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STABILITY AND PERTURBATIONS OF COUNTABLE MARKOV MAPS

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Dedicated to the memory of Bernd O. Stratmann

ABSTRACT. Let T and T_ε , $\varepsilon > 0$, be countable Markov maps such that the branches of T_ε converge pointwise to the branches of T , as $\varepsilon \rightarrow 0$. We study the stability of various quantities measuring the singularity (dimension, Hölder exponent etc.) of the topological conjugacy θ_ε between T_ε and T when $\varepsilon \rightarrow 0$. This is a well-understood problem for maps with finitely-many branches, and the quantities are stable for small ε , that is, they converge to their expected values if $\varepsilon \rightarrow 0$. For the infinite branch case their stability might be expected to fail, but we prove that even in the infinite branch case the quantity $\dim_{\mathbb{H}}\{x : \theta'_\varepsilon(x) \neq 0\}$ is stable under some natural regularity assumptions on T_ε and T (under which, for instance, the Hölder exponent of θ_ε fails to be stable). Our assumptions apply for example in the case of Gauss map, various Lüroth maps and accelerated Manneville-Pomeau maps $x \mapsto x + x^{1+\alpha} \pmod 1$ when varying the parameter α . For the proof we introduce a mass transportation method from the cusp that allows us to exploit thermodynamical ideas from the finite branch case.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $T, S : [0, 1] \rightarrow [0, 1]$ be expanding interval maps with equally many branches, and let $\theta : [0, 1] \rightarrow [0, 1]$ be the topological conjugacy between T and S , that is, $\theta \circ T = S \circ \theta$ and θ is a homeomorphism. These topological conjugacies are often *singular functions* in the sense that the derivative of θ is equal to zero almost everywhere. We are interested in various notions of singularity (dimension of the non-zero derivative set and Hölder exponent amongst others) for the map θ . In particular we investigate what happens to these notions if S is a “perturbation” of T , that is, when T and S are close to each other in a suitable sense.

There are plenty of examples of topological conjugacies θ to be found in the literature. The most classical example is Minkowski’s question-mark function $? : [0, 1] \rightarrow [0, 1]$, which is a topological conjugacy between the Farey map and the tent map (or the Gauss map and alternating Lüroth map), whose study goes back to Denjoy [7] and Salem [28] and more recently to papers of Kesseböhmer and Stratmann [16, 14]. Other works include topological conjugacies between interval maps with affine branches [2, 17, 13, 1, 25], and uniformly expanding maps with finitely many branches by Darst [6], Li, Xiao and Dekking [19], Falconer and Troscheit [10, 32], and the papers of Jordan, Kesseböhmer, Pollicott and Stratmann [12, 16].

We concentrate on conjugacies between interval maps which have an *infinite* number of branches. These maps are known as *countable Markov maps* and they appear in Diophantine approximation, where the key examples are the *Gauss map*, $x \mapsto 1/x \pmod 1$, which generates the continued fraction

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expansion [5, 16], and the various *Lüroth maps*, which generate Lüroth expansions [2, 17, 13]. Moreover, countable Markov maps appear naturally as jump transformations, or “accelerated dynamics”, in the study of non-uniformly hyperbolic dynamical systems such as the intermittent *Manneville-Pomeau maps* [22].

To state our results, let us first fix a little notation (we refer to Section 2 for a more thorough exposition). Let $f_i : [0, 1] \rightarrow [0, 1]$ be C^1 contractions for each $i \in \mathbb{N}$ and where either $f_1(0) = 1$, $f_{i+1}(0) = f_i(1)$ for all $i \in \mathbb{N}$ and $(f_i(0))_{i \in \mathbb{N}}$ is a decreasing sequence with $\lim_{i \rightarrow \infty} f_i(0) = 0$ or we have that $f_1(1) = 1$, $f_{i+1}(1) = f_i(0)$ for all $i \in \mathbb{N}$ and $(f_i(1))_{i \in \mathbb{N}}$ is a decreasing sequence. These maps are the inverse branches of a piecewise differentiable countable Markov map T . In the study of the dynamics of countable Markov maps, certain regularity conditions are often imposed; a typical condition is that the geometric potential $-\log |T'|$ is locally Hölder (there exist $C > 0$ and $0 < \gamma < 1$ such that $\text{var}_n(-\log |T'|) \leq C\gamma^n$) which is helpful for using results from the thermodynamic formalism and proving distortion estimates since it clearly implies $-\log |T'|$ has *summable variations*, that is,

$$\sum_{n=1}^{\infty} \text{var}_n(-\log |T'|) < \infty.$$

This condition is satisfied, for example, for the Gauss map, jump transformations of Manneville-Pomeau maps, and for all α -Lüroth maps.

We will fix such a system $\{T, (f_i)_{i \in \mathbb{N}}\}$ and consider perturbations of the system, in the following sense: For each $k \in \mathbb{N}$ we will consider a system with maps $f_{i,k}$ and T_k satisfying the variation assumption above and where for each $x \in [0, 1]$ we have

$$\lim_{k \rightarrow \infty} f_{i,k}(x) = f_i(x).$$

We need that each of the maps $f_{i,k}$ have the same orientation as the map f_i , for all i . This means the dynamical systems T_k and T are topologically conjugate and we will denote the conjugacy by θ_k . The pointwise convergence of the inverse branches guarantees that as k increases, the conjugacy θ_k converges pointwise to the identity map; an example is shown in Figure 1.

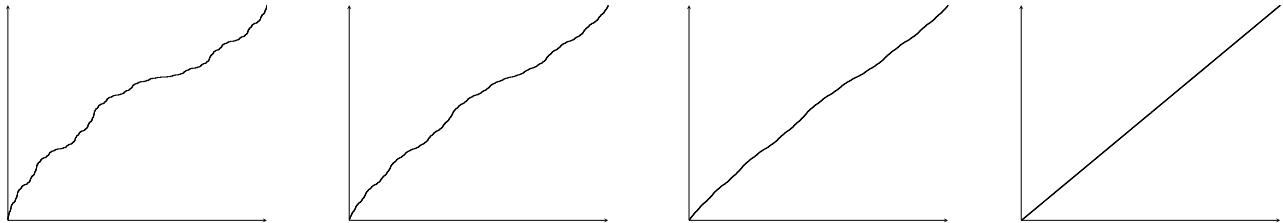


FIGURE 1. The first three graphs show conjugacies θ_k between two countable Markov maps T_k and T , and the last is the identity. The map T is the α_D -Lüroth map for the dyadic partition $\alpha_D = \{[2^{-i}, 2^{-i+1}) : i \in \mathbb{N}\}$ and T_k is the α -Lüroth map for a λ -adic partition $\{[\lambda^{-i}, \lambda^{-i+1}) : i \in \mathbb{N}\}$ for λ with the values 3, 2.5 and 2.1 from left to right respectively (see [25] for the definition of α -Lüroth maps). The maps θ_k approach the identity pointwise when $f_{i,k} \rightarrow f_i$ pointwise.

Let us now study various notions of singularity of θ_k and how they behave as $k \rightarrow \infty$. We will study the following three natural quantities (see Section 2 for definitions):

- (A) Hausdorff dimension of the singularity set: $\dim_{\mathbb{H}}\{x : \theta'_k(x) \neq 0\}$,

- (B) Hölder exponent $\kappa(\theta_k)$ of θ_k , and
- (C) Hausdorff dimension $\dim_{\text{H}}(\mu_T \circ \theta_k)$ of the conjugated measure $\mu_T \circ \theta_k$, where μ_T is the absolutely continuous T -invariant measure for the map T .

These quantities have been studied for interval maps and usually the typical way to study them, in particular property (A), is through thermodynamical formalism. If we have an interval map with finitely-many branches, then under suitable regularity assumptions for the maps T_k and T , where suitable means that they allow the use of thermodynamical tools, these quantities behave continuously as $k \rightarrow \infty$ in the sense that they converge to the values for the identity map:

$$\lim_{k \rightarrow \infty} \dim_{\text{H}}\{x : \theta'_k(x) \neq 0\} = 1, \quad \lim_{k \rightarrow \infty} \kappa(\theta_k) = 1, \quad \text{and} \quad \lim_{k \rightarrow \infty} \dim_{\text{H}}(\mu_T \circ \theta_k) = 1.$$

The key to all these results holding is the convergence of the Lyapunov exponents. In particular, let Σ be the finite shift, Π, Π_k be the natural projections from the shift to $[0, 1]$ corresponding to the maps T_k and T , and define potentials $\varphi_k, \varphi : \Sigma \rightarrow \mathbb{R}$ by

$$\varphi_k(i) = \log |T'_k(\Pi(i))| \quad \text{and} \quad \varphi(i) = \log |T'(\Pi(i))|.$$

Then we will have, by compactness and control of the variations of the functions, that the integral of φ_k converges to the integral of φ . From this we can deduce the above limits. Indeed, since the Hölder exponent is the minimum of the ratio of the two functions, the dimension is entropy (which is fixed) divided by the integral of φ_k (which converges) and the result on the conjugacy can then be deduced using the convergence of Lyapunov exponents or the results from [12]. Note that for $\mu_T \circ \theta_k$ -almost all x , π_k will not have zero derivative, because for μ_T -almost all x , π_k^{-1} will have finite derivative by the absolute continuity of μ_T .

In the infinite branch case we are considering, we have that the quantity (A), the Hausdorff dimension of the singularity set, is still stable, but the quantities given in (B) and (C) fail to be stable as we may not have uniform convergence:

Theorem 1.1. *Suppose T is a countable Markov map with inverse branches f_i such that the potential $-\log |T'|$ is locally Hölder. Let (T_k) be a sequence of countable Markov maps with inverse branches $f_{i,k}$. Assume the following two conditions on the tail and variations:*

- (1) *There exists $0 < t < 1$ with*

$$\sum_{i=1}^{\infty} |f_i[0, 1]|^t < \infty.$$

- (2) *The potentials $-\log |T'_k|$ are locally Hölder with a uniform bound over $k \in \mathbb{N}$ on the sum of the variations:*

$$\sup_{k \in \mathbb{N}} \sum_{n=1}^{\infty} \text{var}_n(-\log |T'_k|) < \infty.$$

If $f_{i,k}(x) \rightarrow f(x)$ as $k \rightarrow \infty$ for any $i \in \mathbb{N}$ and $x \in [0, 1]$, we have

$$\lim_{k \rightarrow \infty} \dim_{\text{H}}\{x : \theta'_k(x) \neq 0\} = 1.$$

Moreover, there exist examples of T_k and T satisfying the assumptions above such that

- (i) *for the Hölder exponents $\kappa(\theta_k)$ of θ_k we have*

$$\lim_{k \rightarrow \infty} \kappa(\theta_k) = 0;$$

(ii) for the Hausdorff dimensions of the conjugated measures $\mu_T \circ \theta_k$ we have

$$\lim_{k \rightarrow \infty} \dim_{\text{H}}(\mu_T \circ \theta_k) = 0.$$

The main reason we observe such a behaviour is that the properties (B) and (C) on the Hölder exponent and Hausdorff dimensions of the conjugated measures are very sensitive to the tail behaviour. Indeed, the constructions for (B) and (C) are based on having very different tail behaviours between T_k and T . On the other hand, property (A) is not too sensitive to any differences between the tails of T_k and T . Indeed, we will see in the proof that we can do a type of “mass transportation” from the cusp for which any difference between the tail behaviours of T_k and T does not matter for the value of the Hausdorff dimension of the singular set; see Section 4.2 for more details.

Condition (1) holds if the countable Markov map T has at most a polynomially fat tail, in the sense that the lengths $|f_i[0, 1]| = O(i^{-p})$ as $i \rightarrow \infty$ for some $p > 1$ (for example the Gauss map $x \mapsto 1/x \pmod{1}$ has this property). Thus (1) yields in particular that the absolutely continuous invariant measure for T has finite entropy, but it is not an equivalent condition. Condition (2) is satisfied if the inverse branches of T_k are linear, i.e., when the maps T_k are α -Lüroth maps for certain partitions α in the notation of [17]. Thus our result gives rather general conditions to have such a perturbation theorem for α -Lüroth maps, provided that the map being perturbed has a thin enough tail.

In the non-linear case, the Gauss map satisfies the condition (1), so the perturbation theorem is valid provided we have a uniform bound (2) over the sums of variations on the family of maps converging to the Gauss map. Furthermore, the conditions in Theorem 1.1 are weak enough for us to apply Theorem 1.1 to the study of a certain family of intermittent maps in non-uniformly hyperbolic dynamics known as the *Manneville-Pomeau maps* $M_\alpha : [0, 1] \rightarrow [0, 1]$,

$$M_\alpha(x) := x + x^{1+\alpha} \pmod{1}, \quad x \in [0, 1],$$

for a parameter $0 < \alpha < \infty$. The jump transformations for M_α give us countable Markov maps that have polynomial tails and satisfy the assumptions of Theorem 1.1 when varying the parameter α for the maps M_α , since this means pointwise convergence of the inverse branches. Thus we obtain the following corollary to Theorem 1.1:

Corollary 1.2. *Let $\alpha > 0$. Then as $\beta \rightarrow \alpha$ we have*

$$\dim_{\text{H}}\{x : \theta'_{M_\beta, M_\alpha}(x) \neq 0\} \rightarrow 1,$$

where $\theta_{M_\beta, M_\alpha}$ is the topological conjugacy between the Manneville-Pomeau maps M_β and M_α .

Corollary 1.2 concerns the topological stability for M_α when varying α . A related area of study for Manneville-Pomeau maps is the measure theoretical *statistical stability*, where the behaviour of the absolutely continuous invariant measure for M_α is studied when varying α , see for example the recent works by Freitas and Todd [11] and Baladi and Todd [3].

1.1. Organisation of the paper. The paper is organised as follows. In Sections 2 and 3 we will give all the necessary background results from dimension theory and thermodynamic formalism. In Section 4 we will give the proof of the first part of Theorem 1.1 for the stability of the Hausdorff dimension of the singularity set. In Section 5 we complete the proof of Theorem 1.1 by constructing examples that show instability for Hölder exponents and Hausdorff dimension of the θ_k pullback measures. In Section 6 we discuss the Manneville-Pomeau example further and prove Corollary 1.2.

2. PRELIMINARIES AND NOTATION

2.1. Interval maps and modeling with a countable shift. A *countable Markov map* $T : [0, 1] \rightarrow [0, 1]$ is defined with the help of its inverse branches. We consider the situation where for each $i \in \mathbb{N}$, there exist maps $f_i : [0, 1] \rightarrow [0, 1]$ which are continuous and strictly decreasing on $[0, 1]$ and differentiable on $(0, 1)$. We further assume that there exists $m \in \mathbb{N}$ and $\xi < 1$ such that for all $(i_1, \dots, i_m) \in \mathbb{N}^m$ we have that $|(f_{i_1} \circ \dots \circ f_{i_m})'(x)| \leq \xi$ for all $x \in (0, 1)$. We will also suppose that $f_1(0) = 1$, $f_i(1) = f_{i+1}(0)$ for all $i \in \mathbb{N}$ and $\lim_{i \rightarrow \infty} f_i(0) = 0$ or alternatively that $f_1(0) = 0$, $f_i(0) = f_{i+1}(1)$ for all $i \in \mathbb{N}$ and $\lim_{i \rightarrow \infty} f_i(0) = 0$. Thus $\bigcup_{i=1}^{\infty} f_i([0, 1]) = (0, 1]$ and if $i \neq j$ then $f_i((0, 1)) \cap f_j((0, 1)) = \emptyset$. We define an expanding map $T : [0, 1] \rightarrow [0, 1]$ by setting

$$T(x) := \begin{cases} f_i^{-1}(x), & \text{if } x \in f_i((0, 1)); \\ 0, & \text{if } x = 0. \end{cases}$$

Given a countable Markov map T with inverse branches f_i , $i \in \mathbb{N}$, it is convenient to model our systems using symbolic dynamics. Let $\Sigma := \mathbb{N}^{\mathbb{N}}$ and let $\sigma : \Sigma \rightarrow \Sigma$ be the usual left-shift transformation. We can relate this to our systems $\{f_i\}, T$ via projections $\pi_T : \Sigma \rightarrow [0, 1]$. We define

$$\pi_T(i_1, i_2, \dots) := \lim_{n \rightarrow \infty} f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n}(0).$$

The factor map π_T allows us to import the thermodynamical formalism from the shift space to measures invariant under T . For a shift invariant measure μ , the push-forward measure $\pi_T \mu := \mu \circ \pi_T^{-1}$ will be T -invariant. Moreover if μ is ergodic for the shift map then $\pi_T \mu$ will be ergodic for T . Thus we can use the symbolic model (Σ, σ) and the geometric model $([0, 1], T)$ interchangeably.

Now if we have a sequence of countable Markov maps T_k with inverse branches $\{f_{i,k}\}$ satisfying the assumptions of Theorem 1.1, we will shorten the notation by letting $\pi_k := \pi_{T_k}$ and $\pi := \pi_T$. Then the topological conjugacy θ_k between T_k and T will satisfy

$$\theta_k(x) = \pi \circ \pi_k^{-1}(x), \quad x \in [0, 1].$$

In other words, the conjugacy map between the systems T and T_k takes the point x with coding given by T and sends it to the point with the same coding, but now understood in terms of T_k .

2.2. Dimension and Hölder/Lyapunov exponents. Let $\dim_{\mathbb{H}} A$ be the Hausdorff dimension of a set $A \subset \mathbb{R}$ and the s -dimensional Hausdorff measures \mathcal{H}^s and the δ -Hausdorff content \mathcal{H}_{δ}^s , see [9] for a definition. For a Radon measure ν on \mathbb{R} , the *Hausdorff dimension of ν* is defined to be

$$\dim_{\mathbb{H}} \nu := \inf\{\dim_{\mathbb{H}} A : \nu(A) > 0\} = \operatorname{ess\,inf}_{x \sim \nu} \underline{\dim}_{\text{loc}}(\nu, x),$$

where $\underline{\dim}_{\text{loc}}(\nu, x)$ is the *lower local dimension* of ν at x , which is defined by

$$\underline{\dim}_{\text{loc}}(\nu, x) := \liminf_{r \searrow 0} \frac{\log \nu(B(x, r))}{\log r}.$$

Definition 2.1 (Hölder exponent). If $\theta : [0, 1] \rightarrow [0, 1]$ is a function, then the *Hölder exponent* $\kappa(\theta)$ of θ is defined to be the infimal $\kappa \geq 0$ such that for some $C > 0$ the following inequality holds:

$$|\theta(x) - \theta(y)| \leq C|x - y|^{\kappa}, \quad x, y \in [0, 1].$$

Now we will consider a fixed measure μ on $[0, 1]$ and countable Markov map T and we will define the notions of Lyapunov exponents and entropy for this measure. Note that the Lyapunov exponent depends upon the mapping T as well as the measure μ .

Definition 2.2 (Lyapunov exponent). The *Lyapunov exponent* of the measure μ is defined to be

$$\lambda(\mu, T) := \int \log |T'| \, d\mu.$$

Similarly, if $I_{\mathbf{i}}^T = \pi_T[\mathbf{i}]$, for $\mathbf{i} \in \mathbb{N}^*$, are the construction intervals generated by the countable Markov map T , the entropy of μ is defined as follows:

Definition 2.3 (Entropy). The *Kolmogorov-Sinai entropy* (with respect to T) of the measure μ is defined to be

$$h(\mu, T) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\mathbf{i} \in \mathbb{N}^n} -\mu(I_{\mathbf{i}}^T) \log \mu(I_{\mathbf{i}}^T).$$

Note that sometimes we also write $h(\mu, T)$ or $\lambda(\mu, T)$ for a measure μ living on Σ and then we just mean the values $h(\pi_T \mu, T)$ and $\lambda(\pi_T \mu, T)$ respectively for the projected measure $\pi_T \mu$. If we want the entropy of such μ with respect to the shift map σ on Σ , we define $h(\mu, \sigma)$ like $h(\mu, T)$ but we replace the intervals $I_{\mathbf{i}}^T$ by the cylinders $[\mathbf{i}]$.

Now, given a countable Markov map T , the Hausdorff dimensions of each of the π_T -projections of an ergodic shift-invariant measure can be computed using the following result:

Proposition 2.4 (Mauldin-Urbański). *If μ is an ergodic T invariant probability measure on $[0, 1]$ and $h(\mu, T) < \infty$, then the Hausdorff dimension of μ is given by*

$$\dim_{\text{H}} \mu = \frac{h(\mu, T)}{\lambda(\mu, T)}.$$

The above result can be found as Theorem 4.4.2 in the book [23] by Mauldin and Urbański.

3. THERMODYNAMICAL FORMALISM FOR THE COUNTABLE MARKOV SHIFT

In this section we present the tools we will need from thermodynamical formalism. We mostly concentrate on the countable Markov shift Σ as this is where we will reformulate the problem, using the theory developed in a much more general setting in D. Mauldin and M. Urbański [23] and the series of works by O. Sarig, see for example [29, 31].

First, recall that a potential φ is said to be *locally Hölder* if there exist constants $C > 0$ and $\delta \in (0, 1)$ such that for all $n \in \mathbb{N}$ the *variations* var_n decay exponentially:

$$\text{var}_n(\varphi) := \sup_{\mathbf{i} \in \mathbb{N}^n} \{|\varphi(\mathbf{j}) - \varphi(\mathbf{k})| : \mathbf{j}, \mathbf{k} \in [\mathbf{i}]\} \leq C\delta^n.$$

Note that since nothing is assumed in the case that $n = 0$, this does *not* imply that φ is bounded.

The *Birkhoff sum* $S_n \varphi$ of a potential $\varphi : \Sigma \rightarrow \mathbb{R}$ is the potential defined by

$$S_n \varphi(\mathbf{i}) := \sum_{k=0}^{n-1} \varphi(\sigma^k(\mathbf{i})).$$

The *pressure* of a locally Hölder potential φ is then the limit

$$P(\varphi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\mathbf{i} \in \mathbb{N}^n} \exp(S_n \varphi(\mathbf{i}^\infty)) \right),$$

where $\mathbf{i}^\infty = \mathbf{iii} \dots$ is the periodic word repeating the word $\mathbf{i} \in \mathbb{N}^n$. Define \mathcal{M}_σ to be the collection of all σ -invariant measures on Σ . A deep and useful result which we will now state is the *variational principle*, which gives a representation of $P(\varphi)$ using the Kolmogorov-Sinai entropy:

Lemma 3.1 (Variational principle). *For any locally Hölder potential φ we have that*

$$P(\varphi) = \sup_{\mu \in \mathcal{M}_\sigma} \left\{ h(\mu, \sigma) + \int \varphi d\mu : \int \varphi d\mu > -\infty \right\}.$$

For a proof, see Theorem 2.1.8 in [23]. If there exists a measure $\mu \in \mathcal{M}_\sigma$ which attains the supremum in Lemma 3.1, then we call μ an *equilibrium state* for a potential φ . In the case of finite pressure more can be said about equilibrium states.

Definition 3.2 (Gibbs measures). Let $\varphi : \Sigma \rightarrow \mathbb{R}$ be a locally Hölder potential. If $P(\varphi)$ is finite, then we call μ_φ a *Gibbs measure* for φ if there exists a constant $C > 0$ such that

$$C^{-1} \exp(S_n \varphi(\mathbf{j}) - nP(\varphi)) \leq \mu_\varphi[\mathbf{i}] \leq C \exp(S_n \varphi(\mathbf{j}) - nP(\varphi))$$

for any $\mathbf{i} \in \mathbb{N}^n$, $\mathbf{j} \in [\mathbf{i}]$ and $n \in \mathbb{N}$.

An example of such a measure is the *Bernoulli measure* μ associated to weights $p_i \in [0, 1]$, $i \in \mathbb{N}$, with $\sum_{i=1}^{\infty} p_i = 1$, which is the equilibrium state for the potential $\varphi(\mathbf{i}) = -\log p_{i_1}$. Then $P(\varphi) = 0$ and

$$\mu[\mathbf{i}] = p_{i_1} \dots p_{i_n} = \exp(S_n \varphi(\mathbf{j})), \quad \text{for } \mathbf{j} \in [\mathbf{i}].$$

The following proposition relates Gibbs measures to equilibrium states.

Proposition 3.3. *Let $\varphi : \Sigma \rightarrow \mathbb{R}$ be a locally Hölder potential. If $P(\varphi) < \infty$ then there exists a unique invariant probability measure, μ_φ which is a Gibbs measure for φ . Moreover, if φ is integrable with respect to μ_φ then μ_φ is the unique equilibrium state for φ .*

For a proof of this result, see Proposition 2.1.9, Theorem 2.2.9 and Corollary 2.7.5 in [23]. The case when φ is not integrable with respect to μ_φ is the subject of the next lemma.

Lemma 3.4. *Let $\varphi : \Sigma \rightarrow \mathbb{R}$ be a locally Hölder potential with $P(\varphi) < \infty$. If φ is not μ_φ integrable, then there exist no equilibrium states for φ .*

Proof. It is a result of Sarig [29, Theorem 7] that the only possible equilibrium state is a fixed point for the Ruelle operator (see [29] for a definition). It is then shown in the proof of [31, Theorem 1] that in the situation where the system satisfies the Big Image Property (see Sarig's paper for the definition; note that it includes the full shift) such measures are Gibbs measures. Thus there cannot exist equilibrium states for φ . \square

All the above thermodynamic definitions can be formulated also for the finite alphabet $\{1, 2, \dots, N\}$, $N \in \mathbb{N}$ and it makes things considerably simpler. For instance, in the finite alphabet case it is known that unique equilibrium states always exist for Hölder potentials and they are Gibbs measures. This makes it convenient to restrict to the finite case and consider approximations for the pressure. Given a locally Hölder potential $\varphi : \Sigma \rightarrow \mathbb{R}$, we write $P_N(\varphi)$ to denote the pressure of φ restricted to the finite shift $\Sigma_N := \{1, 2, \dots, N\}^{\mathbb{N}}$. Then we have the following approximation result, which can be found as Theorem 2.1.5 in [23].

Theorem 3.5 (Finite approximation property). *For any locally Hölder potential φ ,*

$$P(\varphi) = \lim_{N \rightarrow \infty} P_N(\varphi).$$

This theorem will allow us to use results which hold on the full shift with a finite alphabet (or, more generally, on topologically mixing subshifts of finite type). These results can sometimes be extended to the infinite case, but due to the hypotheses needed it is more convenient to use Theorem 3.5 and the results in the finite alphabet case. The first of these results that we will need is the following lemma on the derivative of pressure, which is Proposition 4.10 in [27].

Lemma 3.6 (Derivative of pressure). *Let $\varphi, \psi : \Sigma_N \rightarrow \mathbb{R}$ be Hölder continuous functions and define the analytic function*

$$Z_N(q) := P(q\psi + \varphi).$$

Let μ_q be the Gibbs measure on Σ_N for the potential $q\psi + \varphi$. Then the derivative of Z_N is given by

$$Z'_N(q) = \int \psi \, d\mu_q.$$

Gibbs measures satisfy many statistical theorems similar to ones in probability theory. We will use one of these, namely, the *law of the iterated logarithm*. Before stating this theorem, we recall that a function $\psi : \Sigma_N \rightarrow \mathbb{R}$ is said to be *cohomologous to a constant* if there exists a constant $c \geq 0$ and a continuous function $u : \Sigma_N \rightarrow \mathbb{R}$ such that

$$\psi - c = u - u \circ \sigma.$$

Moreover, ψ is called a *coboundary* if the constant c is equal to 0.

Lemma 3.7 (Law of the iterated logarithm). *Let $\varphi, \psi : \Sigma_N \rightarrow \mathbb{R}$ be Hölder potentials where ψ is not cohomologous to a constant. Then there exists $c(\psi) > 0$ such that for μ_φ -almost every x , we have*

$$\limsup_{n \rightarrow \infty} \frac{S_n \psi(x) - n \int \psi \, d\mu_\varphi}{\sqrt{n \log \log n}} = c(\psi).$$

Proof. This is Corollary 2 in [8]. Note that

$$c(\psi) = \lim_{n \rightarrow \infty} \frac{1}{n} \int (S_n \psi - \int \psi \, d\mu_\varphi)^2 \, d\mu_\varphi$$

and it is shown in Proposition 4.12 of [27] that $c(\psi) \geq 0$, with equality if and only if ψ is cohomologous to a constant. The number $c(\psi)$ is the variance of ψ with respect to μ_φ and is also the second derivative of the pressure function $q \rightarrow P(q\varphi + \psi)$ at $q = 0$. \square

Finally in this section we need the following result in the countable case regarding the behaviour of equilibrium states.

Lemma 3.8. *Let $\varphi : \Sigma \rightarrow (-\infty, 0]$ be locally Hölder such that $P(\varphi) = 0$, and let*

$$s = \sup\{t : P(t\varphi) = \infty\} < \infty.$$

We have that

(1) *there exists a sequence μ_n of compactly supported σ -invariant ergodic measures such that*

$$\lim_{n \rightarrow \infty} h(\mu_n, \sigma) = \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{h(\mu_n, \sigma)}{-\int \varphi d\mu_n} \geq s,$$

(2) *for any $t > s$ there exists $K(t) > 0$ such that if μ is ergodic, φ is integrable with respect to μ and $h(\mu, \sigma) > K(t)$, then*

$$h(\mu, \sigma) + t \int \varphi d\mu < 0.$$

Proof. Let $\epsilon > 0$. We can always find $t \geq \max\{0, s - \epsilon\}$ such that $P(t\varphi) = \infty$. Therefore we can find $N \in \mathbb{N}$ such that

$$P_N(t\varphi) \geq \max\{P((s + \epsilon)\varphi) + 2, 0\} \geq P_N((s + \epsilon)\varphi) + 2.$$

Let $z : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $z(r) = P_N(r\varphi)$ and observe that $z(t) \geq 0$. Since $z(t) \geq z(s + \epsilon) + 2$ and $0 \leq s + \epsilon - t \leq 2\epsilon$ we have by the mean value theorem and the convexity of pressure, $z'(t) \leq -1/\epsilon$. By Lemma 3.6 the equilibrium state μ on Σ_N for $t\varphi$ will satisfy that $\int \varphi d\mu \leq -1/\epsilon$ and $\frac{h(\mu, \sigma)}{\int \varphi d\mu} \geq t$. To complete the proof of the first part for each $n \in \mathbb{N}$ simply take $\epsilon = 1/n$ to find the sequence of measures μ_n .

Now let $t > t_1 > s$. Thus $P(t_1\varphi) < \infty$ and so, by the variational principle, for any ergodic measure μ for which φ is integrable we have

$$t_1 \int \varphi d\mu + h(\mu, \sigma) \leq P(t_1\varphi) < \infty$$

and since, by assumption, $P(\varphi) = 0$ we have that $h(\mu, \sigma) \leq -\int \varphi d\mu$. Thus if $h(\mu, \sigma) \geq -t \int \varphi d\mu$ then

$$-t \int \varphi d\mu + t_1 \int \varphi d\mu \leq P(t_1\varphi).$$

Thus

$$h(\mu, \sigma) \leq -\int \varphi d\mu \leq \frac{P(t_1\varphi)}{t - t_1}.$$

In other words, taking the contrapositive, we have that if $h(\mu, \sigma) > \frac{P(t_1\varphi)}{t - t_1}$ then $h(\mu, \sigma) + t \int \varphi d\mu < 0$, and the proof is complete. \square

4. HAUSDORFF DIMENSION OF THE SINGULARITY SET

In this section we will present the proof of the positive part of Theorem 1.1, that is, the result $\dim_{\text{H}}\{x : \theta'_k(x) \neq 0\} \rightarrow 1$ as $k \rightarrow \infty$.

4.1. Notation. Fix the countable Markov maps T_k and T and define the potentials

$$\varphi_k(\mathbf{i}) := -\log |T'_k(\pi_k(\mathbf{i}))| \quad \text{and} \quad \varphi(\mathbf{i}) := -\log |T'(\pi(\mathbf{i}))|$$

for $\mathbf{i} \in \Sigma$. Recall that by the assumption Theorem 1.1(2) these potentials have uniformly bounded sums of variations. Moreover, they are all locally Hölder.

Let us fix a generation $m \in \mathbb{N}$ and denote by $f_{\mathbf{i},k}$ for $\mathbf{i} \in \mathbb{N}^m$ the inverse branch corresponding to \mathbf{i} of the m -fold composition map $T_k^m = T_k \circ T_k \circ \dots \circ T_k$. We define the branches $f_{\mathbf{i}}$ similarly for the map T^m . Now these maps determine intervals

$$I_{\mathbf{i},k} := f_{\mathbf{i},k}([0, 1]) \quad \text{and} \quad I_{\mathbf{i}} := f_{\mathbf{i}}([0, 1]).$$

We denote the lengths of these intervals by $a_{\mathbf{i},k}$ and $a_{\mathbf{i}}$ respectively.

4.2. Strategy and key differences to the finite branch case. To bound the Hausdorff dimension of the set $\{x : \theta'_k(x) \neq 0\}$ of non-zero derivative for some $k \in \mathbb{N}$, we will find a *compactly supported* ergodic measure μ on the shift space $\mathbb{N}^{\mathbb{N}}$ for which the π_k projection of typical points will not have a derivative. Moreover, we will aim to choose the measure μ such that its Hausdorff dimension is close to 1 when k is large. This will be done in the following steps:

- (1) Our first step is to slightly simplify the problem by ‘iterating’ the potentials φ_k and φ up to a suitable generation $m \in \mathbb{N}$ to obtain new potentials $\psi_k := \frac{1}{m} S_m \varphi_k$ and $\psi := \frac{1}{m} S_m \varphi$ such that the distortion of ψ_k and ψ from analogous potentials coming from systems with linear branches is small. This is possible due to the bounded variations.
- (2) Then, in Lemma 4.1, we then use the absolutely continuous and invariant measure μ_T for T to construct a σ^m Bernoulli measure μ_k^m on $\mathbb{N}^{\mathbb{N}}$ which satisfies both that $-\int \psi_k d\mu_k^m > -\int \psi d\mu_k^m$ and that the π_k projection of μ_k^m has dimension close to 1.
- (3) The construction in Lemma 4.1 is possible due to the pointwise convergence of the inverse branches and the tail/variation assumptions in Theorem 1.1. The idea is based on a technical Lemma 4.3, where we study the perturbed Lyapunov exponents $\underline{\lambda}_m(\varphi_k, \mathbf{i}(n))$ of μ_T with respect to the map T_k . Here if they are too large due to cusp behaviour (which would cause the Hausdorff dimension to be small due to finite entropy), we transport mass from the cusp to avoid that phenomenon.
- (4) The measure μ_k^m constructed in (2) induces canonically a σ -invariant measure $\eta = \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i \mu_k^m$ of the same dimension as μ_k^m for which $\int \varphi_k d\eta > \int \varphi d\eta$. The measure η allows us to apply thermodynamic formalism (Lemmas 4.4 and 4.5) and invoke finite approximation properties (Lemma 4.6) to find a *compactly supported* Gibbs measure μ where $\int \varphi_k d\mu = \int \varphi d\mu$ but $\varphi_k - \varphi$ is not a coboundary, and μ still has dimension close to 1.
- (5) We will then essentially apply the law of iterated logarithms (Lemma 4.7) and the coboundary condition to show that for typical points under the projection of the measure μ the derivative of θ_k does not exist and the dimension of the projection of this measure will be a lower bound for the dimension of the set of points with non-zero derivative. We then show that this dimension tends to 1 as k tends to infinity, which completes the proof.

Step (5) is the same method as used in [12] in the finite branch case, but steps (2) and (3) are different from the finite state case where we can just take the measure to be a suitable equilibrium state. To find a measure which works in this setting we have to introduce the mass transportation property in Lemma 4.1.

4.3. Mass transportation and construction of the Bernoulli measure. Let us begin by constructing the Bernoulli measure μ_k^m .

Lemma 4.1. *For each $0 < \delta < 1/3$ there exists $M(\delta) \in \mathbb{N}$ such that for any $m \geq M(\delta)$ there exists $K(m) \in \mathbb{N}$ such that for any $k \geq K(m)$ there exists a σ^m Bernoulli measure μ_k^m on Σ which satisfies*

$$-\int S_m \varphi_k d\mu_k^m > -\int S_m \varphi d\mu_k^m \quad \text{and} \quad \dim_{\mathbb{H}} \pi_k \mu_k^m = \frac{h(\mu_k^m, T^m)}{-\int S_m \varphi_k d\mu_k^m} \geq \frac{1-3\delta}{1+3\delta}.$$

For the proof of Lemma 4.1, we will need the following two preliminary lemmas. We will let μ_φ be the equilibrium state for $\varphi : \Sigma \rightarrow \mathbb{R}$ (and also recall that $\varphi(\mathbf{i}) = \log |f'_{i_1}(\pi(\sigma(\mathbf{i})))| = -\log |T'(\pi(\mathbf{i}))|$). Note that μ_φ is the absolutely continuous T -invariant measure μ_T for T . Since $P(\varphi) = 0$ we have that $h(\mu_\varphi, T) = -\int \varphi d\mu_\varphi$.

Let us define the following quantities related to the entropy and Lyapunov exponents. For $m \in \mathbb{N}$, $\mathbf{i} \in \mathbb{N}^*$ and a potential f , let us write

$$\bar{\lambda}_m(f, \mathbf{i}) := \sup\{-S_m f(\mathbf{j}) : \mathbf{j} \in [\mathbf{i}]\}$$

and

$$\underline{\lambda}_m(f, \mathbf{i}) := \inf\{-S_m f(\mathbf{j}) : \mathbf{j} \in [\mathbf{i}]\}.$$

For the potential $\varphi = -\log |T'|$, define the numbers

$$\lambda_m := \sum_{\mathbf{i} \in \mathbb{N}^m} \mu_\varphi(I_{\mathbf{i}}) \bar{\lambda}_m(\varphi, \mathbf{i}).$$

Lemma 4.2. *Under the assumptions of Theorem 1.1, we have the following approximations*

(1) *The entropy of the measure μ_φ is given by*

$$h(\mu_\varphi, T) = \lim_{m \rightarrow \infty} \frac{1}{m} \lambda_m.$$

(2) *There exists $C_0 > 0$ such that for any $m \in \mathbb{N}$ and $\mathbf{i} \in \mathbb{N}^m$ we have*

$$\limsup_{k \rightarrow \infty} |\underline{\lambda}_m(\varphi_k, \mathbf{i}) - \underline{\lambda}_m(\varphi, \mathbf{i})| \leq C_0.$$

Proof. (1) By the definition of λ_m we have that

$$0 \leq -\int S_m \varphi d\mu_\varphi \leq \lambda_m \leq -\int S_m \varphi d\mu_\varphi + \sum_{k=1}^{\infty} \text{var}_k(\varphi).$$

The result then follows since

$$m^{-1} \int S_m \varphi d\mu_\varphi = \int \varphi d\mu_\varphi \quad \text{and} \quad h(\mu_\varphi, T) = -\int \varphi d\mu_\varphi.$$

(2) Fix $m \in \mathbb{N}$ and $\mathbf{i} \in \mathbb{N}^m$. Let us first verify that

$$\lim_{k \rightarrow \infty} f_{\mathbf{i},k}(y) = f_{\mathbf{i}}(y)$$

for any $y \in [0, 1]$. We will proceed by induction. For $m = 1$, this is the pointwise convergence assumption for the inverse branches of T_k and T . Now suppose the claim holds for $m - 1$ with $m \geq 2$. Fix $\mathbf{i} \in \mathbb{N}^m$. By the mean value theorem, there exists a point $z \in [0, 1]$ on the interval where the derivative $|f'_{i_1,k}(z)| \leq 1$. Since, according to assumption (2) for Theorem 1.1, we have

$C := \sup_{k \in \mathbb{N}} \sum_{n=1}^{\infty} \text{var}_n(-\log |T'_k|) < \infty$, this yields that $\|f'_{i_1,k}\|_{\infty} \leq e^C$ for all $\mathbf{i} \in \mathbb{N}^m$ and $k \in \mathbb{N}$. The mean value theorem gives

$$|f_{i_1,k}(f_{\sigma \mathbf{i},k}(y)) - f_{i_1,k}(f_{\sigma \mathbf{i}}(y))| \leq e^C |f_{\sigma \mathbf{i},k}(y) - f_{\sigma \mathbf{i}}(y)|,$$

which decays to 0 as $k \rightarrow \infty$ by the induction assumption for $m-1$. This completes the proof as

$$|f_{\mathbf{i},k}(y) - f_{\mathbf{i}}(y)| \leq |f_{i_1,k}(f_{\sigma \mathbf{i},k}(y)) - f_{i_1,k}(f_{\sigma \mathbf{i}}(y))| + |f_{i_1,k}(f_{\sigma \mathbf{i}}(y)) - f_{i_1}(f_{\sigma \mathbf{i}}(y))|$$

and the second term on the right-hand side converges to 0 as $k \rightarrow \infty$ by our assumption on pointwise convergence of inverse branches.

Choose $y_k, y \in [0, 1]$ such that

$$f'_{i_1,k}(y_k) = f_{i_1,k}(1) - f_{i_1,k}(0) \quad \text{and} \quad f'_{\mathbf{i}}(y) = f_{\mathbf{i}}(1) - f_{\mathbf{i}}(0).$$

This is possible by using the mean value theorem again. Then, by what we proved above, we have that the derivatives $f'_{i_1,k}(y_k) \rightarrow f'_{i_1}(y)$ as $k \rightarrow \infty$. Let $\mathbf{v}_k, \mathbf{v} \in [\mathbf{i}]$ be words such that

$$\pi_k(\mathbf{v}_k) = f_{i_1,k}(y_k) \quad \text{and} \quad \pi(\mathbf{v}) = f_{\mathbf{i}}(y).$$

Then by the chain rule

$$|S_m \varphi_k(\mathbf{v}_k) - S_m \varphi(\mathbf{v})| = |\log |f'_{i_1,k}(y_k)| - \log |f'_{\mathbf{i}}(y)||,$$

which converges to 0 as $k \rightarrow \infty$. On the other hand, for any pair $\mathbf{j}, \mathbf{k} \in [\mathbf{i}]$ we have by the triangle inequality

$$|S_m \varphi_k(\mathbf{j}) - S_m \varphi(\mathbf{k})| \leq \sum_{\ell=1}^m \text{var}_{\ell}(\varphi_k) + |S_m \varphi_k(\mathbf{v}_k) - S_m \varphi(\mathbf{v})| + \sum_{\ell=1}^m \text{var}_{\ell}(\varphi).$$

This yields the claim since φ_k and φ have summable variations and by the assumption (2) of Theorem 1.1 the sums for $\sum_{\ell=1}^{\infty} \text{var}_{\ell}(\varphi_k)$ are uniformly bounded over $k \in \mathbb{N}$. \square

Let us now make the choice of $M(\delta)$ for a fixed $0 < \delta < 1$: Write

$$C := \sum_{m=1}^{\infty} \text{var}_m(\varphi) + \sup_{k \in \mathbb{N}} \sum_{m=1}^{\infty} \text{var}_m(\varphi_k) < \infty. \quad (4.1)$$

Since by Lemma 4.2 we have $\frac{1}{m} \lambda_m \rightarrow h(\mu_{\varphi}, \sigma) > 0$, we may choose $M(\delta) \in \mathbb{N}$ such that for any $m \geq M(\delta)$ we have the following properties

(a)

$$\delta \lambda_m > \max\{C_0, 2C\}$$

(b)

$$(1 + \delta) \lambda_m + C \leq m h(\mu_{\varphi}, \sigma) (1 + 2\delta),$$

(c)

$$- \sum_{\mathbf{i} \in \mathbb{N}^m} \mu_{\varphi}([\mathbf{i}]) \log \mu_{\varphi}([\mathbf{i}]) \geq m h(\mu_{\varphi}, \sigma) (1 - \delta),$$

(d)

$$-S_m \varphi(\mathbf{j}) \geq 1 \text{ for all } \mathbf{j} \in \Sigma.$$

where $C_0 > 0$ is the constant from Lemma 4.2(2), and (d) follows from the assumption on the Markov map T that there exists $m \in \mathbb{N}$ and $\xi < 1$ such that for all $(i_1, \dots, i_m) \in \mathbb{N}^m$ we have that $|(f_{i_1} \circ \dots \circ f_{i_m})'(x)| \leq \xi$ for all $x \in (0, 1)$.

Lemma 4.3. *For each $\delta \in (0, 1/3)$, we have that either,*

(1)

$$-\int \varphi d\mu_\varphi < -\int \varphi_k d\mu_\varphi \leq -(1+2\delta) \int \varphi d\mu_\varphi, \quad \text{or,}$$

(2) *For each $m \geq M(\delta)$ and $k \in \mathbb{N}$ there exists a probability vector $(p_{\mathbf{i},k})_{\mathbf{i} \in \mathbb{N}^m}$ and numbers $r_1(k), r_2(k), r_3(k) \in \mathbb{R}$ satisfying $\lim_{k \rightarrow \infty} r_j(k) = 0$ for each $j = 1, 2, 3$ and such that*

(i)

$$\sum_{\mathbf{i} \in \mathbb{N}^m} p_{\mathbf{i},k} \underline{\lambda}_m(\varphi_k, \mathbf{i}) = (1+\delta)\lambda_m + r_1(k);$$

(ii)

$$-\sum_{\mathbf{i} \in \mathbb{N}^m} p_{\mathbf{i},k} \log p_{\mathbf{i},k} = -\sum_{\mathbf{i} \in \mathbb{N}^m} \mu_\varphi([\mathbf{i}]) \log \mu_\varphi([\mathbf{i}]) + r_2(k);$$

(iii)

$$\sum_{\mathbf{i} \in \mathbb{N}^m} p_{\mathbf{i},k} \bar{\lambda}_m(\varphi, \mathbf{i}) = \lambda_m + r_3(k).$$

Proof. Since the measure μ_φ is not an equilibrium state for φ_k , we have

$$-\int \varphi_k d\mu_\varphi > -\int \varphi d\mu_\varphi = h(\mu_\varphi, \sigma)$$

and so if case (1) does not hold, we may assume that

$$-\int \varphi_k d\mu_\varphi > -(1+2\delta) \int \varphi d\mu_\varphi,$$

which yields

$$-m \int S_m \varphi_k d\mu_\varphi > -(1+2\delta)m \int S_m \varphi d\mu_\varphi,$$

by the σ invariance of μ_φ . We put an order on the set of m -tuples $\mathbb{N}^m = \{\mathbf{i}(1), \mathbf{i}(2), \dots\}$ by requiring that $\mu_\varphi([\mathbf{i}(n)]) \geq \mu_\varphi([\mathbf{i}(n+1)])$ and if $\mu_\varphi([\mathbf{i}(n)]) = \mu_\varphi([\mathbf{i}(n+1)])$ we require that the interval $I_{\mathbf{i}(n)}$ is on the right-hand side of $I_{\mathbf{i}(n+1)}$ (recall that these were obtained as a $\pi = \pi_T$ projection of cylinders onto $[0, 1]$). For a fixed $m \geq M(\delta)$ and each $k \in \mathbb{N}$ we define

$$N_k = N_k(m) := \inf \left\{ N \in \mathbb{N} : \sum_{n=1}^N \mu_\varphi([\mathbf{i}(n)]) \underline{\lambda}_m(\varphi_k, \mathbf{i}(n)) \geq (1+\delta)\lambda_m \right\}.$$

Note that N_k cannot be infinite since by the choice of $M(\delta)$ (choice (a)) and by the definition of variations (recall that C is the supremum for the sums of variations of both φ_k and φ), and the

definition of λ_m yields

$$\begin{aligned}
\sum_{n=1}^{\infty} \mu_{\varphi}([\mathbf{i}(n)]) \underline{\lambda}_m(\varphi_k, \mathbf{i}(n)) &\geq \sum_{n=1}^{\infty} \mu_{\varphi}([\mathbf{i}(n)]) \bar{\lambda}_m(\varphi_k, \mathbf{i}(n)) - C \\
&\geq \int -S_m \varphi_k d\mu_{\varphi} - C \\
&\geq (1 + 2\delta) \int -S_m \varphi d\mu_{\varphi} - C \\
&\geq (1 + 2\delta) \sum_{n=1}^{\infty} \mu_{\varphi}([\mathbf{i}(n)]) \bar{\lambda}_m(\varphi, \mathbf{i}(n)) - C \\
&\geq (1 + 2\delta) \lambda_m - 2C \\
&> (1 + \delta) \lambda_m.
\end{aligned}$$

Our first claim is that $N_k \rightarrow \infty$ as $k \rightarrow \infty$. This is proved by contradiction. Suppose that there is a subsequence k_l and a constant $N_0 \in \mathbb{N}$ where $N_{k_l} \leq N_0$ for all $l \in \mathbb{N}$. In this case

$$\sum_{n=1}^{N_0} \mu_{\varphi}([\mathbf{i}(n)]) \underline{\lambda}_m(\varphi_{k_l}, \mathbf{i}(n)) \geq (1 + \delta) \lambda_m.$$

for all $l \in \mathbb{N}$. On the other hand, by Lemma 4.2(2) we have for any $n \in \mathbb{N}$ that

$$\limsup_{k \rightarrow \infty} |\underline{\lambda}_m(\varphi_k, \mathbf{i}(n)) - \underline{\lambda}_m(\varphi, \mathbf{i}(n))| \leq C_0 < \delta \lambda_m$$

since $m \geq M(\delta)$ and we fixed $M(\delta)$ such that $\delta \lambda_m > C_0$ for all $m \geq M(\delta)$ (recall property (a) again). Therefore as

$$\sum_{n=1}^{N_0} \mu_{\varphi}([\mathbf{i}(n)]) \underline{\lambda}_m(\varphi, \mathbf{i}(n)) < \lambda_m,$$

we have

$$\limsup_{l \rightarrow \infty} \sum_{n=1}^{N_0} \mu_{\varphi}([\mathbf{i}(n)]) \underline{\lambda}_m(\varphi_{k_l}, \mathbf{i}(n)) \leq \delta \lambda_m + \sum_{n=1}^{N_0} \mu_{\varphi}([\mathbf{i}(n)]) \underline{\lambda}_m(\varphi, \mathbf{i}(n)) < (1 + \delta) \lambda_m,$$

which is a contradiction. Thus we must have $N_k \rightarrow \infty$ as $k \rightarrow \infty$.

Since $N_k < \infty$ we can define

$$p_{\mathbf{i}(n), k} := \begin{cases} 0, & \text{if } n \geq N_k + 1; \\ \mu_{\varphi}([\mathbf{i}(n)]), & \text{if } 2 \leq n \leq N_k - 1; \\ \frac{(1 + \delta) \lambda_m - \sum_{n=1}^{N_k - 1} \mu_{\varphi}([\mathbf{i}(n)]) \underline{\lambda}_m(\varphi_k, \mathbf{i}(n))}{\underline{\lambda}_m(\varphi_k, \mathbf{i}(N_k))}, & \text{if } n = N_k; \\ 1 - \sum_{n=2}^{\infty} p_{\mathbf{i}(n), k}, & \text{if } n = 1. \end{cases}$$

Let us now define the numbers $r_i(k)$ such that they satisfy properties (i), (ii) and (iii), and then let us also check that they converge to 0 for increasing k .

(i) Define

$$r_1(k) := (p_{\mathbf{i}(1),k} - \mu_\varphi([\mathbf{i}(1)])) \underline{\lambda}_m(\varphi_k, \mathbf{i}(1)).$$

Then by the definition of the weights $p_{\mathbf{i}(n),k}$ we have

$$\begin{aligned} \sum_{n=1}^{\infty} p_{\mathbf{i}(n),k} \underline{\lambda}_m(\varphi_k, \mathbf{i}(n)) &= \sum_{n=1}^{N_k-1} \mu_\varphi([\mathbf{i}(n)]) \underline{\lambda}_m(\varphi_k, \mathbf{i}(n)) \\ &\quad + (p_{\mathbf{i}(1),k} - \mu_\varphi([\mathbf{i}(1)])) \underline{\lambda}_m(\varphi_k, \mathbf{i}(1)) \\ &\quad + p_{\mathbf{i}(N_k),k} \underline{\lambda}_m(\varphi_k, \mathbf{i}(N_k)) \\ &= (1 + \delta) \lambda_m + r_1(k). \end{aligned}$$

(ii) Define

$$\begin{aligned} r_2(k) &:= -p_{\mathbf{i}(1),k} \log p_{\mathbf{i}(1),k} + \mu_\varphi([\mathbf{i}(1)]) \log \mu_\varphi([\mathbf{i}(1)]) \\ &\quad - p_{\mathbf{i}(N_k),k} \log p_{\mathbf{i}(N_k),k} + \sum_{n=N_k}^{\infty} \mu_\varphi([\mathbf{i}(n)]) \log \mu_\varphi([\mathbf{i}(n)]). \end{aligned}$$

Then again

$$-\sum_{n=1}^{\infty} p_{\mathbf{i}(n),k} \log p_{\mathbf{i}(n),k} = -\sum_{n=1}^{\infty} \mu_\varphi([\mathbf{i}(n)]) \log \mu_\varphi([\mathbf{i}(n)]) + r_2(k).$$

(iii) Define

$$\begin{aligned} r_3(k) &:= (p_{\mathbf{i}(1),k} - \mu_\varphi([\mathbf{i}(1)])) \bar{\lambda}_m(\varphi, \mathbf{i}(1)) + p_{\mathbf{i}(N_k),k} \bar{\lambda}_m(\varphi, \mathbf{i}(N_k)) \\ &\quad - \sum_{n=N_k}^{\infty} \mu_\varphi(I_{\mathbf{i}(n)}) \bar{\lambda}_m(\varphi, \mathbf{i}(n)). \end{aligned}$$

Then recalling that λ_m is defined by

$$\lambda_m = \sum_{n=1}^{\infty} \mu_\varphi(I_{\mathbf{i}(n)}) \bar{\lambda}_m(\varphi, \mathbf{i}(n)),$$

we can use the definition of the weights $p_{\mathbf{i}(n),k}$ to obtain the following

$$\begin{aligned} \sum_{n=1}^{\infty} p_{\mathbf{i}(n),k} \bar{\lambda}_m(\varphi, \mathbf{i}(n)) &= \sum_{n=1}^{\infty} \mu_\varphi(I_{\mathbf{i}(n)}) \bar{\lambda}_m(\varphi, \mathbf{i}(n)) \\ &\quad + (p_{\mathbf{i}(1),k} - \mu_\varphi([\mathbf{i}(1)])) \bar{\lambda}_m(\varphi, \mathbf{i}(1)) \\ &\quad + p_{\mathbf{i}(N_k),k} \bar{\lambda}_m(\varphi, \mathbf{i}(N_k)) - \sum_{n=N_k}^{\infty} \mu_\varphi(I_{\mathbf{i}(n)}) \bar{\lambda}_m(\varphi, \mathbf{i}(n)). \\ &= \lambda_m + r_3(k). \end{aligned}$$

By the definition of N_k , observe that

$$0 < p_{\mathbf{i}(N_k),k} = \frac{(1 + \delta) \lambda_m - \sum_{n=1}^{N_k-1} \mu_\varphi([\mathbf{i}(n)]) \underline{\lambda}_m(\varphi_k, \mathbf{i}(n))}{\underline{\lambda}_m(\varphi_k, \mathbf{i}(N_k))} \leq \mu_\varphi([\mathbf{i}(N_k)]).$$

Moreover,

$$p_{\mathbf{i}(1),k} = \mu_\varphi([\mathbf{i}(1)]) + t_k - p_{\mathbf{i}(N_k),k},$$

where we have defined t_k to be the tail of the distribution μ_φ , that is

$$t_k := 1 - \sum_{n=1}^{N_k-1} \mu_\varphi([\mathbf{i}(n)]).$$

Since $N_k \rightarrow \infty$ and so $\mu_\varphi([\mathbf{i}(N_k)]) \rightarrow 0$, we have that as $p_{\mathbf{i}(N_k),k} \leq \mu_\varphi([\mathbf{i}(N_k)])$, both

$$p_{\mathbf{i}(N_k),k} \rightarrow 0 \quad \text{and} \quad t_k \rightarrow 0$$

as $k \rightarrow \infty$. Furthermore, by Lemma 4.2(2) there exists $C_0 > 0$ such that for each $n \in \mathbb{N}$ we have

$$\limsup_{k \rightarrow \infty} |\underline{\lambda}_m(\varphi_k, \mathbf{i}(n)) - \underline{\lambda}_m(\varphi, \mathbf{i}(n))| \leq C_0$$

and $\underline{\lambda}_m(\varphi, \mathbf{i}(n)) < \infty$ for all n . Therefore

$$r_1(k), r_2(k), r_3(k) \rightarrow 0, \quad \text{as} \quad k \rightarrow \infty,$$

and so the lemma is proved. □

Recall that $r_1(k), r_2(k), r_3(k) \rightarrow 0$ and they implicitly depend on m , but the convergence to zero will happen for any fixed $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$ and choose $K(m) \in \mathbb{N}$ such that for any $k \geq K(m)$ we have

$$|r_1(k)|, |r_2(k)|, |r_3(k)| \leq \min\{C, \delta h(\mu_\varphi, \sigma)\},$$

and

$$|r_1(k) - (1 + \delta)r_3(k)| \leq \delta,$$

where C was defined in (4.1).

We are now in a position to prove Lemma 4.1.

Proof of Lemma 4.1. Fix $\delta \in (0, 1/3)$, $m \geq M(\delta)$ and $k \geq k(m)$. We first suppose that we are in the first case of Lemma 4.3. In this case we can fix $\mu_k^m := \mu_\varphi$ which will be σ^k -ergodic since it is Gibbs for σ . We have that

$$- \int S_m \varphi_k d\mu_\varphi > - \int S_m \varphi d\mu_\varphi$$

and

$$\frac{h(\mu_\varphi, \sigma^k)}{- \int S_m \varphi_k d\mu_\varphi} \geq \frac{1}{1 + 2\delta} \geq \frac{1 - 3\delta}{1 + 3\delta}.$$

If we are in the second case of Lemma 4.3, we let μ_k^m be the σ^m Bernoulli measure defined by the weights $(p_{\mathbf{i},k})_{\mathbf{i} \in \mathbb{N}^m}$ from Lemma 4.3. By the properties (i) and (iii) in Lemma 4.3 and the assumption

(d) on $M(\delta)$, we have that

$$\begin{aligned}
-\int S_m \varphi_k d\mu_k^m &\geq \sum_{\mathbf{i} \in \mathbb{N}^m} p_{\mathbf{i},k} \lambda_m(\varphi_k, \mathbf{i}(n)) \\
&= (1 + \delta) \lambda_m + r_1(k) \\
&\geq (1 + \delta) \left(\sum_{\mathbf{i} \in \mathbb{N}^m} p_{\mathbf{i},k} \bar{\lambda}_m(\varphi, \mathbf{i}(n)) \right) + r_1(k) - (1 + \delta) r_3(k) \\
&\geq (1 + \delta) \left(\sum_{\mathbf{i} \in \mathbb{N}^m} p_{\mathbf{i},k} \bar{\lambda}_m(\varphi, \mathbf{i}(n)) \right) - \delta \\
&\geq -(1 + \delta) \int S_m \varphi d\mu_k^m - \delta \\
&> -\int S_m \varphi d\mu_k^m.
\end{aligned}$$

For the dimension we need an estimate in the opposite direction. By property (c) of the choice of $M(\delta)$ we have

$$\begin{aligned}
-\int S_m \varphi_k d\mu_k^m &\leq \sum_{\mathbf{i} \in \mathbb{N}^m} p_{\mathbf{i},k} \lambda_m(\varphi_k, \mathbf{i}(n)) + C \\
&= (1 + \delta) \lambda_m + r_1(k) + C \\
&\leq (1 + 3\delta) (mh(\mu_\varphi, \sigma)).
\end{aligned}$$

We also need an estimate on the entropy. Using property (c) of the choice of $M(\delta)$ once again, we have that

$$\begin{aligned}
h(\mu_k^m, \sigma^m) &= - \sum_{\mathbf{i} \in \mathbb{N}^m} p_{\mathbf{i},k} \log p_{\mathbf{i},k} \\
&= - \sum_{\mathbf{i} \in \mathbb{N}^m} \mu_\varphi([\mathbf{i}]) \log \mu_\varphi([\mathbf{i}]) - r_3(k) \\
&\geq mh(\mu_\varphi, \sigma)(1 - 2\delta).
\end{aligned}$$

Putting these two estimates together, we obtain

$$\frac{h(\mu_k^m, \sigma^m)}{-\int S_m \varphi_k d\mu_k^m} \geq \frac{1 - 3\delta}{1 + 3\delta}.$$

Thus the proof is complete. □

4.4. Applying thermodynamical formalism. Now let us proceed with the proof of Theorem 1.1. Let $\delta > 0$ and fix $m \geq M(\delta)$ (recall the choice of $M(\delta)$ from Lemma 4.1) and write

$$\psi_k := \frac{1}{m} S_m \varphi_k \quad \text{and} \quad \psi := \frac{1}{m} S_m \varphi$$

and define the auxiliary σ invariant measure

$$\eta := \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i \mu_k^m,$$

where μ_k^m is the σ^m Bernoulli measure determined in Lemma 4.1. The measure η satisfies the following properties:

$$\int \varphi_k d\eta = \int \psi_k d\mu_k^m, \quad \int \varphi d\eta = \int \psi d\mu_k^m \quad \text{and} \quad h(\eta, \sigma) = \frac{1}{m}h(\mu_k^m, \sigma)$$

and the dimension

$$s_k := \dim_{\text{H}} \pi_k \eta = \frac{h(\eta, \sigma)}{\int \varphi_k d\eta} = \dim_{\text{H}} \pi_k \mu_k^m.$$

Lemma 4.1 will allow us to deduce the following lower bound on the pressure function

$$q \rightarrow P(q(\varphi_k - \varphi) - t\varphi_k)$$

with a suitable choice of t .

Lemma 4.4. *If $0 < t < s_k$, then*

$$\inf_{q \in \mathbb{R}} P(q(\varphi_k - \varphi) + t\varphi_k) > 0.$$

Proof. By Lemma 4.1, we have

$$-\int \varphi_k d\eta > -\int \varphi d\eta.$$

Thus we have that for all $q \leq 0$ the following property

$$\int [q(\varphi_k - \varphi) + t\varphi_k] d\eta + h(\eta, \sigma) > t \int \varphi_k d\eta + h(\eta, \sigma) > 0.$$

On the other hand, if $q > 0$ we first suppose that the potential φ_k has an equilibrium state ν_k . In this case as $t < s_k \leq 1$ and $\int \varphi_k - \varphi d\nu_k > 0$ we have

$$\int q(\varphi_k - \varphi) + t\varphi_k d\nu_k + h(\eta, \sigma) > t \int \varphi_k d\nu_k + h(\nu_k, \sigma) > 0.$$

Thus by the variational principle,

$$P(q(\varphi_k - \varphi) + t\varphi_k) > \max \left\{ t \int \varphi_k d\eta + h(\eta, \sigma), t \int \varphi_k d\nu_k + h(\nu_k, \sigma) \right\} > 0.$$

If φ_k does not have an equilibrium state then we must have that

$$\sup\{s : P(s\varphi_k) = \infty\} = 1$$

and by assumption

$$0 \leq \sup\{s : P(s\varphi) = \infty\} < 1.$$

Therefore, if we let $1 > s > \max\{\sup\{s : P(s\varphi) = \infty\}, t\}$ and apply the first part of Lemma 3.8 to φ_k and the second part to φ , we can find a compactly supported σ invariant ergodic measure μ such that

$$h(\mu, \sigma) + s \int \varphi_k d\mu \geq 0$$

and

$$h(\mu, \sigma) + s \int \varphi d\mu \leq 0.$$

Therefore $\int \varphi_k d\mu \geq \int \varphi d\mu$ and so for all $q \leq 0$

$$\int q(\varphi_k - \varphi) + t\varphi_k d\mu + h(\mu, \sigma) > 0.$$

□

We can now use the approximation property of pressure to allow us to find suitable measures which are compactly supported. Recall that the finite approximation property was given in Lemma 3.5, and it states that $P(\varphi) = \lim_{N \rightarrow \infty} P_N(\varphi)$, where $P_N(\varphi)$ is the pressure of φ restricted to the finite shift $\{1, 2, \dots, N\}^{\mathbb{N}}$.

Lemma 4.5. *If $0 < t < s_k$, then there exists $N \in \mathbb{N}$ with*

$$\inf\{P_N(q(\varphi_k - \varphi) - t\varphi_k) : q \in \mathbb{R}\} > 0$$

and

$$\lim_{q \rightarrow \infty} P_N(q(\varphi_k - \varphi) - t\varphi_k) = \lim_{q \rightarrow -\infty} P_N(q(\varphi_k - \varphi) - t\varphi_k) = \infty.$$

Proof. First of all by taking ν_k as in the proof of previous Lemma 4.4 we have

$$\int (\varphi_k - \varphi) d\nu_k < 0 \quad \text{and} \quad \int (\varphi_k - \varphi) d\eta > 0.$$

Let us use these measures η and ν_k to construct measures τ_1 and τ_2 satisfying similar properties but supported on a compact set Σ_N for a large enough N as follows. By Birkhoff's ergodic theorem there exist words $\mathbf{i}, \mathbf{j} \in \Sigma$ and indices $n_1, n_2 \in \mathbb{N}$ such that

$$\sigma^{n_1} \mathbf{i} = \mathbf{i}, \quad \sigma^{n_2} \mathbf{j} = \mathbf{j}, \quad S_{n_1}(\varphi_k - \varphi)(\mathbf{i}) > 0, \quad \text{and} \quad S_{n_2}(\varphi_k - \varphi)(\mathbf{j}) < 0.$$

Thus if we let τ_1 and τ_2 be the measures supported on these n_1 and n_2 periodic orbits of \mathbf{i} and \mathbf{j} respectively, then there exists an index $M \in \mathbb{N}$ such that both τ_1, τ_2 are invariant measures on Σ_N for all $N \geq M$ and we will have that

$$\int (\varphi_k - \varphi) d\tau_1 < 0 \quad \text{and} \quad \int (\varphi_k - \varphi) d\tau_2 > 0.$$

Thus if $N \geq M$ and we put

$$q_1 := \frac{t \int \varphi_k d\tau_1}{\int (\varphi_k - \varphi) d\tau_1} \quad \text{and} \quad q_2 := \frac{t \int \varphi_k d\tau_2}{\int (\varphi_k - \varphi) d\tau_2},$$

then by the variational principle there exists $C > 0$ such that $P_N(q(\varphi_k - \varphi) - t\varphi_k) > C$ for all $q \notin [2q_1, 2q_2]$ and

$$\lim_{q \rightarrow \infty} P_N(q(\varphi_k - \varphi) - t\varphi_k) = \lim_{q \rightarrow -\infty} P_N(q(\varphi_k - \varphi) - t\varphi_k) = \infty.$$

On the other hand, by the finite approximation property (Lemma 3.5) and Lemma 4.4 we have that

$$\lim_{n \rightarrow \infty} P_N(q(\varphi_k - \varphi) - t\varphi_k) = P(q(\varphi_k - \varphi) - t\varphi_k) \geq \inf_{q \in \mathbb{R}} P(q(\varphi_k - \varphi) - t\varphi_k) > 0$$

for all $q \in [2q_1, 2q_2]$. Now if for each $n \in \mathbb{N}$ we define the set

$$Q_N := \{q \in [2q_1, 2q_2] : P_N(q(\varphi_k - \varphi) - t\varphi_k) \leq 0\},$$

then $Q_{N+1} \subset Q_N$ for for all $n \in \mathbb{N}$. However, if we can find $q \in \bigcap_{N=1}^{\infty} Q_N$, then $P(q(\varphi_k - \varphi) - t\varphi_k) \leq 0$, which is a contradiction. Thus $\bigcap_{N=1}^{\infty} Q_N = \emptyset$ and since each Q_N is compact there must exist $N \geq M$ such that $Q_N = \emptyset$. For this value of $N \in \mathbb{N}$ we must have that

$$\inf\{P_N(q(\varphi_k - \varphi) - t\varphi_k) : q \in \mathbb{R}\} > 0$$

as claimed. □

Now for the $N \in \mathbb{N}$ constructed in Lemma 4.5, we can formulate a key lemma:

Lemma 4.6. *If $0 < t < s_k$, then there exists $N \in \mathbb{N}$ such that*

- (1) $\varphi_k - \varphi$ is not a coboundary on Σ_N .
- (2) there exists a Gibbs measure μ on Σ_N such that

$$\int \varphi_k d\mu = \int \varphi d\mu \quad \text{and} \quad \frac{h(\mu, \sigma)}{-\int \varphi_k d\mu} > t.$$

Proof. By Lemma 4.5 we know that there exists $N \in \mathbb{N}$ such that

$$\inf\{P_N(q(\varphi_k - \varphi) - t\varphi_k) : q \in \mathbb{R}\} > 0$$

and

$$\lim_{q \rightarrow \infty} P_N(q(\varphi_k - \varphi) - t\varphi_k) = \lim_{q \rightarrow -\infty} P_N(q(\varphi_k - \varphi) - t\varphi_k) = \infty.$$

The restrictions of φ_k and φ to Σ_N are Hölder continuous and so the function $Z_N : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$Z_N(q) := P_N(q(\varphi_k - \varphi) - t\varphi_k)$$

is analytic with

$$Z'_N(q) = \int (\varphi_k - \varphi) d\mu_q$$

by Lemma 3.6, where μ_q is the Gibbs measure on Σ_N for $q(\varphi_k - \varphi) + t\varphi_k$.

Since $\lim_{q \rightarrow \infty} Z_N(q) = \lim_{q \rightarrow -\infty} Z_N(q) = \infty$ we know by the definition of pressure that $\varphi_k - \varphi$ cannot be a coboundary on Σ_N . Therefore, as $\inf\{Z_N(q) : q \in \mathbb{R}\} > 0$, there must exist $q_1 \in \mathbb{R}$ such that $Z'_N(q_1) = 0$. Thus the Gibbs measure $\mu := \mu_{q_1}$ on Σ_N satisfies

$$\int (\varphi_k - \varphi) d\mu = 0$$

and by the variational principle (since $Z_N(q_1) > 0$) we have

$$h(\mu, \sigma) + t \int \varphi_k d\mu > 0.$$

Therefore, we have by the negativity of φ_k that

$$\frac{h(\mu, \sigma)}{-\int \varphi_k d\mu} > t$$

as claimed. □

4.5. Applying the law of iterated logarithms. The key to the proof of the main theorem will be to combine the above result with the following simple application of the law of the iterated logarithm for function differences $f - g$, which are not coboundaries.

Lemma 4.7. *Let $f, g : \Sigma_N \rightarrow \mathbb{R}$ be Hölder continuous potentials such that $f - g$ is not a coboundary and let μ be a Gibbs measure on Σ_N where $\int f d\mu = \int g d\mu$. We then have that*

$$\liminf_{n \rightarrow \infty} e^{S_n(f-g)(x)} = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} e^{S_n(f-g)(x)} = \infty$$

for μ almost all $x \in \Sigma_N$.

Proof. Since $f - g$ is not cohomologous to a constant we can apply the law of the iterated logarithm, Lemma 3.7, to the functions $f - g$ and $g - f$ to conclude that for some positive constants $c_1, c_2 > 0$ the following asymptotic bounds hold:

$$\liminf_{n \rightarrow \infty} \frac{S_n(f - g)(x)}{\sqrt{n \log \log n}} < -c_2 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{S_n(f - g)(x)}{\sqrt{n \log \log n}} > c_1$$

at μ almost every $x \in \Sigma_N$. In particular at these x also

$$\liminf_{n \rightarrow \infty} e^{S_n(f-g)(x)} = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} e^{S_n(f-g)(x)} = \infty.$$

□

Let us now complete the proof of the main theorem.

Proof of Theorem 1.1. For any $0 < \delta < 1/3$ and $m \geq M(\delta)$ by Lemma 4.1, we can find $K = K(m) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \geq K$ there exists a σ^m -invariant ergodic measure μ_k^m on Σ such that

$$\int (\psi_k - \psi) d\mu_k^m > 0 \quad \text{and} \quad \frac{\frac{1}{m} h(\mu_k^m)}{-\int \psi_k d\mu_k^m} = \dim \pi_k \mu_k^m > \frac{1 - 3\delta}{1 + 3\delta}.$$

Thus by Lemma 4.6 applied to $t = (1 - 2\delta)/(1 + 2\delta)$ and for the $N \in \mathbb{N}$ given by that result, $\varphi_k - \varphi$ is not a coboundary on Σ_N and we can find a Gibbs measure μ supported on a compact set of Σ (i.e. Σ_N embedded into Σ) such that

$$\int \psi_k d\mu = \int \psi d\mu \quad \text{and} \quad \dim \mu \circ \pi_k > \frac{1 - 3\delta}{1 + 3\delta}.$$

Therefore, by Lemma 4.7, we may also assume that at μ almost all $x \in \Sigma$ we have

$$\liminf_{n \rightarrow \infty} e^{S_n(\psi_k - \psi)(x)} = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} e^{S_n(\psi_k - \psi)(x)} = \infty.$$

Fix one such $x \in \Sigma$. Recall that the projections $\pi_k, \pi : \Sigma \rightarrow [0, 1]$ map cylinder sets from Σ onto T_k and T construction intervals respectively and the conjugacy θ_k between T_k and T satisfies

$$\theta_k(\pi_k(x)) = \pi(x).$$

Now for each $n \in \mathbb{N}$, let us define a word $y = y(n) \in \mathbb{N}^{n+1}$ by

$$y := \begin{cases} x|_n 3, & \text{if } x_{n+1} = 1; \\ x|_n 4, & \text{if } x_{n+1} = 2; \\ x|_n 1, & \text{if } x_{n+1} \geq 3. \end{cases}$$

Then $\pi_k(y) \in I_{x_1, \dots, x_n}^{(T_k)}$ and so $\theta_k(\pi_k(y)) \in I_{x_1, \dots, x_n}^{(T)}$, where we emphasise the interval map T_k or T used. Therefore, for all $n \in \mathbb{N}$ the distances

$$|\pi_k(x) - \pi_k(y)| \leq |I_{x_1, \dots, x_n}^{(T_k)}| = e^{S_n \psi_k(x)}$$

and

$$|\theta_k(\pi_k(x)) - \theta_k(\pi_k(y))| \leq |I_{x_1, \dots, x_n}^{(T)}| = e^{S_n \psi(x)}.$$

Moreover, we have the lower bound

$$|\pi_k(x) - \pi_k(y)| \geq \begin{cases} |I_{x_1, \dots, x_n 2}^{(T_k)}|, & \text{if } x_{n+1} = 1; \\ |I_{x_1, \dots, x_n 3}^{(T_k)}|, & \text{if } x_{n+1} = 2; \\ |I_{x_1, \dots, x_n 2}^{(T_k)}|, & \text{if } x_{n+1} \geq 3. \end{cases}$$

so in all cases there is $c_k = c_k(x) > 0$ independent of n satisfying

$$|\pi_k(x) - \pi_k(y)| \geq c_k e^{S_n \psi_k(x)}.$$

Similarly, for a suitable $c = c(x) > 0$ independent of n the images satisfy

$$|\theta_k(\pi_k(x)) - \theta_k(\pi_k(y))| \geq c e^{S_n \psi(x)}$$

Thus as the numbers c_k and c are independent of n we obtain by our choice of x that

$$\liminf_{n \rightarrow \infty} \frac{|\theta_k(\pi_k(x)) - \theta_k(\pi_k(y))|}{|\pi_k(x) - \pi_k(y)|} \leq \liminf_{n \rightarrow \infty} c_k^{-1} e^{S_n(\psi_k - \psi)(x)} = 0$$

and

$$\limsup_{n \rightarrow \infty} \frac{|\theta_k(\pi_k(x)) - \theta_k(\pi_k(y))|}{|\pi_k(x) - \pi_k(y)|} \geq \limsup_{n \rightarrow \infty} c e^{S_n(\psi_k - \psi)(x)} = \infty.$$

Thus the derivative of θ_k at $\pi_k(x)$ cannot exist. Since x was μ typical, this means that $\mu \circ \pi_k$ gives full mass to the set of y where $\theta'_k(y)$ does not exist. Therefore, for all $k \geq K$ we have

$$\dim_{\text{H}}\{y \in [0, 1] : \theta'_k(y) \text{ does not exist}\} \geq \dim \pi_k \mu > \frac{1 - 3\delta}{1 + 3\delta}.$$

The proof of Theorem 1.1 is therefore complete, since $1/3 > \delta > 0$ was chosen arbitrarily. \square

5. HÖLDER EXPONENTS AND THE DIMENSION OF THE CONJUGATED MEASURE

In this section we will finish the proof of Theorem 1.1. To do this we need to give example of countable Markov maps T_k and T satisfying the conditions of Theorem 1.1 but where we have that the Hölder exponents $\kappa(\theta_k)$ of θ_k satisfy

$$\lim_{k \rightarrow \infty} \kappa(\theta_k) = 0.$$

We also need to give a similar example of countable Markov maps T_k and T satisfying the conditions of Theorem 1.1 but where we have the Hausdorff dimensions of the conjugated measures $\mu_T \circ \theta_k$ satisfy

$$\lim_{k \rightarrow \infty} \dim_{\text{H}}(\mu_T \circ \theta_k) = 0.$$

Both of the examples we give below come from the class of α -Lüroth maps, which were introduced in [17], so let us briefly recall the definition.

Definition 5.1 (α -Lüroth maps). We start with a decreasing sequence of real numbers $(t_k)_{k \geq 1}$ with $t_1 = 1$ and $0 < t_k < 1$ for $k \geq 2$, and having the property that $\lim_{k \rightarrow \infty} t_k = 0$ and let $\alpha := \{A_n := (t_{n+1}, t_n] : n \in \mathbb{N}\}$. We also denote the length of A_n by $a_n := a_n(\alpha)$. Then the α -Lüroth map L_α is defined to be the countable Markov map with inverse branches that map the unit interval affinely onto each partition element A_n .

Two particular examples we will use below come from the partitions α_L , defined by $t_n := 1/n$, and α_D , which is given by $t_n := 2^{-(n-1)}$.

5.1. Hölder exponents. We start with the map $T := L_{\alpha_D}$ as described above. Then we modify the partition α_D to obtain a sequence of α -Lüroth maps that converge pointwise to T , in the following way. Let α_k be the partition where $a_n(\alpha_k) = a_n(\alpha_D)$ for all $n \notin \{k, k+1\}$, and we modify the point $t_{k+1}(\alpha_D)$ in order to obtain the lengths $a_k(\alpha_k) = 2^{-k^2}$ and $a_{k+1} = 2^{-k} + 2^{-(k+1)} - 2^{-k^2}$. Then the conjugacy map θ_k between T_k and T is exactly the map studied in [17], where in particular it was shown in [17, Lemma 2.3] that the Hölder exponent of θ_k is given by

$$\kappa(\theta_k) = \inf \left\{ \frac{\log a_n(\alpha_D)}{\log a_n(\alpha_k)} : n \in \mathbb{N} \right\}.$$

Therefore, for our example, we see that the Hölder exponent of θ_k is given by $1/k$.

5.2. Hausdorff dimension of the conjugated measure. In this section we look at the dimension of the conjugated measure $\mu_T \circ \theta_k$. In this case we choose T to be the α_L -Lüroth map, so $a_n(\alpha_L) = 1/(n(n+1))$ for all $n \in \mathbb{N}$ and μ_T is the Lebesgue measure. Therefore we have that the Lyapunov exponent and the entropy

$$\lambda(\mu_T, T) = h(\mu_T, T) = \sum_{i=1}^{\infty} -a_i \log a_i < +\infty.$$

Now for each $k \in \mathbb{N}$ we make a modification to the partition α_L to obtain a sequence of partitions α_k as follows. Fix the first k elements of the partition, and then for $i > k$ let the partition elements have size

$$a_i(\alpha_k) = \frac{1}{(k+1)2^{i-k}}.$$

Letting $T_k := L_{\alpha_k}$, and the conjugacy between T_k and T again be denoted by θ_k , the conditions of Theorem 1.1 are readily seen to hold as $-\log |T_k'|$ is a piecewise constant function and the tail t_i decays exponentially. However, for each k we have that

$$h(\mu_T \circ \theta_k, T_k) = h(\mu_T, T) < +\infty,$$

but the maps T_k are constructed such that

$$\lambda(\mu_T \circ \theta_k, T_k) = \sum_{i=1}^{\infty} -a_i(\alpha_L) \log(a_i(\alpha_k)) = +\infty.$$

Thus for each k , due to Proposition 2.4, we have that $\dim_{\text{H}} \mu_T \circ \theta_k = 0$

6. STABILITY OF MANNEVILLE-POMEAU MAPS

Let us now prove Corollary 1.2 to Theorem 1.1. Fix $\alpha, \beta > 0$ with $\alpha \neq \beta$ and let \widehat{M}_α and \widehat{M}_β be the jump transformations of M_α and M_β . That is, if $r_\alpha(x) \in \mathbb{N}$ is the first hitting time to the interval between $[b_\alpha, 1]$, where b_α is the solution to the equation $x + x^{1+\alpha} = 1$ on $(0, 1)$, then

$$\widehat{M}_\alpha(x) := M_\alpha^{r_\alpha(x)}(x)$$

and similarly for \widehat{M}_β . Now the topological conjugacy $\theta_{\alpha, \beta}$ between M_α and M_β agrees with the topological conjugacy between \widehat{M}_α and \widehat{M}_β . Therefore, in order to prove Corollary 1.2, we need to establish the assumptions on Theorem 1.1 when $\beta \rightarrow \alpha$.

(a) Pointwise convergence of the inverse branches of the induced maps can be established since when $\beta \rightarrow \alpha$, we have that $M_\beta(x) \rightarrow M_\alpha(x)$ and the hitting times $r_\beta(x) \rightarrow r_\alpha(x)$ for a fixed $x \in [0, 1]$.

(b) Now for the tail behaviour, that is, condition (1) in Theorem 1.1, we will cite Sarig [30] and in particular the proof of Proposition 1 there, where it is proved that if f_i are the inverse branches of \widehat{M}_α , then for any $0 < \alpha < \infty$ there exists $t(\alpha) > 0$ with

$$\sum_{i=1}^{\infty} |f_i[0, 1]|^{t(\alpha)} < \infty.$$

(c) Finally, the variations will be uniformly bounded. Fix any $\varepsilon > 0$ such that $\alpha - \varepsilon > 0$. For $\beta > 0$, write

$$\varphi_\beta(\mathbf{i}) := -\log |\widehat{M}'_\beta(\pi_{\widehat{M}_\beta}(\mathbf{i}))|,$$

where we recall that $\pi_{\widehat{M}_\beta}$ maps cylinders $[\mathbf{i}]$ onto intervals $I_{\mathbf{i}}^{\widehat{M}_\beta}$. Then to check the uniform bound (2) in Theorem 1.1 on variations, we will need to establish

$$\sup_{\beta \in I(\alpha)} \sum_{n=1}^{\infty} \text{var}_n(\varphi_\beta) < \infty,$$

where $I(\alpha) := [\alpha - \varepsilon, \alpha + \varepsilon] \subset (0, \infty)$ as this yields the assumption (2) in Theorem 1.1 for all sequences \widehat{M}_{β_k} , where $\beta_k \rightarrow \alpha$ as $k \rightarrow \infty$. To do this, we just need to check that the mapping $\beta \mapsto \sum_{n=1}^{\infty} \text{var}_n(\varphi_\beta)$ is bounded by a continuous function since the supremum is over a compact interval $I(\alpha)$. This follows from Nakaishi's work [26, Lemmas 2.1 and 2.2] where the following estimate can be established:

$$|\varphi_\beta(\mathbf{j}) - \varphi_\beta(\mathbf{k})| \leq C(\beta)n^{-p(\beta)}$$

for $\mathbf{i} \in \mathbb{N}^n$ and $\mathbf{j}, \mathbf{k} \in [\mathbf{i}]$ and so $\text{var}_n(\varphi_\beta) \leq C(\beta)n^{-p(\beta)}$. Here the constants $C(\beta) > 0$ and $p(\beta) > 1$ depend continuously on the parameter β . Hence $\sum_{n=1}^{\infty} \text{var}_n(\varphi_\beta) \leq C(\beta)\zeta(p(\beta))$, where ζ is the Riemann zeta function. Thus the sum is bounded by a continuous function of β , which is what we wanted.

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