hyperelliptic curve (and [5] for the general theory).

Outline. Section 2 concerns affine groups of automorphisms of hyperelliptic curves, the permutation action on the set of roots R and the character γ . The main theorem here is Theorem 2.4, which is Theorem 1.1 with an explicit description of the quotient map. Section 3 proves some general facts about $H_{\text{ét}}^1$ for quotient curves, in particular showing that $H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)$ is the unique representation V of G with rational character for which $\dim V^H = 2 \text{genus}(C/H)$ for every subgroup $H < G$. Then Section 4 applies this to hyperelliptic curves, proving Theorem 1.2 (=4.1), with a slightly cumbersome representation-theoretic computation. An alternative approach would be to use the Weil–Serre description of the G -action on étale cohomology in terms of an equivariant Riemann–Hurwitz formula [12], [10, Ch. VI, §4], but that seems to be an equally long computation. Section 5 proves Theorem 1.3 (=5.1).

Remark 1.4. To obtain an explicit description for $H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)$, we only use that it is the unique representation V of G with rational character for which $\dim V^H = 2 \text{genus}(C/H)$ for every subgroup $H < G$ (Theorem 3.3). Thus Theorem 1.2 would also hold for any other cohomology theory with these properties.

¹meaning $\Phi G = G\Phi$ as sets

Notation. Throughout the paper, we use the following notation.

- C/k hyperelliptic curve $Y^2 = c \prod_{r \in R} (X - r)$ with the right-hand side in $k[x]$, $R \subset \bar{k}$, $\text{char } k \neq 2$; genus ≤ 1 is allowed.
- G a finite group of affine automorphisms of C/k , acting by $g(X) = \alpha(g)X + \beta(g)$, $g(Y) = \gamma(g)Y$ for $g \in G$.
- $H^1(C) = H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l) \otimes \mathbb{C}$ (choosing some embedding $\mathbb{Q}_l \hookrightarrow \mathbb{C}$).
- $\mathbf{1}, \oplus, \ominus$ trivial representation, direct sum and direct difference ($V \ominus W$ is well-defined up to isomorphism because the category of $\mathbb{C}[G]$ -modules is semisimple).
- $\lfloor \cdot \rfloor$ the floor function.
- ζ_m primitive m th root of unity.

2. QUOTIENTS OF HYPERELLIPTIC CURVES

We begin with affine actions and invariant functions on \mathbb{A}^1 , and then move to hyperelliptic curves. We refer the reader interested in arbitrary (non-affine) automorphism groups of hyperelliptic curves to [11] and [2].

Lemma 2.1. *Let k be a field, and $G \subset \text{Aut } \mathbb{A}_k^1$ a finite group of affine linear transformations*

$$X \mapsto g(X) = \alpha(g)X + \beta(g).$$

Then

- (1) $G \simeq T \rtimes C_m$, where T is the subgroup of translations. If $\text{char } k = 0$, then T is trivial. If $\text{char } k = p > 0$, then $p \nmid m$, and T is an elementary abelian p -group. Moreover, T is an $\mathbb{F}_p(\zeta_m)$ -vector space, with C_m acting on T by multiplication by m th roots of unity.
- (2) Every $g \in G \setminus T$ has a unique fixed point $a \in \bar{k}$; moreover, $a \in k$.
- (3) If $m = 1$, then every G -orbit on $\mathbb{A}^1(\bar{k})$ is regular².
- (4) If $m \neq 1$, then there is a unique non-regular G -orbit on $\mathbb{A}^1(\bar{k})$. It is a regular T -orbit, and consists of the fixed points of all $g \in G \setminus T$.
- (5) The field of invariant rational functions $k(X)^G$ is generated by

$$I = \prod_{g \in G} (\alpha(g)X + \beta(g)),$$

the unique G -invariant polynomial of degree $|G|$ with constant coefficient 0 and leading coefficient $(-1)^{|G|-1}$.

- (6) If $O \subset \bar{k} = \mathbb{A}^1(\bar{k})$ is a regular G -orbit, then

$$\prod_{r \in O} (X - r) = (-1)^{|G|-1} (I - \prod_{r \in O} r).$$

²Recall that an action of a finite group G on a set X is *regular* if $X \simeq G$ as a G -set, equivalently the action is transitive and non-trivial elements of G have no fixed points.

- (7) If $U \subset \bar{k} = \mathbb{A}^1(\bar{k})$ is a non-regular G -orbit, let $I_T = \prod_{g \in T} g(X)$ and $\lambda = \prod_{r \in U} r$. Then

$$\prod_{r \in U} (X - r) = I_T - \lambda,$$

the unique monic polynomial of degree $|G|/m$ that is G -invariant up to scalars.

Proof. (1) The elements of G with $\alpha(g) = 1$ form a group of translations T , which is naturally an additive subgroup of k . As G is finite, T is trivial if $\text{char } k = 0$. The map $g \mapsto \alpha(g)$ embeds $A = G/T \hookrightarrow k^\times$, so it is a cyclic group, $A \simeq C_m$.

Suppose $\text{char } k = p > 0$. Then $p \nmid m$. As $|T|$ and $|A|$ are coprime, the extension of A by T is split. Take a generator g of $A = C_m$ with $\alpha(g) = \zeta_m$. It conjugates a translation $x \mapsto x + v$ to $x \mapsto x + \zeta_m v$. This makes T into an $\mathbb{F}_p(\zeta_m)$ -vector space, with the asserted conjugation action by C_m .

(2) If $\alpha(g) \neq 1$, then the map $X \mapsto \alpha(g)X + \beta(g)$ has a unique fixed point $a = \beta(g)/(1 - \alpha(g))$ on \mathbb{A}^1 , and it is k -rational.

(3) Clear, since translations have no fixed points.

(4) Let g be a generator of C_m and $a \in k$ its fixed point. The stabiliser of a is precisely C_m (as it meets T trivially), so a has orbit of length $|T|$; it is a regular T -orbit. Every other point in this orbit has a conjugate stabiliser that meets C_m trivially, by the uniqueness of fixed points. Thus, these stabilisers cover $(m-1)|T|$ elements of G , which must be the whole of $G \setminus T$. Hence, the orbit of a accounts for all fixed points of elements of G , in other words every other G -orbit is regular.

(5) $I(X)$ is clearly G -invariant, and has the right degree $|G|$. Its constant coefficient is 0, and the leading coefficient is

$$\prod_{g \in G} \alpha(g) = \left(\prod_{i=0}^{m-1} \zeta_m^i \right)^{|T|} = \begin{cases} (-1)^{|T|} = -1, & \text{if } 2|m \\ 1^{|T|} = 1, & \text{if } 2 \nmid m \end{cases} = (-1)^{|G|-1}.$$

(6) The right-hand side is G -invariant. So is the left-hand side, since every $g \in G$ permutes its roots and $\alpha(g)^{|G|} = 1$. The claim follows from (5), by comparing the leading and the constant terms.

(7) Because U is a G -orbit, the polynomial $\prod_{x \in U} (X - r)$ is G -invariant up to scalars. (And it is a unique such polynomial of this degree, because a non-regular orbit is unique if it exists.) By part (6) with $G = T$,

$$\prod_{r \in U} (X - r) = I_T - \lambda.$$

□

Proposition 2.2. *Let k be a field with $\text{char } k \neq 2$, and C/k a hyperelliptic curve*

$$C : Y^2 = c \prod_{r \in R} (X - r), \quad R \subset \bar{k}.$$

Let $G \subset \text{Aut}_k C$ be a subgroup of affine automorphisms

$$g(X) = \alpha(g)X + \beta(g), \quad g(Y) = \gamma(g)Y \quad (g \in G).$$

Write K and \bar{G} for the kernel and the image of the map $X \mapsto g(X)$ from G to $\text{Aut}_k \mathbb{A}^1$.

Then:

- (1) G is a central extension of \bar{G} by K , and

$$G = \bar{G} \iff |K| = 1 \iff \text{hyperelliptic involution} \notin G.$$

$$G \neq \bar{G} \iff |K| = 2 \iff \text{hyperelliptic involution} \in G.$$
- (2) $\bar{G} \simeq T \rtimes_{\alpha} C_m$ with $T = \ker \alpha$ the subgroup of translations of \bar{G} .
- (3) If $\text{char } k = 0$, then T is trivial.
- (4) If $\text{char } k = p > 0$, then $p \nmid m$, and T is an elementary abelian p -group. Moreover, T is an $\mathbb{F}_p(\zeta_m)$ -vector space, with C_m acting on T by multiplication by m th roots of unity.
- (5) α is a primitive character of $\bar{G}/T \simeq C_m$, and $\gamma^2 = \alpha^{|R|}$. The group \bar{G} acts naturally on R , and either
 - (a) $|R| \equiv 0 \pmod{m}$ and $\gamma^2 = \mathbf{1}$; as a \bar{G} -set, R is a union of regular orbits; or
 - (b) $|R| \equiv 1 \pmod{m}$ and $\gamma^2 = \alpha \neq \mathbf{1}$; as a \bar{G} -set, R is a union of regular orbits and one non-regular orbit $R_0 \simeq \bar{G}/C_m$, which is the set of fixed points of non-trivial elements of \bar{G} .
- (6) If $|R|$ is even, then $\gamma(g)/\alpha(g)^{|R|/2} = \pm 1$ for $g \in G$, and it is -1 if and only if g swaps the two points at infinity of C .

Proof. (1) Clear. (2),(3),(4) follow directly from Lemma 2.1. (5) Since automorphisms preserve the equation of C up to a constant, \bar{G} permutes the elements of R , and $\gamma^2 = \alpha^{|R|}$. The rest follows from the following lemma and Lemma 2.1 (2),(3). (6) The coordinates of the chart at infinity for C are $u = 1/X$ and $v = Y/X^{|R|/2}$, and the two points at infinity are $u = 0, v = \pm\sqrt{c}$ on this chart, when $|R|$ is even. The claim follows. \square

Lemma 2.3. *Suppose G is a group of the form $G = \mathbb{F}_p^r \rtimes C_m$ ($p \nmid m$) such that no non-trivial elements $g \in C_m$ and $h \in \mathbb{F}_p^r$ commute. Let R be a faithful G -set on which elements $\text{id} \neq g \in \mathbb{F}_p^r$ have no fixed points, and elements $\text{id} \neq g \in C_m$ have at most one fixed point. Then*

- (1) $p^r \equiv 1 \pmod{m}$;
- (2) R is a union of G -regular orbits plus at most one non-regular orbit $R_0 \simeq G/C_m$, consisting of fixed points of non-trivial elements of G on R .

Proof. (1) From the non-commutativity assumption, it follows that the orbits of C_m on $\mathbb{F}_p^r \setminus \{\text{id}\}$ have length m .

(2) Since non-trivial elements of \mathbb{F}_p^r have no fixed points, R is a union of regular \mathbb{F}_p^r -orbits. Now suppose that R has a non-regular orbit of G . It is a union of m/d regular \mathbb{F}_p^r -orbits for some $d > 1$, which C_m permutes transitively. As d and $|\mathbb{F}_p^r|$ are coprime, elements of C_k must have fixed

points in each one of them. But they have at most one fixed point in total, and so $d = m$, such an orbit is $\simeq G/C_m$, and it is necessarily unique (and clearly consists of fixed points of non-trivial elements of G). Hence

$$R \simeq G \amalg \cdots \amalg G \quad \text{or} \quad R \simeq G \amalg \cdots \amalg G \amalg G/C_m.$$

In the first case $|R| \equiv 0 \pmod{m}$, and in the second case $|R| \equiv p^r \equiv 1 \pmod{m}$. \square

Theorem 2.4. *Let C be a hyperelliptic curve over a field k with $\text{char } k \neq 2$,*

$$C : Y^2 = cf(X); \quad f(X) = \prod_{r \in R} (X - r),$$

and $G \subset \text{Aut}_k C$ a group of automorphisms acting via

$$g(X) = \alpha(g)X + \beta(g), \quad g(Y) = \gamma(g)Y \quad (g \in G).$$

Let $I = \prod_{g \in G} g(X)$, and write R/G for the set of orbits of G on the roots of f through the X -coordinate action.

- (1) *If G contains the hyperelliptic involution then $C/G \simeq \mathbb{P}_k^1$ with field of rational functions $k(C/G) = k(\sqrt{I})$.*
- (2) *If G acts trivially on the Y -coordinate, then*

$$C/G : y^2 = c(-1)^{|R|-|R/G|} \prod_{O \in R/G} (x - \prod_{r \in O} r)$$

with $x = I$ and $y = Y$.

- (3) *Otherwise,*

$$C/G : y^2 = c(-1)^\delta (x - \lambda^m) \prod_{\substack{O \in R/G \\ O \neq U}} (x - \prod_{r \in O} r)$$

with $x = I$ and $y = (I_T - \lambda)^{\lfloor m/2 \rfloor} Y$. Here U is the set of $a \in k$ that are fixed points of some non-trivial $g \in G$ acting through the X -coordinate,

$$m = \frac{|G|}{|U|}, \quad \lambda = \prod_{a \in U} a, \quad \delta = (m-1)(|R/G|-1), \quad I_T = \prod_{g \in \ker \alpha} g(X).$$

Proof. (1) Write ι for the hyperelliptic involution. We clearly have $C/\langle \iota \rangle \simeq \mathbb{P}_k^1$, and $k(C/\langle \iota \rangle) = k(X)$. Its quotient by $G/\langle \iota \rangle$ is again \mathbb{P}_k^1 , and its field of rational functions is $k(\sqrt{I})$ by Lemma 2.1(5).

- (2) The polynomials Y and I are G -invariant, and

$$k(C/G) = k(C)^G = k(I, Y)$$

by degree considerations. To find the relation between I and Y , first note that as $\gamma^2 = 1$, R is a union of regular \bar{G} -orbits by Proposition 2.2. Because

$Y^2/c = f(X) \in k(X)$ is G -invariant, by Lemma 2.1(6),

$$\begin{aligned} f(X) &= \prod_{r \in R} (X - r) = \prod_{O \in R/G} (-1)^{|G|-1} (I - \prod_{r \in O} r) \\ &= (-1)^{|R|-|R/G|} \prod_{O \in R/G} (I - \prod_{r \in O} r), \end{aligned}$$

which gives the required identity between $x = I$ and $y = Y$.

(3) Write $G = T \rtimes C_m$ as in Proposition 2.2(1,2). First, suppose that T is trivial, so that $G = C_m$ with $m > 1$. We have two cases:

Case $|T|=1$, m odd. Since $\gamma \neq \mathbf{1}$, we are in case (5b) of Proposition 2.2. So $\gamma = \alpha^{-(m-1)/2}$ (the unique character that squares to $\gamma^2 = \alpha$) and there is a unique G -invariant root $\lambda \in R$. Hence elements $g \in G$ must act on the curve C as

$$g(X) = \alpha(g)(X - \lambda) + \lambda, \quad g(Y) = \alpha(g)^{-(m-1)/2} Y,$$

and the polynomials

$$I = (X - \lambda)^m + \lambda^m, \quad J = (X - \lambda)^{(m-1)/2} Y$$

are easily seen to be G -invariant (and the first one is I by Lemma 2.1(5)). As $[k(X) : k(I)] \leq m$ and $k(X, Y) = k(X, J)$, the polynomials I and J generate $k(C)^G$ by degree considerations. They satisfy the relation

$$J^2 = Y^2(X - \lambda)^{m-1} = c f(X) \frac{I - \lambda^m}{X - \lambda} = c (I - \lambda^m) \frac{f(X)}{X - \lambda}.$$

Finally, since $f(X)$ has all roots but λ coming in G -regular orbits, by Lemma 2.1(6),

$$\frac{f(X)}{X - \lambda} = \prod_{\substack{O \in R/G \\ O \neq \{\lambda\}}} (I - \prod_{r \in O} r),$$

which gives the required equation

$$J^2 = c(I - \lambda^m) \prod_{\substack{O \in R/G \\ O \neq \{\lambda\}}} (I - \prod_{r \in O} r).$$

Case $|T|=1$, m even. If we were in case (5b) of Proposition 2.2, then α would have order m and γ order $2m$; in that case, $\ker \alpha \neq \{\text{id}\}$ and G contains the hyperelliptic involution — a contradiction. Hence we must be in case (5a), in other words R is a union of regular \bar{G} -orbits. Note that $\gamma = \alpha^{m/2}$ since $\gamma^2 = \mathbf{1}$ and $\gamma \neq \mathbf{1}$.

By Lemma 2.1(3), there is a unique fixed point $\lambda \in k$ of $G = C_m$, with $\lambda \notin R$ by the above. The polynomials

$$I = -(X - \lambda)^m + \lambda^m, \quad J = (X - \lambda)^{m/2} Y$$

are G -invariant (and the first one is I by Lemma 2.1(5)), and generate $k(C)^G$ by degree considerations. Using Lemma 2.1(6) as before, we get the required

relation

$$\begin{aligned} J^2 &= (X - \lambda)^m c f(x) = c (-I + \lambda^m) \prod_{O \in R/G} -(I - \prod_{r \in O} r) \\ &= (-1)^{1+|R/G|} c (I - \lambda^m) \prod_{O \in R/G} (I - \prod_{r \in O} r). \end{aligned}$$

Case $|T| \neq 1$. Finally, suppose that T is non-trivial. First, we can compute C/T by using (2) and then $C/G = (C/T)/C_m$ using the $|T| = 1$ cases. We have

$$k(C/T) = k(C)^T = k(Y, I_T), \quad I_T = \prod_{g \in T} g(X),$$

and the equation for C/T is

$$C/T : Y^2 = c \prod_{O \in R/T} (I_T - \prod_{r \in O} r).$$

The group C_m acts on this curve by affine transformations of the form

$$g(I_T) = \alpha_T(g)I_T + \beta_T(g), \quad g(Y) = \gamma(g)Y \quad (g \in C_m),$$

and

$$\alpha_T(g) = \alpha(g)^{|T|} = \alpha(g),$$

since $|T| \equiv 1 \pmod{m}$ by Lemma 2.3(1) and Proposition 2.2(5). In particular, C_m acts non-trivially on the Y -coordinate of C/T and does not contain the hyperelliptic involution of C/T . By Lemma 2.1 (7), $I_T - \lambda$ is the unique monic polynomial of degree $|G|/m$ that is G -invariant up to scalars. In other words, on the I_T -coordinate λ is the unique fixed point for the C_m -action. Hence, by the $|T| = 1$ cases we find the quotient

$$(C/T)/C_m : y^2 = c (-1)^\delta (x - \lambda^m) \prod_{\substack{O \in (R/T)/C_m \\ O \neq \{\lambda\}}} (x - \prod_{r \in O} r),$$

where

$$\begin{aligned} \delta &= (m-1)(|(R/T)/C_m| - 1) = (m-1)(|R/G| - 1), \\ x &= \prod_{g \in C_m} g(I_T) = \prod_{g \in C_m} g\left(\prod_{t \in T} t(X)\right) = I, \\ y &= (I_T - \lambda)^{\lfloor m/2 \rfloor} Y, \end{aligned}$$

using that R is a union of regular T -orbits by Proposition 2.2 (5) with $G = T$. \square

Corollary 2.5. *Suppose C and G are as in Theorem 2.4, and let r be the number of regular orbits of G on R . Then*

$$\text{genus}(C/G) = \begin{cases} 0 & \text{if } \iota \in G, \\ \lfloor \frac{r}{2} - \frac{1}{2} \rfloor = \lfloor \frac{|R|}{2|G|} - \frac{1}{2} \rfloor & \text{if } \gamma = \mathbf{1}, \\ \lfloor \frac{r}{2} \rfloor = \lfloor \frac{|R|}{2|G|} \rfloor & \text{if } \iota \notin G, \gamma \neq \mathbf{1}. \end{cases}$$

Proof. Apply Theorem 2.4, and recall that the genus of a hyperelliptic curve given by a polynomial of degree n is $\lfloor \frac{n-1}{2} \rfloor$. \square

3. ÉTALE COHOMOLOGY OF QUOTIENT CURVES

In this section, C/k is any non-singular projective curve with a finite group of automorphisms $G < \text{Aut}_k C$. We will show that the G -action on the étale cohomology group $H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)$ is determined by the genera of the quotients of C by subgroups of G .

Recall that for every prime l there are canonical isomorphisms

$$H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l) \simeq H_{\text{ét}}^1(\text{Jac}(C_{\bar{k}}), \mathbb{Q}_l) \simeq (V_l \text{Jac}(C))^*,$$

where, as usual,

$$V_l \text{Jac}(C) = (\varprojlim \text{Jac}(C)[l^n]) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

is the vector space associated to the Tate module, and $*$ denotes \mathbb{Q}_l -linear dual. If $G < \text{Aut}_k C$ is a group of automorphisms, these become naturally $\mathbb{Q}_l G$ -representations, and the isomorphisms respect this structure.

Theorem 3.1. *Let $\pi : C \rightarrow D$ be a Galois cover of non-singular projective curves over a field k , with Galois group G , and let l be a prime number.*

- (1) *If $l \neq \text{char } k$, then $H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)$ is a $\mathbb{Q}_l G$ -representation with rational character.*
- (2) *There is an isomorphism of \mathbb{Q}_l -vector spaces*

$$H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)^G \simeq H_{\text{ét}}^1(D_{\bar{k}}, \mathbb{Q}_l).$$

- (3) *If $\Phi : C \rightarrow C$ is a morphism of curves that commutes with π , then the isomorphism in (2) commutes with the action on Φ .*
- (4) *If $l \neq \text{char } k$, then $\text{genus}(D) = \frac{1}{2} \dim H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)^G$.*

Proof. (1) By the Lefschetz fixed-point formula [8, Thm. 25.1], the trace of any $\sigma \in G$ on H^1 is an integer, namely 2 minus the number of fixed points of σ on C .

- (2) The projection π induces pushforward and pullback on divisors,

$$\pi_* : \text{Div}(C) \longrightarrow \text{Div}(D), \quad \pi^* : \text{Div}(D) \longrightarrow \text{Div}(C).$$

The composition $\pi_* \pi^*$ is multiplication by $|G|$, while $\pi^* \pi_*$ is the trace map $\text{Tr} : P \mapsto \sum_{g \in G} P^g$. These maps descend to $\text{Pic}^0(C) = \text{Jac}(C)$ and its l^n -torsion $\text{Jac}(C)[l^n]$. The image of Tr on $V_l \text{Jac}(C)$ is the group of G -invariants, and

$$(V_l \text{Jac}(C))^G \simeq \text{Tr}(V_l \text{Jac}(C)) \simeq \pi_*(V_l \text{Jac}(C)) \simeq V_l \text{Jac}(D),$$

as multiplication by $|G|$ is an isomorphism on V_l , and hence π^* is injective and π_* is surjective.

- (3) Clear from the construction in (2).
- (4) Clear from (2) and the genus formula $\dim H_{\text{ét}}^1(D_{\bar{k}}, \mathbb{Q}_l) = 2 \text{genus}(D)$. \square

Lemma 3.2. *Let G be a finite group. A representation V of G with rational character is uniquely determined by $\dim V^H$ for all cyclic subgroups $H < G$.*

Proof. A suitable multiple of V is realisable over \mathbb{Q} ([9] Ch. 12, Prop. 34), and for such representations the claim is proved in [9] Ch. 13, Cor. to Thm. 30'. \square

Theorem 3.3. *Let C/k be a curve, and $l \neq \text{char } k$ a prime number. Suppose G is a finite group acting as automorphisms of C/k (not necessarily faithfully). Then $H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)$ is the unique \mathbb{Q}_l -representation V of G such that*

- (a) *all character values of G on V are rational, and*
- (b) *$\dim V^H = 2 \text{genus}(C/H)$ for every subgroup $H < G$.*

Proof. Uniqueness follows from Lemma 3.2. Theorem 3.1(1,4) shows that $V = H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)$ satisfies (a) and (b) when the action is faithful. Therefore, if $q : G \rightarrow \text{Aut } C$ is any action, then

$$\dim V^H = \dim V^{q(H)} = 2 \text{genus}(C/q(H)) = 2 \text{genus}(C/H),$$

and $q(g)$ has rational trace on V for every $g \in G$. \square

Remark 3.4. In Theorem 3.3(b), ‘subgroup’ may be replaced by ‘cyclic subgroup’, since these are sufficient for Lemma 3.2. Note also that the assertion concerns only the character of V , so the result remains true after any extension of the coefficient field (e.g. to $\bar{\mathbb{Q}}_l$ or \mathbb{C}).

4. ÉTALE COHOMOLOGY OF HYPERELLIPTIC CURVES

In this section we prove the following theorem that describes the action of automorphisms on the first étale cohomology group of a hyperelliptic curve.

Theorem 4.1. *Let k be a field of characteristic $\neq 2$, and C/k a hyperelliptic curve given by*

$$Y^2 = c \prod_{r \in R} (X - r), \quad R \subset \bar{k}.$$

Let $G \subset \text{Aut}_k C$ be an affine group of automorphisms acting as

$$g(X) = \alpha(g)X + \beta(g), \quad g(Y) = \gamma(g)Y \quad (g \in G).$$

Then for every prime $l \neq \text{char } k$ and every embedding $\mathbb{Q}_l \hookrightarrow \mathbb{C}$,

$$H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l) \otimes \mathbb{C} \simeq V \oplus \epsilon$$

as a complex representation of G , where

$$V = \tilde{\gamma} \otimes (\mathbb{C}[R] \oplus \mathbf{1}), \quad \epsilon = \begin{cases} 0 & \text{if } |R| \text{ is odd,} \\ \det V & \text{if } |R| \text{ is even,} \end{cases}$$

and $\tilde{\gamma} : G \rightarrow \mathbb{C}^\times$ is any one-dimensional representation with $\ker \tilde{\gamma} = \ker \gamma$. The representation $\mathbf{1} \oplus \epsilon$ is the permutation action of G on the points at infinity of C .

Proof. When $|R| = 1$, the cohomology group $H_{\text{ét}}^1(C_{\bar{k}}, \mathbb{Q}_l)$ is zero, and the theorem is true. Assume $|R| \geq 2$, so that

$$\ker(G \rightarrow \text{Aut } \mathbb{A}_{\bar{k}}^1) = \ker(G \rightarrow \text{Aut } R).$$

By Theorem 3.3 (and Remark 3.4), it suffices to prove that $V \ominus \epsilon$ has rational character and that $\dim(V \ominus \epsilon)^H = 2 \text{genus}(C/H)$ for every $H < G$. This is a purely representation-theoretic statement, proved below in Theorem 4.2. By Lemma 2.1 and Proposition 2.2, Theorem 4.2 applies here with G , R , $\tilde{\gamma}$ and with κ the hyperelliptic involution if it is in G and $\kappa = \text{id}$ otherwise. Theorem 4.2 shows that $V \ominus \epsilon$ has rational character, and that

$$\dim(V \ominus \epsilon)^H = \begin{cases} 0 & \text{if hyperelliptic involution} \in H \\ 2 \lfloor \frac{|R|}{2|H|} - \frac{1}{2} \rfloor & \text{if } \gamma(H) = 1 \\ 2 \lfloor \frac{|R|}{2|H|} \rfloor & \text{otherwise.} \end{cases}$$

By Corollary 2.5, this is precisely $2 \text{genus}(C/H)$.

The claim about $\mathbf{1} \oplus \epsilon$ is clear when $|R|$ is odd (as there is one point at infinity, fixed by G), and follows from Proposition 2.2 (6) and Lemma 4.9 when $|R|$ is even. \square

Theorem 4.2. *Let G be a finite group acting on a set R , and $\gamma : G \rightarrow \mathbb{C}^\times$ a one-dimensional representation, satisfying the following*

- *The kernel of the action of G on R is generated by an element κ of order 1 or 2.*
- *Write $\bar{G} = G/\langle \kappa \rangle$. Then $\bar{G} \simeq T \rtimes C_m$, where T is an $\mathbb{F}_p(\zeta_m)$ -vector space with $p \nmid m$, and some generator of C_m acts on T by $v \mapsto \zeta_m v$.*
- *Every $g \in T \setminus \{\text{id}\}$ has no fixed points on R , and every $g \in G \setminus \{\text{id}, \kappa\}$ has at most one fixed point.*
- *If $\kappa \neq \text{id}$ then $\gamma(\kappa) = -1$.*
- *γ is trivial on T , and γ^2 is an $|R|$ th power of some one-dimensional representation α of $C_m = \bar{G}/T$ of exact order m .*

For $H < G$ define

$$\mathcal{F}(H) = \begin{cases} 0 & \text{if } \text{id} \neq \kappa \in H & \text{(Case I)} \\ 2 \lfloor \frac{|R|}{2|H|} - \frac{1}{2} \rfloor & \text{if } \gamma(H) = 1 & \text{(Case II)} \\ 2 \lfloor \frac{|R|}{2|H|} \rfloor & \text{otherwise} & \text{(Case III),} \end{cases}$$

and let

$$V = \gamma \otimes (\mathbb{C}[R] \ominus \mathbf{1}), \quad \epsilon = \begin{cases} 0 & \text{if } |R| \text{ is odd,} \\ \det V & \text{if } |R| \text{ is even.} \end{cases}$$

Then $V \ominus \epsilon$ is the unique representation with rational character for which

$$\dim(V \ominus \epsilon)^H = \mathcal{F}(H) \quad \text{for all } H < G. \quad (*)$$

Proof. Lemma 3.2 proves uniqueness. By Lemma 4.5(2), $V \ominus \epsilon$ has rational character, and it remains to prove (*). When $H = G$, this is shown in Lemmas 4.6, 4.7, 4.8, in Case I, II and III, respectively. To establish (*) for

general $H < G$, we can invoke the lemmas with H in place for G and with R and γ restricted to H . (Observe that every $H < G$ satisfies the conditions of the theorem.) \square

In the remainder of this section we prove the ingredients of Theorem 4.2.

Notation 4.3. Let $G, \bar{G} = T \rtimes C_m, R, \gamma, \kappa, \mathcal{F}, V$ and ϵ be as in Theorem 4.2. Write r for the number of regular orbits of \bar{G} on R , and $\tilde{r} = 0$ if r is even and 1 if r is odd (i.e. the last binary digit of r).

Lemma 4.4. *We have*

- (1) $(\mathbb{C}[\bar{G}/C_m] \ominus \mathbf{1})^{\oplus m} \simeq \mathbb{C}[\bar{G}] \ominus \mathbb{C}[\bar{G}/T]$ as representations of \bar{G} .
- (2) The representations $\mathbb{C}[\bar{G}]$ and $\mathbb{C}[\bar{G}/C_m] \ominus \mathbf{1}$ of \bar{G} are invariant under twisting by one-dimensional representations of $\bar{G}/T \simeq C_m$.
- (3) For every one-dimensional representation ψ of G/T , the G -representations $\psi \otimes \mathbb{C}[\bar{G}]$ and $\psi \otimes (\mathbb{C}[\bar{G}/C_m] \ominus \mathbf{1})$ have rational characters.

Proof. (1) The group $\bar{G}/T \simeq C_m$ acts on 1-dimensional characters of T by conjugation, and from the action of C_m on T we see that every non-trivial character has trivial stabiliser. Hence, by Clifford theory (or see [9, §8.2]), every irreducible representation of \bar{G} is either a lift of a 1-dimensional one of \bar{G}/T , or is m -dimensional and is induced from a 1-dimensional representation of T . In particular, $\mathbb{C}[\bar{G}] \ominus \mathbb{C}[\bar{G}/T]$ is the sum of the m -dimensional irreducibles, each with multiplicity m .

Since $\bar{G} \simeq T \rtimes C_m$,

$$\bar{G}/C_m \simeq T \quad \text{as } T\text{-sets,}$$

and so the restriction $\text{Res}_T(\mathbb{C}[\bar{G}/C_m] \ominus \mathbf{1})$ contains every non-trivial one-dimensional representation of T . So, by Frobenius reciprocity, $\mathbb{C}[\bar{G}/C_m] \ominus \mathbf{1}$ contains every m -dimensional irreducible of \bar{G} . By comparing dimensions, each occurs with multiplicity one, and the claim follows.

(2) The twist-invariance is clear for $\mathbb{C}[\bar{G}]$ and $\mathbb{C}[\bar{G}/T]$, and so follows for $\mathbb{C}[\bar{G}/C_m] \ominus \mathbf{1}$ from (1).

(3) Generally, for every rational character ρ of G which is invariant under C_m -twists, $\psi \otimes \rho$ is again rational. Indeed, ψ^2 clearly kills κ and factors through C_m , and so

$$\psi(g)^2 \rho(g) = \rho(g) \quad \text{for all } g \in G.$$

Thus, either $\psi(g)\rho(g) = 0$ or $\psi(g) = \pm 1$, and $\psi(g)\rho(g)$ is rational in both cases. \square

Lemma 4.5. *We also have*

- (1) *Either*
 - (a) $|R| \equiv 0 \pmod{m}$ and $\gamma^2 = \mathbf{1}$; as a \bar{G} -set, R is a union of regular orbits; or
 - (b) $|R| \equiv 1 \pmod{m}$ and $\gamma^2 = \alpha \neq \mathbf{1}$; as a \bar{G} -set, R is a union of regular orbits and one non-regular orbit $\simeq \bar{G}/C_m$.
- (2) *The representations V and $V \ominus \epsilon$ have rational characters.*

Proof. (1) The \bar{G} -structure of R follows from Lemma 2.3. By assumption, $\gamma^2 = \alpha^{|R|}$, and this is $\mathbf{1}$ or α , respectively.

(2) In case 1(a), the representation $\mathbb{C}[R] \ominus \mathbf{1}$ is realisable over \mathbb{Q} and γ has order ≤ 2 , so $V = \gamma \otimes (\mathbb{C}[R] \ominus \mathbf{1})$ is also realisable over \mathbb{Q} . In case 1(b), $V = \gamma \otimes (\mathbb{C}[R] \ominus \mathbf{1})$ has rational character by Lemma 4.4 (3). Finally, $\epsilon = 0$ or $\epsilon = \det V$, each of which is also rational. \square

Lemma 4.6 (Case I). *If $\kappa \neq \text{id}$, then*

$$\dim(\gamma \otimes \mathbb{C}[R])^G = \dim \gamma^G = \dim \epsilon^G = 0.$$

In particular, $\mathcal{F}(G) = \dim(V \ominus \epsilon)^G$ in this case.

Proof. Because $\kappa \neq \text{id}$, $\gamma(\kappa) = -1$. Since κ acts trivially on $\mathbb{C}[R]$, both γ and $\gamma \otimes \mathbb{C}[R]$ have trivial G -invariants, as does $\epsilon = \gamma^{|R|-1} \det \mathbb{C}[R]$ when $|R|$ is even. \square

Lemma 4.7 (Case II). *If $\kappa = \text{id}$ and $\gamma = \mathbf{1}$, then*

$$\mathcal{F}(G) = r - 2 + \tilde{r}, \quad \dim(\gamma \otimes \mathbb{C}[R])^G = r, \quad \dim \gamma^G = 1 \quad \text{and} \quad \dim \epsilon^G = 1 - \tilde{r}.$$

In particular, $\mathcal{F}(G) = \dim(V \ominus \epsilon)^G$ in this case.

Proof. First of all, $G = \bar{G}$ since $\kappa = \text{id}$. Next, as $\gamma = \mathbf{1}$, R is a union of G -regular orbits by Lemma 4.5(1). Now,

$$\begin{aligned} \mathcal{F}(G) &= 2 \left[\frac{r|G|}{2|G|} - \frac{1}{2} \right] = 2 \left[\frac{r-1}{2} \right] = \begin{cases} r-2, & \text{if } 2 \nmid r \\ r-1, & \text{if } 2 \mid r \end{cases} = r - 2 + \tilde{r}, \\ \dim(\gamma \otimes \mathbb{C}[R])^G &= \dim \mathbb{C}[R]^G = |R/G| = r, \\ \dim \gamma^G &= 1. \end{aligned}$$

If $|R|$ is even, then

$$\epsilon = \det V = \gamma^{|R|-1} \det \mathbb{C}[R] = \det \mathbb{C}[R] = (\det \mathbb{C}[G])^r.$$

This is non-trivial if and only if r is odd³ (in which case $|G|$ is even as $|R|$ is even). So $\dim \epsilon^G = 1 - \tilde{r}$, as claimed.

On the other hand, if $|R|$ is odd, then $\epsilon = 0$. As R is a union of regular G -orbits, there must be an odd number of them, so that $\tilde{r} = 1$ and once again $\dim \epsilon^G = 1 - \tilde{r}$. \square

Lemma 4.8 (Case III). *Suppose $\kappa = \text{id}$ and $\gamma \neq \mathbf{1}$. Then*

- (1) $\mathcal{F}(G) = r - \tilde{r}$.
- (2) R is a union of regular G -orbits if and only if $|R|$ and m are both even.
- (3) We have

$$\dim(\gamma \otimes \mathbb{C}[R])^G = r, \quad \dim \gamma^G = 0 \quad \text{and} \quad \dim \epsilon^G = \tilde{r}.$$

In particular, $\mathcal{F}(G) = \dim(V \ominus \epsilon)^G$ in this case.

³For a group G , recall that $\det \mathbb{C}[G]$ is the trivial character unless G has a non-trivial cyclic 2-Sylow subgroup, in which case $\det \mathbb{C}[G]$ is of order 2.

Proof. First of all, $G = \tilde{G} = T \rtimes C_m$, since $\kappa = \text{id}$, and $m > 1$ as $\gamma \neq \mathbf{1}$. By Lemma 4.5(1), R decomposes as a G -set as

$$R = G^{\text{II}r} \quad \text{or} \quad R = G^{\text{II}r} \amalg G/C_m.$$

(1) By definition of \mathcal{F} ,

$$\mathcal{F}(G) = 2 \left\lfloor \frac{|R|}{2|G|} \right\rfloor = 2 \left\lfloor \frac{r|G| + \delta}{2|G|} \right\rfloor,$$

with $\delta = 0$ or $\delta = |G/C_m|$. As $\delta < |G|$, we have $\mathcal{F}(G) = 2 \lfloor \frac{r}{2} \rfloor = r - \tilde{r}$.

(2) If m is odd, then $\gamma \neq \mathbf{1} \Rightarrow \gamma^2 \neq \mathbf{1}$, so R has an irregular orbit by Lemma 4.5(1b). If m is even, then every regular orbit has even size while $|G/C_m| = |T|$ is odd, so the parity of $|R|$ is determined by whether there is an irregular orbit.

(3) Clearly $\dim \gamma^G = 0$. By Lemma 4.5,

$$\gamma \otimes \mathbb{C}[G] \simeq \mathbb{C}[G] \quad \text{and} \quad \gamma \otimes (\mathbb{C}[G/C_m] \ominus \mathbf{1}) = \mathbb{C}[G/C_m] \ominus \mathbf{1},$$

and it follows that either

$$\gamma \otimes \mathbb{C}[R] = \mathbb{C}[G]^r \quad \text{or} \quad \gamma \otimes \mathbb{C}[R] = \mathbb{C}[G]^r \oplus \mathbb{C}[G/C_m] \oplus \gamma \ominus \mathbf{1},$$

depending on whether R is a union of regular orbits or not. Each $\mathbb{C}[G]$ summand has 1-dimensional G -invariants, and their dimensions add up to r , while

$$\dim(\mathbb{C}[G/C_m])^G + \dim \gamma^G - \dim \mathbf{1}^G = 1 + 0 - 1 = 0.$$

This proves the first claim, and it remains to show that $\dim \epsilon^G = \tilde{r}$.

Suppose $|R|$ is even, so that $\epsilon = \det V$. If m is odd, then G has odd order while $\det V$ has rational character by Lemma 4.5(2), so $\epsilon = \det V = \mathbf{1}$. On the other hand, R has an irregular orbit by (2), and all orbits are of odd size, so r is odd. Hence $\tilde{r} = 1 = \dim \epsilon^G$.

If m is even, then R is a union of regular orbits by (2), and $\det \mathbb{C}[G] = \eta$, the non-trivial character of C_m of order 2. Moreover, $\gamma = \eta$ because $\gamma \neq \mathbf{1}$ but $\gamma^2 = \mathbf{1}$ by Lemma 4.5(1). Therefore

$$\begin{aligned} \epsilon = \det V &= \det(\gamma \otimes (\mathbb{C}[R] \ominus \mathbf{1})) = \gamma^{-1} \otimes \det(\gamma \otimes \mathbb{C}[G]^{\oplus r}) \\ &= \gamma^{-1} \otimes \det(\mathbb{C}[G]^{\oplus r}) = \eta^{r-1}, \end{aligned}$$

and so $\dim \epsilon^G = \tilde{r}$.

Finally suppose $|R|$ is odd, so that $\epsilon = 0$ and we need to show that r is even. By (2), R has an irregular orbit, so $\gamma^2 = \alpha$ by Lemma 4.5(1), which has order m . Hence m must be odd, as G has no 1-dimensional representation of order $2m$. Thus every G -orbit has odd size, and $r \equiv |R| - 1 \pmod{2}$ is even. \square

Lemma 4.9. *If R is even, then $\epsilon = \alpha^{|R|/2} \otimes \gamma^{-1}$.*

Proof. We need show that $\gamma^{|R|-1} \det(\mathbb{C}[R]) = \alpha^{|R|/2} \otimes \gamma^{-1}$. As R is even and $\gamma^2 = \alpha^{|R|}$ by assumption (cf. Thm. 4.2), this is equivalent to

$$\det(\mathbb{C}[R]) = \alpha^{(|R|-1)|R|/2}.$$

Both sides are rational characters (that is of order 1 or 2) of \bar{G} ; this is clear for $\det(\mathbb{C}[R])$, and follows from the fact that $|R| \equiv 0, 1 \pmod{m}$ for the right-hand side (Lemma 4.5), and m is the order α . Moreover, if $|R| \equiv 1 \pmod{m}$ then m is odd as R is even, so both characters are trivial. Therefore we may assume that $R = \mathbb{C}[\bar{G}]^{\oplus r}$, a union of r regular orbits (Lemma 4.5 again). If r is even, then the left-hand side is trivial, and so is the right-hand side, as $|R|/2$ is a multiple of m .

Finally, suppose r is odd and $|R| = rm|T|$, in particular m is even. Let $\eta = \alpha^{m/2}$ be the non-trivial character of order 2 of \bar{G} . Both $\det(\mathbb{C}[R])$ and $\alpha^{(|R|-1)|R|/2}$ are odd powers of η in this case (see footnote above), and the claim follows. \square

5. DESCENDING MORPHISMS

In this section we describe how certain morphisms descend to quotients of hyperelliptic curves. Our motivation comes from the arithmetic of hyperelliptic curves over finite and local fields, and the question of how the Frobenius automorphism acts on the quotient curve. See §6 for an example.

Let k be a field of characteristic $p > 2$, and let C/k be a hyperelliptic curve. Let $G \subset \text{Aut}_k C$ be an affine group of automorphisms, given by

$$g(X) = \alpha(g)X + \beta(g), \quad g(Y) = \gamma(g)Y \quad (g \in G)$$

as before, and C/G be the quotient curve given explicitly in Theorem 2.4. We say that a morphism $\Phi : C \rightarrow C$ *normalises* G if for every $g \in G$ there is $g' \in G$ for which $g\Phi = \Phi g'$.

Theorem 5.1. *Suppose $\text{char } k = p > 2$, G does not contain the hyperelliptic involution, and $\Phi : C \rightarrow C$ is a morphism of the form*

$$\Phi X = aX^q + b, \quad \Phi Y = dY^q$$

that normalises G , with q a power of p . Then Φ descends⁴ to a morphism $\Psi : C/G \rightarrow C/G$ given by

$$\begin{aligned} \Psi x &= a^{|G|} x^q + \prod_{g \in G} (\alpha(g)b + \beta(g)), \\ \Psi y &= \begin{cases} d y^q & \text{if } \gamma = \mathbf{1}, \\ a^{\lfloor m/2 \rfloor |G|/m} d y^q & \text{if } \gamma \neq \mathbf{1}, \end{cases} \end{aligned}$$

where m is the prime-to- p part of $|G|$, and x, y and the model for C/G are those of Theorem 2.4.

Proof. We may assume that k is algebraically closed. The quotient map $C \rightarrow C/G$ corresponds to a field inclusion

$$k(C/G) = k(x, y) \quad \hookrightarrow \quad k(X, Y) = k(C).$$

The morphism Φ preserves $k(C/G) = k(C)^G$ as it normalises G , so it descends to a morphism $\Psi : C/G \rightarrow C/G$. On the level of functions, Ψ is just

⁴that is $\pi\Phi = \Psi\pi$ where $\pi : C \rightarrow C/G$ is the quotient map

the restriction of Φ to $k(C/G)$. We now describe the action of Ψ on the generators x and y explicitly. Note that for every polynomial $h(X)$,

$$\Phi \cdot h(X) = h(aX^q + b) = h((\sqrt[q]{a}X + \sqrt[q]{b})^q) = (h^{(1/q)}(\sqrt[q]{a}X + \sqrt[q]{b}))^q, \quad (\dagger)$$

where $h^{(1/q)}(X)$ denotes the polynomial obtained from $h(X)$ by raising every coefficient to the power $1/q$.

Action on $x = I(X)$. Recall from Theorem 2.4 that $x = I(X) = \prod_{g \in G} g(X)$. Since it is G -invariant, so is

$$\Phi \cdot I(X) = (I^{(1/q)}(\sqrt[q]{a}X + \sqrt[q]{b}))^q.$$

This has a unique q th root, namely $I^{(1/q)}(\sqrt[q]{a}X + \sqrt[q]{b})$, which must therefore be G -invariant as well. As it has the same degree as $I(X)$, by Lemma 2.1(5),

$$I^{(1/q)}(\sqrt[q]{a}X + \sqrt[q]{b}) = uI(X) + v$$

for some $u, v \in k$. Comparing the leading and the constant coefficients, we see that $u = a^{|G|/q}$ and $v = I^{(1/q)}(\sqrt[q]{b})$. Thus

$$\Psi(x) = (ux + v)^q = a^{|G|}x^q + I(b).$$

Action on y . We have two cases:

Case $\gamma = 1$. Here $y = Y$, and so

$$\Psi(y) = d y^q.$$

Case $\gamma \neq 1$. In this case $y = (I_T(X) - \lambda)^{\lfloor m/2 \rfloor} Y$, where $I_T(X) - \lambda$ is the unique monic polynomial of degree $|G|/m$ that is G -invariant up to scalars (see Lemma 2.1 (7)). Because Φ normalises G ,

$$g \cdot \Phi \cdot (I_T(X) - \lambda) = \Phi \cdot g' \cdot (I_T(X) - \lambda) = \text{scalar} \cdot \Phi \cdot (I_T(X) - \lambda),$$

so $\Phi \cdot (I_T(X) - \lambda)$ is also G -invariant up to scalars. By (\dagger) , it is a q th power,

$$\Phi \cdot (I_T(X) - \lambda) = (I_T^{(1/q)}(\sqrt[q]{a}X + \sqrt[q]{b}) - \sqrt[q]{\lambda})^q,$$

and $I_T^{(1/q)}(\sqrt[q]{a}X + \sqrt[q]{b}) - \sqrt[q]{\lambda}$ must be G -invariant up to scalars as well. But it has the same degree as $I_T(X)$, so by uniqueness we must have

$$\Phi \cdot (I_T(X) - \lambda) = \text{constant} \cdot (I_T(X) - \lambda)^q.$$

Comparing the leading terms, we see that the constant is $a^{\deg I_T(X)} = a^{|G|/m}$. Hence

$$\Psi \cdot y = a^{\lfloor m/2 \rfloor |G|/m} (I_T(X) - \lambda)^{q \lfloor m/2 \rfloor} d Y^q = a^{\lfloor m/2 \rfloor |G|/m} d y^q.$$

□

6. AN EXAMPLE

To illustrate the results of this article, let us identify the local Galois representation attached to a specific hyperelliptic curve over a local field. This requires a compatibility statement for étale cohomology (for curves whose Jacobian has potentially good reduction this is (‡) below); it appears to be well-known, but seems not to have been phrased in terms of explicit actions on points ([3] §2.3 or [4] §2.10, paragraph ‘Naïvely, ...’). We will explain this compatibility in a forthcoming work [6]. The actual result (§6.2) can alternatively be obtained as in [7, Ex. 3.1], which bypasses (‡).

6.1. Setting. Let Z/\mathbb{Q}_3 be the hyperelliptic curve of genus 3 given by

$$Z : y^2 = x^8 + 3^4.$$

Let ζ be a primitive 8th root of 1 and let $\alpha = \sqrt{3}\zeta$ be a root of $x^8 + 3^4$, so that the other roots are $\zeta^i\alpha$ for $i = 1, \dots, 7$. Let $F = \mathbb{Q}_3(\zeta, \alpha)$ be the splitting field of $x^8 + 3^4$. It is a C_4 -extension of \mathbb{Q}_3 with ramification and residue degrees 2, so that in particular $\mathbb{Q}_3(\sqrt{3})^{\text{nr}} = F^{\text{nr}}$, the maximal unramified extension of F . Finally, let $\phi \in \text{Gal}(F^{\text{nr}}/\mathbb{Q}_3(\sqrt{3}))$ be the (arithmetic) Frobenius element and let $\tau \in \text{Gal}(F^{\text{nr}}/\mathbb{Q}_3^{\text{nr}})$ be the element of order 2. Thus ϕ gives a Frobenius element of $F^{\text{nr}}/\mathbb{Q}_3$ and τ generates its inertia group.

6.2. Result. We claim that the Galois action on $H^1(Z) = H_{\text{ét}}^1(Z_{\mathbb{Q}_3}, \mathbb{Q}_l) \otimes \mathbb{C}$ for $l \neq 3$ factors through $F^{\text{nr}}/\mathbb{Q}_3$ and that, with respect to a suitable basis,

$$\phi^{-1} \mapsto \begin{pmatrix} \sqrt{-3} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{-3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1+\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1-\sqrt{2} \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

In particular, the representation is tamely ramified with conductor exponent 4 and local polynomial $1 + 3T^2$, so that the Euler factor at $p = 3$ of the L -series of Z/\mathbb{Q} is $\frac{1}{1+3^{1-2s}}$.

6.3. Galois action on the semistable model. The curve Z acquires good reduction over F , since the substitution

$$s(x) = \alpha x, \quad s(y) = \alpha^4 y$$

clearly transforms it to the model $\mathcal{C}/O_F : y^2 = x^8 - 1$, which has the 8th roots of unity as roots of the right-hand side. We will write C for its special fibre

$$C : y^2 = x^8 - 1 \quad \text{over } k = \mathbb{F}_9.$$

The group

$$\text{Gal}(F^{\text{nr}}/\mathbb{Q}_3) = \langle \tau, \phi \rangle \simeq C_2 \times \hat{\mathbb{Z}}$$

acts naturally on $C(\bar{k})$ (see [6]). For an element $\sigma \in \text{Gal}(F^{\text{nr}}/\mathbb{Q}_3)$ this action is given by the composition

$$C(\bar{k}) \xrightarrow{\text{lift}} \mathcal{C}(O_{F^{\text{nr}}}) \xrightarrow{s^{-1}} Z(F^{\text{nr}}) \xrightarrow{\sigma} Z(F^{\text{nr}}) \xrightarrow{s} \mathcal{C}(O_{F^{\text{nr}}}) \xrightarrow{\text{reduce}} C(\bar{k}).$$

In our example,

$$\sigma : (x, y) \longmapsto \left(\frac{\alpha^\sigma}{\alpha} x^\sigma, \left(\frac{\alpha^\sigma}{\alpha} \right)^4 y^\sigma \right) \pmod{\mathfrak{m}},$$

where \mathfrak{m} is the maximal ideal of F^{nr} . Observe⁵ that $\alpha^\tau = -\alpha$ and $\alpha^\phi = \zeta\alpha$, so, in particular,

$$\tau : (x, y) \mapsto (-x, y) \quad \text{and} \quad \phi : (x, y) \mapsto (\bar{\zeta}^{-1}x^3, -y^3),$$

where $\bar{\zeta}$ denotes the image of ζ in \mathbb{F}_9 .

Define morphisms $g, \Phi : C \rightarrow C$ by the above formulae for τ, ϕ on points. By the Néron-Ogg-Shafarevich criterion, the natural Galois action on the étale cohomology group $H^1(Z)$ factors through $\text{Gal}(F^{\text{nr}}/\mathbb{Q}_3)$. By [6], there is an isomorphism of \mathbb{Q}_l -vector spaces

$$H^1(Z) \simeq H^1(C), \quad (\ddagger)$$

under which the action of τ and ϕ on $H^1(Z)$ translates to the natural geometric action of g and Φ on $H^1(C)$. We are now in a position to apply the results of §4 and §5 to explicitly determine the representation $H^1(Z)$.

6.4. $H^1(Z)$ as inertia representation. To determine the action of the inertia group on $H^1(Z)$ we apply Theorem 4.1 to the curve C with the automorphism group $G = \langle g \rangle \simeq C_2$ (with the action described above) and $\tilde{\gamma} = \mathbf{1}$. Write η for the non-trivial 1-dimensional representation of G . The roots of $x^8 - 1$ come in four regular G -orbits,

$$\{1, -1\}, \quad \{\bar{\zeta}, -\bar{\zeta}\}, \quad \{\bar{\zeta}^2, -\bar{\zeta}^2\}, \quad \{\bar{\zeta}^3, -\bar{\zeta}^3\},$$

so the theorem shows that, as a G -module,

$$H^1(Z) \simeq H^1(C) \simeq \mathbb{C}[G]^{\oplus 4} \ominus \mathbf{1} \ominus \mathbf{1} \simeq \mathbf{1}^{\oplus 2} \oplus \eta^{\oplus 4}.$$

In other words g has eigenvalues 1 and -1 with multiplicities 2 and 4, respectively, in its action on $H^1(Z)$, as claimed.

6.5. Counting fixed points. To describe $H^1(Z)$ as a full $\text{Gal}(\bar{\mathbb{Q}}_3/\mathbb{Q}_3)$ -module, we will exploit the identifications

$$H^1(Z) \simeq H^1(C) \quad \text{and} \quad H^1(Z)^{C_2} \simeq H^1(C)^{C_2} \simeq H^1(C/C_2).$$

To be precise, first note that as Φ commutes with g , it preserves the $\mathbf{1}$ - and the η -isotypical components of $C_2 = \langle g \rangle$, and that $H^1(C)$ is completely determined by the eigenvalues of Φ on them. (The action of Φ is known to be semisimple, although in our case this will be clear as its eigenvalues will turn out to be distinct.) By (\ddagger) , the eigenvalues of ϕ on the inertia invariants $H^1(Z)^{\langle \tau \rangle}$ agree with those of Φ on $H^1(C)^{C_2}$. These are, by Theorem 3.1 (2,3), the eigenvalues of Ψ on $H^1(C/C_2)$, where Ψ is the induced morphism on C/C_2 .

⁵Writing ζ_{16} for the 16th root of unity $= \frac{\alpha}{\sqrt[3]{3}}$, we clearly have $\alpha^\tau = -\sqrt[3]{3}\zeta_{16} = -\alpha$ and $\alpha^\phi = \zeta_{16}^3\sqrt[3]{3} = \zeta\alpha$, from the definitions of τ and ϕ .

By Theorem 2.4, the quotient C/C_2 is the genus 1 curve

$$C/C_2 : y^2 = (x+1)(x+\bar{\zeta}^2)(x+\bar{\zeta}^4)(x+\bar{\zeta}^6) = x^4 - 1$$

and, by Theorem 5.1, Φ descends to

$$\begin{aligned} \Psi : C/C_2 &\longrightarrow C/C_2 \\ (x, y) &\longmapsto (-\bar{\zeta}^2 x^3, -y^3). \end{aligned}$$

From the Lefschetz fixed point formula, the inverse characteristic polynomial of Ψ on $H^1(C/C_2)$ is

$$\det(1 - \Psi^{-1}T \mid H^1(C/C_2)) = 1 - aT + 3T^2$$

for some $a \in \mathbb{Z}$, and its value at $T = 1$ is the number of fixed points of Ψ on the curve. To find it explicitly, first count $\bar{\mathbb{F}}_3$ -solutions to the system

$$y^2 = x^4 - 1, \quad x = -\bar{\zeta}^2 x^3, \quad y = -y^3.$$

Starting from the last equation,

$$\begin{aligned} y = 0 &\implies x^4 = 1, x = -\bar{\zeta}^2 x^3 &\implies &\text{no solutions;} \\ y = \pm \bar{\zeta}^2 &\implies x^4 = \bar{\zeta}^4 + 1 = 0, x = -\bar{\zeta}^2 x^3 &\implies &x = 0, y = \pm \bar{\zeta}^2. \end{aligned}$$

Finally, to see the action on the points at infinity ∞_{\pm} , let $s = \frac{1}{x}, t = \frac{y}{x^2}$. The equation of the curve becomes

$$t^2 = 1 - s^4, \quad \infty_{\pm} = (0, \pm 1),$$

and the transformation $\Psi : (x, y) \mapsto (-\bar{\zeta}^2 x^3, -y^3)$ on this chart is

$$s = \frac{1}{x} \mapsto \frac{-1}{\bar{\zeta}^2 x^3} = \bar{\zeta}^2 s^3, \quad t = \frac{y}{x^2} \mapsto \frac{-y^3}{(-\bar{\zeta}^2 x^3)^2} = \frac{y^3}{x^6} = t^3.$$

It fixes both ∞_+ and ∞_- . Overall, Ψ has 4 fixed points, and its inverse characteristic polynomial on $H^1(C/C_2)$ is therefore $1 + 3T^2$. Hence the eigenvalues of Φ on this subspace are $\pm \frac{1}{\sqrt{-3}}$, as claimed.

Similarly, counting $a_i =$ the number of fixed points of Φ^i on C itself, we find the sequence to be $(4, 20, 28, 92, 244, 692, \dots)$. Thus, by the Lefschetz fixed point formula again, the inverse characteristic polynomial of Φ on the full space $H^1(C)$ is

$$\exp\left(\sum_{i \geq 1} \frac{a_i}{i} T^i\right)(1 - T)(1 - 3T) = (1 + 3T^2)(1 + 2T^2 + 9T^4).$$

The first factor lives, as we have seen, on the $\mathbf{1}$ -component of $C_2 = \langle g \rangle$, and so the second factor lives on the η -component. In other words, the eigenvalues of Φ on the -1 -eigenspace of g on $H^1(C)$ are $\pm \frac{1}{1 \pm \sqrt{2}}$, as claimed.

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