A METRIZABLE TOPOLOGY ON THE CONTRACTING BOUNDARY OF A GROUP

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Abstract. The 'contracting boundary' of a proper geodesic metric space consists of equivalence classes of geodesic rays that behave like rays in a hyperbolic space. We introduce a geometrically relevant, quasi-isometry invariant topology on the contracting boundary. When the space is the Cayley graph of a finitely generated group we show that our new topology is metrizable.

1. Introduction

There is a long history in geometry of attaching a 'boundary at infinity' or 'ideal boundary' to a space. When a group acts geometrically on a space we might wonder to what extent the group and the boundary of the space are related. In the setting of (Gromov) hyperbolic groups this relationship is very strong: the boundary is determined by the group, up to homeomorphism. In particular, the boundaries of all Cayley graphs of a hyperbolic group are homeomorphic, so it makes sense to call any one of these boundaries the boundary *of the group*. This is not true, for example, in the case of a group acting geometrically on a non-positively curved space: Croke and Kleiner [19] gave an example of a group acting geometrically on two different CAT(0) spaces with non-homeomorphic visual boundaries, so there is not a well-defined visual boundary associated to the group.

Charney and Sultan [14] sought to rectify this problem by defining a 'contracting boundary' for CAT(0) spaces. Hyperbolic boundaries and visual boundaries of CAT(0) spaces can be constructed as equivalence classes of geodesic rays emanating from a fixed basepoint. These represent the metrically distinct ways of 'going to infinity'. Charney and Sultan's idea was to restrict attention to ways of going to infinity in hyperbolic directions: They consider equivalence classes of geodesic rays that are 'contracting', which is a way of quantifying how hyperbolic such rays are. They topologize the resulting set using a direct limit construction, and show that this topology is preserved by quasi-isometries. However, their construction has drawbacks: basically, it has too many open sets. In general it is not first countable.

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In this paper we define a bordification of a proper geodesic metric space by adding a contracting boundary with a quasi-isometry invariant topology. When the space is a Cayley graph of a finitely generated group, we prove that the topology on the boundary is metrizable, which is a significant improvement over the direct limit topology. (See Example 1.1 for a motivating example.) Furthermore, our topology more closely resembles the topology of the boundary of a hyperbolic space, which we hope will make it easier to work with.

Our contracting boundary consists of equivalence classes of 'contracting quasi-geodesics'. The definition of contraction we use follows that of Arzhantseva, Cashen, Gruber, and Hume [5]; this is weaker than that of Charney and Sultan, so our construction applies to more general spaces. For example, we get contracting quasi-geodesics from cyclic subgroups generated by non-peripheral elements of relatively hyperbolic groups [23], pseudo-Anosov elements of mapping class groups [32, 24], fully irreducible free group automorphisms [2], and generalized loxodromic elements of acylindrically hyperbolic groups [36, 20, 8, 39]. On CAT(0) spaces the two definitions agree, so our boundary is the same as theirs *as a set*, but our topology is coarser.

Cordes [15] has defined a 'Morse boundary' for proper geodesic metric spaces by applying Charney and Sultan's direct limit construction to the set of equivalence classes of Morse geodesic rays. This boundary has been further studied by Cordes and Hume [18], who relate it to the notion of 'stable subgroups' introduced by Durham and Taylor [25]; for a recent survey of these developments[1], see Cordes [16]. It turns out that our notion of contracting geodesic is equivalent to the Morse condition, and our contracting boundary agrees with the Morse boundary as a set, but, again, our topology is coarser.

If the underlying space is hyperbolic then all of these boundaries are homeomorphic to the Gromov boundary. At the other extreme, all of these boundaries are empty in spaces with no hyperbolic directions. In particular, it follows from work of Drutu and Sapir [23] that groups that are *wide*, that is, no asymptotic cone contains a cut point, will have empty contracting boundary. This includes groups satisfying a law: for instance, solvable groups or bounded torsion groups.

The boundary of a proper hyperbolic space can be topologized as follows. If ζ is a point in the boundary, an equivalence class of geodesic rays issuing from the chosen basepoint, we declare a small neighborhood of ζ to consist of boundary points η such that if $\alpha \in \zeta$ and $\beta \in \eta$ are representative geodesic rays then β closely fellow-travels α for a long time. In proving that this topology is invariant under quasi-isometries, hyperbolicity is used at two key points. The first is that quasi-isometries take geodesic rays uniformly close to geodesic rays. In general a quasi-isometry only takes a geodesic ray to a quasi-geodesic ray, but hyperbolicity implies that this is within bounded distance of a geodesic ray, with bound depending only on the quasi-isometry and hyperbolicity constants. The second use of hyperbolicity is to draw a clear distinction between fellow-travelling and not, which is used to show that the time for which two geodesics fellow-travel is roughly preserved by quasi-isometries. If α and β are non-asymptotic geodesic rays issuing from a common

[1]In even more recent developments, Behrstock [6] produces interesting examples of right-angled Coxeter groups whose Morse boundaries contain a circle, and Charney and Murray [13] give conditions that guarantee that a homeomorphism between Morse boundaries of CAT(0) spaces is induced by a quasi-isometry.

basepoint in a hyperbolic space, then closest point projection sends β to a bounded subset $\alpha([0, T_0])$ of α, and there is a transition in the behavior of β at time T_0. For $t < T_0$ the distance from $\beta(t)$ to α is bounded and the diameter of the projection of $\beta([0, t])$ to α grows like t. After this time β escapes *quickly* from α, that is, $d(\beta(t), \alpha)$ grows like $t - T_0$, and the diameter of the projection of $\beta([T_0, t])$ is bounded.

We recover the second point for non-hyperbolic spaces using the contraction property. Our definition of a *contracting* set Z, see Definition 3.2, is that the diameter of the projection of a ball tangent to Z is bounded by a function of the radius of the ball whose growth rate is less than linear. Essentially this means that sets far from Z have large diameter compared to the diameter of their projection. In contrast to the hyperbolic case, it is not true, in general, that if α is a contracting geodesic ray and β is a geodesic ray not asymptotic to α then β has bounded projection to α. However, we *can* still characterize the escape of β from α by the relation between the growth of the projection of $\beta([0, t])$ to α and the distance from $\beta(t)$ to α. The main technical tool we introduce is a divagation estimate that says if α is contracting and β is a quasi-geodesic then β cannot wander slowly away from α; if it is to escape, it must do so quickly. More precisely, once β exceeds a threshold distance from α, depending on the quasi-geodesic constants of β and the contraction function for α, then the distance from $\beta(t)$ to α grows superlinearly compared to the growth of the projection of $\beta([0, t])$ to α. In fact, for the purpose of proving that fellow-travelling time is roughly preserved by quasi-isometries it will be enough to know that the this relationship is at least a fixed linear function.

The first point cannot be recovered, and, in fact, the topology as described above, using only geodesic rays, is not quasi-isometry invariant for non-hyperbolic spaces [12]. Instead, we introduce a finer topology that we call the *topology of fellow-travelling quasi-geodesics*. The idea is that η is close to ζ if all *quasi*-geodesics tending to η closely fellow-travel quasi-geodesics tending to ζ for a long time. See Definition 5.3 for a precise definition. Using our divagation estimates we show that this topology is quasi-isometry invariant.

The use of quasi-geodesic rays in our definition is quite natural in the setting of coarse geometry, since then the rays under consideration do not depend on the choice of a particular metric within a fixed quasi-isometry class. Geodesics, on the other hand, are highly sensitive to the choice of metric, and it is only the presence of a very strong hypothesis like global hyperbolicity that allows us to define a quasi-isometry invariant boundary topology using geodesics alone.

Example 1.1. Consider $H := \langle a, b \mid [a, b] = 1\rangle * \langle c\rangle$, which can be thought of as the fundamental group of a flat, square torus wedged with a circle. Let X be the universal cover, with basepoint o above the wedge point.

Connected components of the preimage of the torus are Euclidean planes isometrically embedded in X. Geodesic segments contained in such a plane behave more like Euclidean geodesics than hyperbolic geodesics. In fact, a geodesic ray α based at o is contracting if and only if there exists a bound B_α such that α spends time at most time B_α in any one of the planes. Let $\alpha(\infty)$ denote the equivalence class of this ray as a point in the contracting boundary.

In Charney and Sultan's topology, if $(\alpha^n)_{n\in\mathbb{N}}$ is a sequence of contracting geodesic rays with the B_{α^n} unbounded, then $(\alpha^n(\infty))$ is not a convergent sequence in the contracting boundary. Murray [33] uses this fact to show that the contracting boundary is not first countable.

In the topology of fellow-travelling quasi-geodesics it will turn out that $(\alpha^n(\infty))$ converges if and only if there exists a contracting geodesic α in X such that the projections of the α^n to geodesics in the Bass-Serre tree of H (with respect to the given free product splitting of H) converge to the projection of α.

From another point of view, H is hyperbolic relative to the Abelian subgroup $A := \langle a, b\rangle$. We show in Theorem 7.6 that this implies that there is a natural map from the contracting boundary of H to the Bowditch boundary of the pair (H, A), and, with the topology of fellow-travelling quasi-geodesics, that this map is a topological embedding. The embedding statement cannot be true for Charney and Sultan's topology, since it is not first countable.

After some preliminaries in Section 2, we define the contraction property and recall/prove some basic technical results in Section 3 concerning the behavior of geodesics relative to contracting sets. In Section 4 we extend these results to quasi-geodesics, and derive the key divagation estimates, see Corollary 4.3 and Lemma 4.6.

In Section 5 introduce the topology of fellow-travelling quasi-geodesics and show that it is first countable, Hausdorff, and regular. In Section 6 we prove that it is also quasi-isometry invariant.

We compare other possible topologies in Section 7.

In Section 8 we consider the case of a finitely generated group. In this case we prove that the contracting boundary is second countable, hence metrizable.

We also prove a weak version of North-South dynamics for the action of a group on its contracting boundary in Section 9, in the spirit of Murray's work [33].

Finally, in Section 10 we show that the contracting boundary of an infinite, finitely generated group is non-empty and compact if and only if the group is hyperbolic.

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2. Preliminaries

Let X be a metric space with metric d. For $Z \subset X$, define:

- $N_r Z := \{x \in X \mid \exists z \in Z,\, d(z, x) < r\}$
- $N_r^c Z := \{x \in X \mid \forall z \in Z,\, d(z, x) \geq r\}$
- $\bar{N}_r Z := \{x \in X \mid \exists z \in Z,\, d(z, x) \leq r\}$
- $\bar{N}_r^c Z := \{x \in X \mid \forall z \in Z,\, d(z, x) > r\}$

For $L \geq 1$ and $A \geq 0$, a map $\phi\colon (X, d_X) \to (X', d_{X'})$ is an (L, A)*–quasi-isometric embedding* if for all $x, y \in X$:

$$\frac{1}{L} d_X(x, y) - A \leq d_{X'}(\phi(x), \phi(y)) \leq L d_X(x, y) + A$$

If, in addition, $\bar{N}_A \phi(X) = X'$ then ϕ is an (L, A)*–quasi-isometry*. A *quasi-isometry inverse* $\bar{\phi}$ of a quasi-isometry $\phi\colon X \to X'$ is a quasi-isometry $\bar{\phi}\colon X' \to X$ such that the compositions $\phi \circ \bar{\phi}$ and $\bar{\phi} \circ \phi$ are both bounded distance from the identity map on the respective space.

A *geodesic* is an isometric embedding of an interval. A *quasi-geodesic* is a quasi-isometric embedding of an interval. If $\alpha\colon I \to X$ is a quasi-geodesic, we often use α_t to denote $\alpha(t)$, and conflate α with its image in X. When I is of the form $[a, b]$ or $[a, \infty)$ we will assume, by precomposing α with a translation of the domain, that $a = 0$. We use $\alpha + \beta$ and $\bar{\alpha}$ to denote concatenation and reversal, respectively.

A metric space is *geodesic* if every pair of points can be connected by a geodesic.

A metric space if *proper* if closed balls are compact.

It is often convenient to improve quasi-geodesics to be continuous, which can be accomplished by the following lemma.

Lemma 2.1 (Taming quasi-geodesics [11, Lemma III.H.1.11])**.** *If X is a geodesic metric space and $\gamma\colon [a,b] \to X$ is an (L,A)–quasi-geodesic then there exists a continuous $(L, 2(L+A))$–quasi-geodesic γ' such that $\gamma_a = \gamma'_a$, $\gamma_b = \gamma'_b$ and the Hausdorff distance between γ and γ' is at most $L+A$.*

Proof. Define γ' to agree with γ at the endpoints and at integer points of $[a,b]$, and then connect the dots by geodesic interpolation. □

A subspace Z of a geodesic metric space X is A*–quasi-convex* for some $A \geq 0$ if every geodesic connecting points in Z is contained in $\bar{N}_A Z$.

If f and g are functions then we say $f \preceq g$ if there exists a constant $C > 0$ such that $f(x) \leq Cg(Cx+c)+C$ for all x. If $f \preceq g$ and $g \preceq f$ then we write $f \asymp g$.

We will give a detailed account of the contracting property in the next section, but let us first take a moment to recall alternate characterizations, which will prove useful later in the paper.

A subspace Z of a metric space X is μ*–Morse* for some $\mu : [1,\infty) \times [0,\infty) \to \mathbb{R}$ if for every $L \geq 1$ and every $A \geq 0$, every (L,A)–quasi-geodesic with endpoints in Z is contained in $\bar{N}_{\mu(L,A)}Z$. We say Z is *Morse* if there exists μ such that it is μ–Morse. It is easy to see that the property of being Morse is invariant under quasi-isometries. In particular, a subset of a finitely generated group G is Morse in one Cayley graph of G if and only if it is Morse in every Cayley graph of G. Thus, we can speak of a *Morse subset* of G without specifying a finite generating set.

A set Z is called t*–recurrent*[2], for $t \in (0, 1/2)$, if for every $C \geq 1$ there exists $D \geq 0$ such that if p is a path with endpoints x and y on Z such that the ratio of the length of p to the distance between its endpoints is at most C, then there exists a point $z \in Z$ such that $d(p,z) \leq D$ and $\min\{d(z,x), d(z,y)\} \geq td(x,y)$. The set Z is called *recurrent* if it is t–recurrent for every $t \in (0,1/2)$.

Theorem 2.2. *Let Z be a subset of a geodesic metric space X. The following are equivalent:*

1. *Z is Morse.*
2. *Z is contracting.*
3. *Z is recurrent.*
4. *There exists $t \in (0,1/2)$ such that Z is t-recurrent.*

Moreover, each of the equivalences are 'effective', in the sense that the defining function of one property determines the defining functions of each of the others.

Proof. The equivalence of (1) and (2) is proved in [5]. That (3) implies (4) is obvious. The implications '(2) implies (3)' and '(4) implies (1)' are proved in [3] for the case that Z is a quasi-geodesic, but their proofs go through with minimal change for arbitrary subsets Z. □

[2]This characterization was introduced in [22] with $t = 1/3$ for Z a quasi-geodesic. The idea is that a short curve must pass near the 'middle third' of the subsegment of Z connecting its endpoints. The property, again only for quasi-geodesics, but for variable t, is called 'middle recurrence' in [3].

3. Contraction

Definition 3.1. We call a function ρ *sublinear* if it is non-decreasing, eventually non-negative, and $\lim_{r\to\infty}\rho(r)/r = 0$.

Definition 3.2. Let X be a proper geodesic metric space. Let Z be a closed subset of X, and let $\pi_Z\colon X \to 2^Z : x \mapsto \{z \in Z \mid d(x,z) = d(x,Z)\}$ be closest point projection to Z. Then, for a sublinear function ρ, we say that Z is *ρ–contracting* if for all x and y in X:

$$d(x,y) \leq d(x,Z) \implies \operatorname{diam} \pi_Z(x) \cup \pi_Z(y) \leq \rho(d(x,Z))$$

We say Z is *contracting* if there exists a sublinear function ρ such that Z is ρ–contracting. We say a collection of subsets $\{Z_i\}_{i\in\mathcal{I}}$ is *uniformly contracting* if there exists a sublinear function ρ such that for every $i \in \mathcal{I}$ the set Z_i is ρ–contracting.

We shorten π_Z to π when Z is clear from context.

Let us stress that the closest point projection map is set-valued, and there is no bound on the diameter of image sets other than that implied by the definition.

In a tree every convex subset is ρ–contracting where ρ is identically 0. More generally, in a hyperbolic space a set is contracting if and only if it is quasi-convex. In fact, in this case more is true: the contraction function is bounded in terms of the hyperbolicity and quasi-convexity constants. We call a set *strongly contracting* if it is contracting with bounded contraction function.

The more general Definition 3.2 was introduced by Arzhantseva, Cashen, Gruber, and Hume to characterize Morse geodesics in small cancellation groups [4].

The concept of strong contraction (sometimes simply called 'contraction' in the literature) has been studied before, notably by Minsky [32] to describe axes of pseudo-Anosov mapping classes in Teichmüller space, by Bestvina and Fujiwara [9] to describe axes of rank-one isometries of CAT(0) spaces (see also Sultan [40]), and by Algom-Kfir [2] to describe axes of fully irreducible free group automorphisms acting on Outer Space.

Masur and Minsky [31] introduced a different notion of contraction that requires the existence of constants A and B such that:

$$d(x,y) \leq d(x,Z)/A \implies \operatorname{diam} \pi_Z(x) \cup \pi_Z(y) \leq B$$

This is satisfied, for example, by axes of pseudo-Anosov elements in the mapping class group (as opposed to Teichmüller space). Some authors refer to this property as 'contraction', eg [7, 24, 1]. It is not hard to show that this version implies the version in Definition 3.2 with the contraction function ρ being logarithmic.

We now recall some further results about contracting sets in a geodesic metric space X.

Lemma 3.3 ([5, Lemma 6.3])**.** *Given a sublinear function ρ and a constant $C \geq 0$ there exists a sublinear function $\rho' \asymp \rho$ such that if $Z \subset X$ and $Z' \subset X$ have Hausdorff distance at most C and Z is ρ–contracting then Z' is ρ'–contracting.*

Theorem 3.4 (Geodesic Image Theorem [5, Theorem 7.1])**.** *For $Z \subset X$, there exists a sublinear function ρ so that Z is ρ–contracting if and only if there exists a sublinear function ρ' and a constant κ_ρ so that for every geodesic segment γ, with endpoints denoted x and y, if $d(\gamma, Z) \geq \kappa_\rho$ then $\operatorname{diam}\pi(\gamma) \leq \rho'(\max\{d(x,Z),\, d(y,Z)\})$. Moreover ρ' and κ_ρ depend only on ρ and vice-versa, with $\rho' \asymp \rho$.*

An easy consequence is that there exists a κ'_ρ such that if γ is a geodesic segment with endpoints at distance at most κ_ρ from a ρ–contracting set Z then $\gamma \subset \bar{N}_{\kappa'_\rho}(Z)$.

The following is a special case of [5, Proposition 8.1].

Lemma 3.5. *Given a sublinear function ρ and a constant $C \geq 0$ there exists a constant B such that if α and β are ρ–contracting geodesics such that their initial points α_0 and β_0 satisfy $d(\alpha_0, \beta_0) = d(\alpha, \beta) \leq C$ then $\alpha \cup \beta$ is B–quasi-convex.*

The next two lemmas are easy-to-state generalizations of results that are known for strong contraction. The proofs are rather tedious, due to the weak hypotheses, so we postpone them until after Lemma 3.8.

Lemma 3.6. *Given a sublinear function ρ there is a sublinear function $\rho' \asymp \rho$ such that every subsegment of a ρ–contracting geodesic is ρ'–contracting.*

Lemma 3.7. *Given a sublinear function ρ there is a sublinear function $\rho' \asymp \rho$ such that if α and β are ρ–contracting geodesic rays or segments such that $\gamma := \bar{\alpha} + \beta$ is geodesic, then γ is ρ'–contracting.*

Given $C \geq 0$ a *geodesic C–almost triangle* is a trio of geodesics $\alpha^i \colon [a_i, b_i] \to X$, for $i \in \{0, 1, 2\}$ and $a_i \leq 0 \leq b_i \in \mathbb{R} \cup \{-\infty, \infty\}$, such that for each $i \in \{0, 1, 2\}$, with scripts taken modulo 3, we have:

- $b_i < \infty$ if and only if $a_{i+1} > -\infty$.
- If b_i and a_{i+1} are finite then $d(\alpha^i_{b_i}, \alpha^{i+1}_{a_{i+1}}) \leq C$.
- If b_i and a_{i+1} are not finite then $\alpha^i_{[0,\infty)}$ and $\bar{\alpha}^{i+1}_{[0,\infty)} = \alpha^{i+1}_{(-\infty,0]}$ are asymptotic.

Lemma 3.8. *Given a sublinear function ρ and constant $C \geq 0$ there is a sublinear function $\rho' \asymp \rho$ such that if α, β, and γ are a geodesic C–almost triangle and α and β are ρ–contracting then γ is ρ'–contracting.*

Proof. First suppose α, β, and γ are segments. By Lemma 3.5, there exists a B depending only on ρ and C such that $\alpha \cup \beta$ is B–quasi–convex. Thus, we can replace $\alpha \cup \beta$ by a single geodesic segment δ whose endpoints are C–close to the endpoints of γ. Furthermore, δ is a union of two subsegments, one of which has endpoints within distance B of α, and the other of which has endpoints within distance B of β. Consequently, by Theorem 3.4 there exists B' so that these two subsegments are B'–Hausdorff equivalent to subsegments of α and of β, respectively. Applying Lemma 3.6, Lemma 3.3, and Lemma 3.7, there is a $\rho'' \asymp \rho$ depending on ρ and B' such that δ is ρ''–contracting. Theorem 3.4 implies that since γ and δ are close at their endpoints, they stay close along their entire lengths, so their Hausdorff distance is determined by ρ'' and C, hence by ρ and C. Applying Lemma 3.3 again, we conclude γ is ρ'–contracting with $\rho' \asymp \rho'' \asymp \rho$ depending only on ρ and C.

In the case of an ideal triangle, where not all three sides are segments, replace C by $\max\{C, \kappa_\rho\}$. Theorem 3.4 implies that if, say, γ and $\bar{\alpha}$ have asymptotic tails then the set of points γ that come κ_ρ–close to α is unbounded. Truncate the triangle at such a C–close pair of points. Doing the same for other ideal vertices, we get a C–almost triangle to which we can apply the previous argument and conclude that a subsegment of γ is ρ'–contracting. Since γ comes κ_ρ–close to α on an unbounded set, we can repeat the argument for larger and larger almost triangles approximating α, β, γ, and find that every subsegment of γ is contained in a ρ'–contracting subsegment, which implies that γ itself is ρ'–contracting. ∎

Definition 3.9. If Z is a subset of $\mathbb{R}$ define the *interval of* Z, $\mathrm{invl}(Z)$, to be the smallest closed interval containing Z. If $\gamma\colon I \to X$ is a geodesic and Z is a subset of γ let $\mathrm{invl}(Z) := \gamma(\mathrm{invl}(\gamma^{-1}(Z)))$.

Proof of Lemma 3.6. Let $\gamma\colon I \to X$ be a ρ–contracting geodesic. Let $J := [j_0, j_1]$ be a subinterval of I. Let $\rho'' \asymp \rho$ be the function given by Theorem 3.4, and let κ'_ρ be the constant defined there. We claim it suffices to take $\rho'(r) := 2(2\kappa'_\rho + \rho''(2r) + \rho(2r))$.

First we show that if $\pi_{\gamma_I}(x)$ misses γ_J then $\pi_{\gamma_J}(x)$ is relatively close to one of the endpoints of γ_J. This is automatic if $\mathrm{diam}\,\gamma_J \leq \rho(d(x,\gamma_J))$, so assume not. With this assumption, $\pi_{\gamma_I}(x)$ cannot contain points on both sides of γ_J, that is, if $\gamma^{-1}(\pi_{\gamma_I}(x))$ contains a point less than j_0 then it does not also contain one greater than j_1, and vice versa. Suppose that $\gamma^{-1}(\pi_{\gamma_I}(x))$ is contained in $(-\infty, j_0)$. Let β be a geodesic from x to a point y in $\pi_{\gamma_J}(x)$. There exists a first time s such that $d(\beta_s, \gamma_I) = \kappa_\rho$. By Theorem 3.4, $\mathrm{diam}\,\pi_{\gamma_I}(\beta|_{[0,s]}) \leq \rho''(d(x,\gamma_I))$. Suppose that $\gamma_{j_0} \in \mathrm{invl}(\pi_{\gamma_I}(\beta|_{[0,s]}))$. Then there is a first time $s' \in [0,s]$, such that $\pi_I(\beta_{s'})$ contains a point in $\gamma_{[j_0,\infty)}$. By the assumption on the diameter of γ_J, we actually have $\pi_{\gamma_I}(\beta_{s'}) \cap \gamma_J \neq \emptyset$, so $y \in \pi_{\gamma_J}(\beta_{s'}) \subset \pi_{\gamma_I}(\beta_{s'}) \subset \pi_{\gamma_I}(\beta|_{[0,s]})$ and $\mathrm{diam}\,\gamma_{j_0} \cup \pi_{\gamma_J}(x) \leq \rho''(d(x,\gamma_I))$. Otherwise, if $\gamma_{j_0} \notin \mathrm{invl}(\pi_{\gamma_I}(\beta|_{[0,s]}))$, then let $t > s$ be the first time such that $\gamma_{j_0} \in \mathrm{invl}\,\pi_{\gamma_I}(\beta|_{[s,t]})$. Again, $y \in \pi_{\gamma_I}(\beta_t)$. Since the points of β after β_s, are contained in $\bar{N}_{\kappa'_\rho}\gamma$, for all small $E > 0$ we have $\mathrm{diam}\,\pi_{\gamma_I}\beta_{t-E} \cup \pi_{\gamma_I}\beta_t \leq E + 2\kappa'_\rho$. Therefore, $d(\gamma_{j_0}, y) \leq d(y, \pi_{\gamma_I}(\beta_{t-E})) \leq E + 2\kappa'_\rho$, for all sufficiently small E. We conclude:

$$\mathrm{diam}\,\gamma_{j_0} \cup \pi_{\gamma_J}(x) \leq \max\{2\kappa'_\rho, \rho''(d(x,\gamma_I))\} \tag{1}$$

Now suppose x and y are points such that $d(x,y) \leq d(x,\gamma_J)$. Note that $d(y,\gamma_J) \leq 2d(x,\gamma_J)$. We must show $\mathrm{diam}\,\pi_{\gamma_J}(x) \cup \pi_{\gamma_J}(y)$ is bounded by a sublinear function of $d(x,\gamma_J)$. There are several cases, depending on whether $\pi_{\gamma_I}(x)$ and $\pi_{\gamma_I}(y)$ hit γ_J.

Case 1: $\gamma_J \cap \pi_{\gamma_I}(x) \neq \emptyset$ and $\gamma_J \cap \pi_{\gamma_I}(y) \neq \emptyset$. In this case $\pi_{\gamma_J}(x) \subset \pi_{\gamma_I}(x)$, and likewise for y, so:

$$\mathrm{diam}\,\pi_{\gamma_J}(x) \cup \pi_{\gamma_J}(y) \leq \mathrm{diam}\,\pi_{\gamma_I}(x) \cup \pi_{\gamma_I}(y) \leq \rho(d(x,\gamma_I)) = \rho(d(x,\gamma_J))$$

Case 2: $\gamma^{-1}(\pi_{\gamma_I}(x)) < j_0$ and $\gamma^{-1}(\pi_{\gamma_I}(y)) < j_0$. By (1) twice:

$$\begin{aligned}\mathrm{diam}\,\pi_{\gamma_J}(x) \cup \pi_{\gamma_J}(y) &\leq 2\max\{2\kappa'_\rho, \rho''(d(x,\gamma_I)), \rho''(d(y,\gamma_I))\}\\ &\leq 2\max\{2\kappa'_\rho, \rho''(2d(x,\gamma_J))\}\end{aligned}$$

Case 3: $\gamma^{-1}(\pi_{\gamma_I}(y)) < j_0$ and $\gamma_J \cap \pi_{\gamma_I}(x) \neq \emptyset$. In this case $\pi_{\gamma_J}(x) \subset \pi_{\gamma_I}(x)$ and $d(x,y) \leq d(x,\gamma_J) = d(x,\gamma_I)$, so $\mathrm{diam}\,\pi_{\gamma_I}(y) \cup \pi_{\gamma_J}(x) \leq \rho(d(x,\gamma_J))$. By hypothesis, $\gamma_{j_0} \in \mathrm{invl}(\pi_{\gamma_I}(y) \cup \pi_{\gamma_J}(x))$, and by (1): $\mathrm{diam}\,\gamma_{j_0} \cup \pi_{\gamma_J}(y) \leq \max\{2\kappa'_\rho, \rho''(2d(x,\gamma_J))\}$. Thus:

$$\mathrm{diam}\,\pi_{\gamma_J}(x) \cup \pi_{\gamma_J}(y) \leq \max\{\rho(d(x,\gamma_J)), 2\kappa'_\rho, \rho''(2d(x,\gamma_J))\}$$

Case 4: $\gamma^{-1}(\pi_{\gamma_I}(x)) < j_0$ and $\pi_{\gamma_I}(y) \cap \gamma_{[j_0,\infty)} \neq \emptyset$. If $j_1 - j_0 \leq 2\rho(2d(x,\gamma_J))$ then there is nothing more to prove, so assume not. Let β be a geodesic from x to y. For all $z \in \beta$:

$$d(x,z) + d(z,y) = d(x,y) \leq d(x,\gamma_J) \leq d(x,z) + d(z,\gamma_J)$$

This implies $d(z,y) \leq d(z,\gamma_J)$. Let z be the first point on β such that $\gamma^{-1}(\pi_{\gamma_I}(z))$ contains a point greater than or equal to j_0. By the hypothesis on $|J|$, $\gamma^{-1}(\pi_{\gamma_I}(z)) <$

j_1. This means $\operatorname{diam}\pi_{\gamma_J}(z)\cup\pi_{\gamma_J}(y)$ is controlled by one of the previous cases, and it suffices to control $\operatorname{diam}\pi_{\gamma_J}(x)\cup\pi_{\gamma_J}(z)$.

We know from (1) that $\pi_{\gamma_J}(x)$ is $\max\{2\kappa'_\rho, \rho''(d(x,\gamma_J))\}$-close to γ_{j_0}, so it suffices to control $\operatorname{diam}\gamma_{j_0}\cup\pi_{\gamma_J}(z)$. Take a point $w\neq z$ on β before z such that $d(z,w)\leq d(z,\gamma_I)$. By hypothesis, $\gamma_{j_0}\in\operatorname{invl}\pi_{\gamma_I}(w)\cup\pi_{\gamma_I}(z)$, but $\operatorname{diam}\pi_{\gamma_I}(w)\cup\pi_{\gamma_I}(z)\leq\rho(d(z,\gamma_I))=\rho(d(z,\gamma_J))\leq\rho(2d(x,\gamma_J))$.

Up to symmetric arguments, this exhausts all the cases. □

Proof of Lemma 3.7. Let α and β be ρ–contracting geodesic segments or rays with $\alpha_0=\beta_0$ such that $\gamma:=\bar\alpha+\beta$ is geodesic.

First suppose that x is a point such that $\pi_\gamma(x)\cap\alpha\neq\emptyset$ and $\pi_\gamma(x)\cap\beta\neq\emptyset$. Let δ be a geodesic from x to $\alpha_0=\beta_0$. Recall from Theorem 3.4 that once δ enters the κ_ρ–neighborhood of either α or β then it cannot leave the κ'_ρ–neighborhood. Thus, δ intersects at most one of $\bar N_{\kappa_\rho}\alpha\backslash N_{2\kappa'_\rho}\alpha_0$ or $\bar N_{\kappa_\rho}\beta\backslash N_{2\kappa'_\rho}\beta_0$. Without loss of generality, suppose δ does not intersect $\bar N_{\kappa_\rho}\beta\setminus N_{2\kappa'_\rho}\beta_0$. Let t be the first time such that $d(\delta_t,\beta)=\kappa_\rho$. Then $d(\delta_t,\beta_0)\leq 2\kappa'_\rho$ and, by Theorem 3.4, there is a sublinear $\rho''\asymp\rho$ such that $\operatorname{diam}\pi_\beta(\delta|_{[0,t]})\leq\rho''(d(x,\beta))=\rho''(d(x,\gamma))$. In particular, this means $\operatorname{diam}\pi_\beta(x)\cup\beta_0\leq\operatorname{diam}\pi_\beta(\delta)\leq 4\kappa'_\rho+\rho''(d(x,\gamma))$. Now let δ' be a geodesic from x to a point $x'\in\pi_\beta(x)$, and project δ' to α. Since $\bar\alpha+\beta$ is geodesic, $\operatorname{diam}\pi_\alpha(x)\cup\alpha_0\leq\operatorname{diam}\pi_\alpha\delta'\leq\rho''(\max\{d(x,\alpha),d(x',\alpha)\})$ by Theorem 3.4. We have already established that $d(x',\alpha)\leq 4\kappa'_\rho+\rho''(d(x,\gamma))$. Since ρ'' grows sublinearly, $d(x,\alpha)>4\kappa'_\rho+\rho''(d(x,\gamma))$ except for $d(x,\alpha)$ less than some bound depending only on ρ and ρ''. We conclude that there is a sublinear function $\rho'''\asymp\rho$ depending only on ρ such that $\operatorname{diam}\pi_\alpha(x)\cup\alpha_0\leq\rho'''(d(x,\gamma))$ and $\operatorname{diam}\pi_\beta(x)\cup\beta_0\leq\rho'''(d(x,\gamma))$, hence $\operatorname{diam}\pi_\gamma(x)\leq 2\rho'''(d(x,\gamma))$.

Now suppose x, $y\in X$ are points such that $d(x,y)\leq d(x,\gamma)$. There are several cases according to where $\pi_\gamma(x)$ and $\pi_\gamma(y)$ lie.

Case 1: $\pi_\gamma(x)\cap\alpha\neq\emptyset\neq\pi_\gamma(y)\cap\alpha$. Then $d(x,y)\leq d(x,\gamma)=d(x,\alpha)$, so contraction for α implies $\operatorname{diam}\pi_\alpha(x)\cup\pi_\alpha(y)\leq\rho(d(x,\alpha))=\rho(d(x,\gamma))$. There are four sub-cases to check, according to whether $\pi_\gamma(x)$ and $\pi_\gamma(y)$ hit β. These are easy to check, with the worst bound being $\operatorname{diam}\pi_\gamma(x)\cup\pi_\gamma(y)\leq\rho(d(x,\gamma))+2\rho'''(2d(x,\gamma))$.

Case 2: $\pi_\gamma(x)\cap\beta=\emptyset=\pi_\gamma(y)\cap\alpha$. Let δ be a geodesic from x to y. Let w be the first point on δ such that $\pi_\gamma(w)\cap\beta\neq\emptyset$. Then $d(w,\alpha)=d(w,\beta)=d(w,\gamma)\leq 2d(x,\gamma)$ and $d(y,\beta)=d(y,\gamma)\leq 2d(x,\gamma)$. We can apply that α is ρ-contracting to the pair x,w since $d(x,w)\leq d(x,y)\leq d(x,\gamma)=d(x,\alpha)$. Likewise, we can apply that β is ρ-contracting to w,y since $d(x,w)+d(w,y)=d(x,y)\leq d(x,\gamma)\leq d(x,w)+d(w,\gamma)$ so $d(w,y)\leq d(w,\gamma)$. We conclude:

$$\begin{aligned}\operatorname{diam}\pi_\gamma(x)\cup\pi_\gamma(y)&\leq\operatorname{diam}\pi_\alpha(x)\cup\pi_\alpha(w)+\operatorname{diam}\pi_\gamma(w)\\&\qquad+\operatorname{diam}\pi_\beta(w)\cup\pi_\beta(y)\\&\leq\rho(d(x,\alpha))+2\rho'''(d(w,\gamma))+\rho(d(w,\beta))\\&\leq\rho(d(x,\gamma))+2\rho'''(2d(x,\gamma))+\rho(2d(x,\gamma))\end{aligned}$$

By symmetry these two cases cover all possibilities, so it suffices to define $\rho'(r):=2\rho(2r)+2\rho'''(2r)$. □

4. Contraction and Quasi-geodesics

In this section we explore the behavior of a quasi-geodesic ray based at a point in a contracting set Z. The main conclusion is that such a ray can stay close to Z for an arbitrarily long time, but once it exceeds a certain threshold distance depending on the quasi-geodesic constants and the contraction function then the ray must escape Z at a definite linear rate.

Definition 4.1. Given a sublinear function ρ and constants $L \geq 1$ and $A \geq 0$, define:

$$\kappa(\rho, L, A) := \max\{3A, 3L^2, 1 + \inf\{R > 0 \mid \forall r \geq R,\, 3L^2\rho(r) \leq r\}\}$$

Define:

$$\kappa'(\rho, L, A) := (L^2 + 2)(2\kappa(\rho, L, A) + A)$$

Remark. For the rest of the paper κ and κ' always refer to the functions defined in Definition 4.1. We use them frequently and without further reference.

This definition implies that for $r \geq \kappa(\rho, L, A)$ we have:

$$r - L^2\rho(r) - A \geq \frac{1}{3}r \geq L^2\rho(r) \tag{2}$$

An inspection of the proof of [5, Theorem 7.1] gives that $\kappa(\rho, 1, 0) \geq \kappa_\rho$ and $\kappa'(\rho, 1, 0) \geq \kappa'_\rho$, so the results of the previous section still hold using $\kappa(\rho, 1, 0)$ and $\kappa'(\rho, 1, 0)$. Enlarging the constants lets us give unified proofs for geodesics and quasi-geodesics.

Theorem 4.2 (Quasi-geodesic Image Theorem)**.** *Let $Z \subset X$ be ρ–contracting. Let $\beta\colon [0, T] \to X$ be a continuous (L, A)–quasi-geodesic segment. If $d(\beta, Z) \geq \kappa(\rho, L, A)$ then:*

$$\operatorname{diam} \pi(\beta_0) \cup \pi(\beta_T) \leq \frac{L^2 + 1}{L^2}\,(A + d(\beta_T, Z)) + \frac{L^2 - 1}{L^2} d(\beta_0, Z) + 2\rho(d(\beta_0, Z))$$

The proof generalizes the proof of the Geodesic Image Theorem to work for quasi-geodesics. We typically apply the result when $d(\beta_T, Z) = \kappa(\rho, L, A)$, in which case the theorem says that for fixed ρ, L, and A the projection diameter of β is bounded in terms of $d(\beta_0, Z)$. In particular, when β is geodesic, or, more generally, when $L = 1$, the bound is sublinear in $d(\beta_0, Z)$, and we recover a version of the Geodesic Image Theorem. With a little more work we can prove this stronger statement for quasi-geodesics as well. Although we do not need it in this paper, the stronger version may be of independent interest, so we include a proof at the end of this section (see Theorem 4.8).

Proof of Theorem 4.2. Let $t_0 := 0$. For each $i \in \mathbb{N}$ in turn, let t_{i+1} be the first time such that $d(\beta_{t_i}, \beta_{t_{i+1}}) = d(\beta_{t_i}, Z)$, or set $t_{i+1} = T$ if no such time exists. Let j be the first index such that $d(\beta_{t_j}, \beta_T) \leq d(\beta_{t_j}, Z)$.

$$\begin{aligned}
T &= T - t_j + \sum_{i=0}^{j-1}(t_{i+1} - t_i) \\
&\geq \frac{1}{L}\left(d(\beta_{t_j}, \beta_T) - d(\beta_{t_j}, Z) + \sum_{i=0}^{j}(d(\beta_{t_i}, Z) - A)\right) \\
&\geq \frac{1}{L}\left(-d(\beta_T, Z) + \sum_{i=0}^{j}(d(\beta_{t_i}, Z) - A)\right)
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\frac{T}{L} - A &\leq d(\beta_0, \beta_T) \\
&\leq d(\beta_0, Z) + \operatorname{diam} \pi(\beta_0) \cup \pi(\beta_T) + d(Z, \beta_T) \\
&\leq d(\beta_0, Z) + d(\beta_T, Z) + \sum_{i=0}^{j} \rho(d(\beta_{t_i}, Z))
\end{aligned}$$

Combining these gives:

$$\begin{aligned}
&\sum_{i=1}^{j} \big(d(\beta_{t_i}, Z) - L^2\rho(d(\beta_{t_i}, Z)) - A\big) \\
&\qquad\qquad \leq d(\beta_T, Z) + L^2\left(A + d(\beta_0, Z) + d(\beta_T, Z)\right) \\
&\qquad\qquad\qquad - \big(d(\beta_0, Z) - L^2\rho(d(\beta_0, Z)) - A\big)
\end{aligned}$$

By (2), the left-hand side is at least $L^2 \sum_{i=1}^{j} \rho(d(\beta_{t_i}, Z))$, so:

$$\begin{aligned}
\operatorname{diam} \pi(\beta_0) \cup \pi(\beta_T) &\leq \sum_{i=0}^{j} \rho(d(\beta_{t_i}, Z)) \\
&\leq \frac{L^2+1}{L^2}\left(A + d(\beta_T, Z)\right) + \frac{L^2-1}{L^2} d(\beta_0, Z) + 2\rho(d(\beta_0, Z)) \qquad \square
\end{aligned}$$

Corollary 4.3. *Let Z be ρ–contracting and let β be a continuous (L, A)–quasi-geodesic ray with $d(\beta_0, Z) \leq \kappa(\rho, L, A)$. There are two possibilities:*

1. *The set $\{t \mid d(\beta_t, Z) \leq \kappa(\rho, L, A)\}$ is unbounded and β is contained in the $\kappa'(\rho, L, A)$–neighborhood of Z.*
2. *There exists a last time T_0 such that $d(\beta_{T_0}, Z) = \kappa(\rho, L, A)$ and:*

$$(\star) \qquad \forall t, \quad d(\beta_t, Z) \geq \frac{1}{2L}(t - T_0) - 2(A + \kappa(\rho, L, A))$$

Proof. Let $\kappa := \kappa(\rho, L, A)$. Let $[a, b]$ be a maximal interval such that $d(\beta_t, Z) \geq \kappa$ for $t \in [a, b]$ and $d(\beta_a, Z) = d(\beta_b, Z) = \kappa$.

For $t \in [a, b]$ we have $d(\beta_t, Z) \leq \kappa + L \cdot (b - a)/2 + A$. Since β is quasi-geodesic:

$$(b - a) \leq L(A + d(\beta_a, \beta_b)) \leq L(A + 2\kappa + \operatorname{diam} \pi(\beta_a) \cup \pi(\beta_b))$$

Theorem 4.2 implies:

$$\begin{aligned}\operatorname{diam}\pi(\beta_a)\cup\pi(\beta_b) &\leq \frac{L^2+1}{L^2}\,(A+\kappa)+\frac{L^2-1}{L^2}\kappa+\frac{2\kappa}{3L^2}\\ &= \frac{L^2+1}{L^2}A+\frac{6L^2+2}{3L^2}\kappa\end{aligned}$$

Putting these estimates together yields:

$$d(\beta_t, Z) < (L^2+2)(2\kappa+A) = \kappa'(\rho, L, A)$$

Thus, once β leaves the $\kappa'(\rho, L, A)$–neighborhood of Z it can never return to the $\kappa(\rho, L, A)$–neighborhood of Z. If $\{t \mid d(\beta_t, Z) \leq \kappa\}$ is unbounded then β never leaves the $\kappa'(\rho, L, A)$–neighborhood of Z.

Suppose now that there does exist some last time T_0 such that $d(\beta_{T_0}, Z) = \kappa$. Any segment $\beta_{[T_0,t]}$ stays outside $N_\kappa Z$, so apply Theorem 4.2 to see:

$$\begin{aligned}\frac{t-T_0}{L}-A &\leq d(\beta_t, \beta_{T_0})\\ &\leq d(\beta_t, Z)+\operatorname{diam}\pi(\beta_t)\cup\pi(\beta_{T_0})+\kappa\\ &\leq \frac{6L^2-1}{3L^2}d(\beta_t, Z)+\frac{L^2+1}{L^2}(A+\kappa)+\kappa\end{aligned}$$

Thus:

$$d(\beta_t, Z) \geq \frac{3L}{6L^2-1}(t-T_0)-\frac{6L^2+3}{6L^2-1}(A+\kappa) \qquad \square$$

Lemma 4.4. *Suppose α is a continuous, ρ–contracting (L, A)–quasi-geodesic and β is a continuous (L, A)–quasi-geodesic ray such that $d(\alpha_0, \beta_0) \leq \kappa(\rho, L, A)$. If there are r, $s \in [0, \infty)$ such that $d(\alpha_r, \beta_s) \leq \kappa(\rho, L, A)$ then $d_{Haus}(\alpha_{[0,r]}, \beta_{[0,s]}) \leq \kappa'(\rho, L, A)$. If $\alpha_{[0,\infty)}$ and $\beta_{[0,\infty)}$ are asymptotic then their Hausdorff distance is at most $\kappa'(\rho, L, A)$.*

Proof. Corollary 4.3 (1) reduces the asymptotic case to the bounded case and shows that β is contained in $\bar{N}_{\kappa'(\rho,L,A)}\alpha$.

For the other direction, suppose that (a, b) is a maximal open subinterval of the domain of α such that $\alpha_{(a,b)} \cap \pi_\alpha(\beta \cap \bar{N}_{\kappa(\rho,L,A)}\alpha) = \emptyset$. For subsegments of β contained in $\bar{N}_{\kappa(\rho,L,A)}\alpha$ the projection to α has jumps of size at most $2\kappa(\rho, L, A)$. For subsegments of β outside $\bar{N}_{\kappa(\rho,L,A)}\alpha$ the largest possible gap in the projection is bounded by Theorem 4.2 by:

$$\begin{aligned}d(\alpha_a, \alpha_b) &\leq \frac{L^2+1}{L^2}(A+\kappa(\rho, L, A))+\frac{L^2-1}{L^2}\kappa(\rho, L, A)+2\rho(\kappa(\rho, L, A))\\ &\leq \frac{L^2+1}{L^2}(A+\kappa(\rho, L, A))+\frac{L^2-1}{L^2}\kappa(\rho, L, A)+\frac{2\kappa(\rho, L, A)}{3L^2}\\ &= \frac{L^2+1}{L^2}A+\frac{6L^2+2}{6L^2}\cdot 2\kappa(\rho, L, A)\end{aligned} \tag{3}$$

This is greater than $2\kappa(\rho, L, A)$.

In either case, for $c \in (a, b)$ we have:

$$d(\alpha_c, \beta) \leq \kappa(\rho, L, A) + \min\{d(\alpha_a, \alpha_c),\, d(\alpha_b, \alpha_c)\} \tag{4}$$
$$\leq \kappa(\rho, L, A) + A + L\frac{b-a}{2}$$
$$\leq \kappa(\rho, L, A) + A + L\frac{LA + Ld(\alpha_a, \alpha_b)}{2}$$

Substitute (3) into (4) and observe that the resulting bound is less than $\kappa'(\rho, L, A)$, which was defined to be $(L^2 + 2)(A + 2\kappa(\rho, L, A))$. □

Lemma 4.5. *If α is a ρ–contracting geodesic ray and β is a continuous (L, A)–quasi-geodesic ray asymptotic to α with $\alpha_0 = \beta_0$ then β is ρ'–contracting where $\rho' \asymp \rho$ depends only on ρ, L, and A.*

Proof. Lemma 4.4 says the Hausdorff distance between α and β is bounded in terms of ρ, L, and A, so the claim follows from Lemma 3.3. □

The next lemma gives the key divagation estimate, which gives us lower bounds on fellow-travelling distance.

Lemma 4.6. *Let α be a ρ-contracting geodesic ray, and let β be a continuous (L, A)–quasi-geodesic ray with $\alpha_0 = \beta_0 = o$. Given some R and J, suppose there exists a point $x \in \alpha$ with $d(x, o) \geq R$ and $d(x, \beta) \leq J$. Let y be the last point on the subsegment of α between o and x such that $d(y, \beta) = \kappa(\rho, L, A)$. There is a constant $M \leq 2$ and a function $\lambda(\phi, p, q)$ defined for sublinear ϕ, $p \geq 1$, and $q \geq 0$ such that λ is monotonically increasing in p and q and:*

$$d(x, y) \leq MJ + \lambda(\rho, L, A)$$

Thus:

$$d(o, y) \geq R - MJ - \lambda(\rho, L, A)$$

Proof. If $d(x, \beta) \leq \kappa(\rho, L, A)$ then $y = x$ and we are done. Otherwise, let a be the last time such that β_a is $\kappa(\rho, L, A)$–close to α between o and x, and let $y' \in \alpha$ be the last point of α with $d(\beta_a, y') = \kappa(\rho, L, A)$. Note $d(y, x) \leq d(y', x)$.

Now let b be the first time such that $d(\beta_b, x) = J$. The subsegment $\beta_{[a,b]}$ stays outside $N_{\kappa(\rho,L,A)}\alpha$. Pick a geodesic from β_b to x and let w be the first point such that $d(w, \alpha) = \kappa(\rho, 1, 0)$. Pick $z \in \pi(\beta_b)$ and $v \in \pi(w)$, and let $W := d(\beta_b, w)$, $Y := d(y', z)$, $Z := d(z, v)$, and $X := d(v, x)$, see Figure 1.

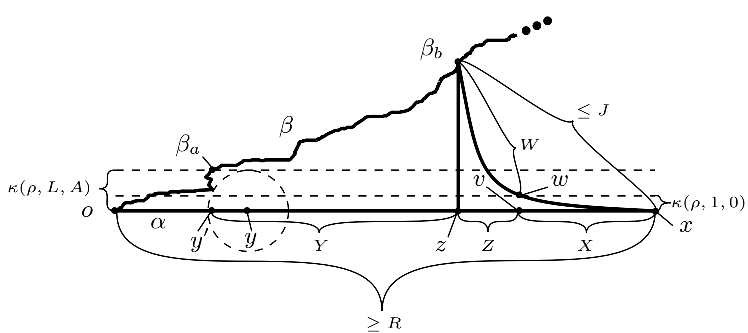

FIGURE 1. Setup for Lemma 4.6

We have $W \geq d(\beta_b, \alpha) - Z - \kappa(\rho, 1, 0)$, and $X \leq J - W + \kappa(\rho, 1, 0)$, so:

$$\begin{aligned} d(y', x) &\leq X + Y + Z \\ &\leq Y + Z + J - d(\beta_b, \alpha) + Z + 2\kappa(\rho, 1, 0) \end{aligned}$$

Apply Theorem 4.2 to the subsegment of γ between γ_a to γ_b to bound Y. Apply Theorem 4.2 to the subsegment of the chosen geodesic from β_b to x between β_b and w to bound Z. (Note in the latter case that we are applying Theorem 4.2 to a geodesic, so use $L = 1$ and $A = 0$ for this case.) Combining these bounds for Y and Z with the bound on $d(x, y')$ above yields:

$$d(y', x) \leq J + 6\kappa(\rho, 1, 0) + \frac{L^2+1}{L^2}(A + \kappa(\rho, L, A)) - \frac{1}{L^2} d(\beta_b, \alpha) + 6\rho(d(\beta_b, \alpha))$$

Now use the facts that $\rho(d(\beta_b, \alpha)) \leq \frac{d(\beta_b,\alpha)}{3L^2}$ and $d(\beta_b, \alpha) \leq J$ to achieve:

$$\begin{aligned} d(y, x) \leq d(y', x) &\leq \frac{L^2+1}{L^2} J + 6\kappa(\rho, 1, 0) + \frac{L^2+1}{L^2}(A + \kappa(\rho, L, A)) \\ &\leq 2J + 6\kappa(\rho, 1, 0) + 2(A + \kappa(\rho, L, A)) \end{aligned}$$

Set $M := 2$ and $\lambda(\phi, p, q) := 6\kappa(\phi, 1, 0) + 2(q + \kappa(\phi, p, q))$. □

Here is an application of Lemma 4.6 that we will use in Section 9.

Lemma 4.7. *Given a sublinear function ρ and constants $L \geq 1$, $A \geq 0$ there exist constants $L' \geq 1$ and $A' \geq 0$ such that if α is a ρ-contracting geodesic ray or segment and β is a continuous (L, A)–quasi-geodesic ray not asymptotic to α with $\alpha_0 = \beta_0 = o$, then we obtain a continuous (L', A')–quasi-geodesic by following α backward until α_{s_0}, then following a geodesic from α_{s_0} to β_{t_0}, then following β, where β_{t_0} is the last point of β at distance $\kappa(\rho, L, A)$ from α, and where α_{s_0} is the last point of α at distance $\kappa(\rho, L, A)$ from β_{t_0}.*

Proof. Define $\kappa := \kappa(\rho, L, A)$ and M and $\lambda := \lambda(\rho, L, A)$ from Lemma 4.6. Recall $M \leq 2 \leq 2L$. It suffices to take $A' := \left(\frac{(4L+1)\kappa+\lambda}{4L} + A\right)$ and $L' := 4L$. Since we have constructed a concatenation of three quasi-geodesic segments, it suffices to check that points on different segments are not too close together. Since $A' > A + \kappa$ we may ignore the short middle segment. Thus, we need to check for $s \geq s_0$ and $t \geq t_0$ that $d(\alpha_s, \beta_t) \geq \frac{s-s_0+t-t_0+\kappa}{L'} - A'$.

For such s and t, let $x := \alpha_s$, $y := \alpha_{s_0}$, and $z := \beta_t$. By Lemma 4.6, $s - s_0 = d(x, y) \leq Md(x, z) + \lambda < 2Ld(x, z) + \lambda$. Choose some point $z' \in \pi_\alpha(z)$. By Corollary 4.3 ($\star$) we have $d(z, x) \geq d(z, z') \geq \frac{t-t_0}{2L} - 2(A + \kappa)$. Now average these two lower bounds for $d(x, z)$:

$$\begin{aligned} d(\alpha_s, \beta_t) = d(x, z) &\geq \frac{1}{2}\left(\frac{s-s_0}{2L} - \frac{\lambda}{2L} + \frac{t-t_0}{2L} - 2(A+\kappa)\right) \\ &\geq \frac{s - s_0 + t - t_0 + \kappa}{4L} - \left(\frac{\lambda}{4L} + \frac{4L+1}{4L}\kappa + A\right) \end{aligned}$$ □

To close this section we give the stronger formulation of the Quasi-geodesic Image Theorem:

Theorem 4.8. *Given a sublinear function ρ and constants $L \geq 1$ and $A \geq 0$ there is a sublinear function ρ' such that if Z is ρ–contracting and $\beta \colon [0, T] \to X$ is a continuous (L, A)–quasi-geodesic segment with $d(\beta, Z) = d(\beta_T, Z) = \kappa(\rho, L, A)$ then* $\operatorname{diam} \pi(\beta_0) \cup \pi(\beta_T) \leq \rho'(d(\beta_0, Z))$.

Proof. Define $\rho'(r) := \sup_\beta \operatorname{diam} \pi(\beta_0) \cup \pi(\beta_T)$ where the supremum is taken over all continuous (L, A)–quasi-geodesic segments β such that $d(\beta, Z) = \kappa(\rho, L, A)$ is realized at one endpoint of β and the other endpoint is at distance at most r from Z. Suppose that ρ' is not sublinear, so suppose $\limsup_{r\to\infty} \rho'(r)/r = 2\epsilon > 0$. Then there exists a sequence $(r_i) \to \infty$ such that for each i there exists a continuous (L, A)–quasi-geodesic segment $\beta^{(i)} \colon [0, T_i] \to X$ with $d(\beta^{(i)}_{T_i}, Z) = \kappa(\rho, L, A)$ and $d(\beta^{(i)}_0, Z) \leq r_i$ and $\operatorname{diam} \pi(\beta^{(i)}_0) \cup \pi(\beta^{(i)}_{T_i}) \geq \epsilon r_i$, so that $\operatorname{diam} \pi(\beta^{(i)}_0) \cup \pi(\beta^{(i)}_{T_i}) \geq \epsilon d(\beta^{(i)}_0, Z)$.

For $n \in \mathbb{N}$ define κ_n large enough so that for all $r \geq \kappa_n$ we have $r - L^2\rho(r) - A \geq \frac{1}{3}r \geq nL^2\rho(r)$ (recalling (2), $\kappa_1 = \kappa(\rho, L, A)$). The proof of Theorem 4.2 shows that if a continuous (L, A)–quasi-geodesic segment stays outside the κ_n–neighborhood of Z then:

$$\begin{aligned}\operatorname{diam} \pi(\beta_0) \cup \pi(\beta_T) &\leq \frac{L^2+1}{nL^2}(A + d(\beta_T, Z)) + \frac{L^2-1}{nL^2} d(\beta_0, Z) + \frac{n+1}{n}\rho(d(\beta_0, Z)) \\ &\leq \frac{1}{n}\left(2A + 2d(\beta_T, Z) + d(\beta_0, Z)\right) \end{aligned} \tag{5}$$

For $\epsilon > 0$ as above, choose $n \in \mathbb{N}$ large enough that $n\epsilon > 2$. For all sufficiently large i we have that $\operatorname{diam} \pi(\beta^{(i)}_0) \cup \pi(\beta^{(i)}_{T_i}) > 2A + 2\kappa_1 + \kappa_n$. By (5) for $n = 1$, we have $d(\beta^{(i)}_0, Z) > \kappa_n$. Let $s_i > 0$ be the first time such that $d(\beta^{(i)}_{s_i}, Z) = \kappa_n$.

$$\begin{aligned}\epsilon d(\beta^{(i)}_0, Z) &\leq \operatorname{diam} \pi(\beta^{(i)}_{T_i}) \cup \pi(\beta^{(i)}_0) \\ &\leq \operatorname{diam} \pi(\beta^{(i)}_{T_i}) \cup \pi(\beta^{(i)}_{s_i}) + \operatorname{diam} \pi(\beta^{(i)}_{s_i}) \cup \pi(\beta^{(i)}_0) \\ &\leq (2A + 2\kappa_1 + \kappa_n) + \frac{1}{n}\left(2A + 2\kappa_n + d(\beta^{(i)}_0, Z)\right) \\ &\leq (2A + 2\kappa_1 + \kappa_n) + \frac{\epsilon}{2}\left(2A + 2\kappa_n + d(\beta^{(i)}_0, Z)\right)\end{aligned}$$

Solving for $d(\beta^{(i)}_0, Z)$, we find that it is bounded, independent of i. By (5) for $n = 1$, this would bound $\operatorname{diam} \pi(\beta^{(i)}_0) \cup \pi(\beta^{(i)}_{T_i})$, independent of i, whereas we have assumed $\operatorname{diam} \pi(\beta^{(i)}_0) \cup \pi(\beta^{(i)}_{T_i}) \geq \epsilon r_i \to \infty$. This is a contradiction, so we conclude $\lim_{r\to\infty} \rho'(r)/r = 0$. □

5. The contracting boundary and the topology of fellow-travelling quasi-geodesics

Definition 5.1. Let X be a proper geodesic metric space with basepoint o. Define $\partial_c X$ to be the set of contracting quasi-geodesic rays based at o modulo Hausdorff equivalence.

Lemma 5.2. *For each $\zeta \in \partial_c X$:*

- *The set of contracting geodesic rays in ζ is non-empty.*
- *There is a sublinear function:*

$$\rho_\zeta(r) := \sup_{\alpha, x, y} \operatorname{diam} \pi_\alpha(x) \cup \pi_\alpha(y)$$

Here the supremum is taken over geodesics $\alpha \in \zeta$ and points x and y such that $d(x, y) \leq d(x, \alpha) \leq r$.

- *Every geodesic in ζ is ρ_ζ–contracting.*

Proof. By definition, ζ is an equivalence class of contracting quasi-geodesic rays, so there exists some ρ'–contracting (L, A)–quasi-geodesic ray $\beta \in \zeta$ based at o. Since X is proper, a sequence of geodesic segments connecting o to β_i for $i \in \mathbb{N}$ has a subsequence that converges to a geodesic α'. By Theorem 3.4, all of these geodesic segments, hence α' as well, are contained in a bounded neighborhood of β, with bound depending only on ρ', so there do exist geodesics asymptotic to β. Furthermore, Corollary 4.3 implies that geodesic rays asymptotic to β have uniformly bounded Hausdorff distance from β, with bound depending on ρ', L, and A. By Lemma 3.3, all such geodesics are ρ''–contracting for some $\rho'' \asymp \rho'$ depending on ρ', L, and A.

The function ρ_ζ is non-decreasing and bounds projection diameters by definition. The fact that there exists a sublinear function ρ'' such that all geodesics in ζ are ρ''–contracting implies $\rho_\zeta \leq \rho''$, so ρ_ζ is also sublinear. □

Definition 5.3. Let X be a proper geodesic metric space. Take $\zeta \in \partial_c X$. Fix a geodesic ray $\alpha^\zeta \in \zeta$. For each $r \geq 1$ define $U(\zeta, r)$ to be the set of points $\eta \in \partial_c X$ such that for all $L \geq 1$ and $A \geq 0$ and every continuous (L, A)–quasi-geodesic ray $\beta \in \eta$ we have $d(\beta, \alpha^\zeta \cap N_r^c o) \leq \kappa(\rho_\zeta, L, A)$.

Informally, $\eta \in U(\zeta, r)$ means that inside the ball of radius r about the basepoint quasi-geodesics in η fellow-travel α^ζ just as closely as quasi-geodesics in ζ do. Alternatively, quasi-geodesics in η do not escape from α^ζ until after they leave the ball of radius r about the basepoint.

Definition 5.4. Define the *topology of fellow-travelling quasi-geodesics* on $\partial_c X$ by:

$$\mathcal{FQ} := \{U \subset \partial_c X \mid \forall \zeta \in U, \exists r \geq 1, U(\zeta, r) \subset U\}$$

The contracting boundary equipped with this topology is denoted $\partial_c^{\mathcal{FQ}} X$.

We do not assume that the sets $U(\zeta, r)$ are open in the topology $\mathcal{FQ}$. Indeed, from the definition it is not even clear that $U(\zeta, r)$ is a neighborhood of ζ, but we will show that this is the case.

Proposition 5.5. *$\mathcal{FQ}$ is a topology on $\partial_c X$, and for each $\zeta \in \partial_c X$ the collection $\{U(\zeta, n) \mid n \in \mathbb{N}\}$ is a neighborhood basis at ζ.*

Corollary 5.6. *$\partial_c^{\mathcal{FQ}} X$ is first countable.*

Observation 5.7. Suppose $\eta \notin U(\zeta, r)$. By definition, for some L and A there exists a continuous (L, A)–quasi-geodesic $\beta \in \eta$ such that $d(\beta, \alpha^\zeta \cap N_r^c o) > \kappa(\rho_\zeta, L, A)$. Since $o \in \beta$, this is not possible if $\kappa(\rho_\zeta, L, A) \geq r$. Thus, in light of Definition 4.1, the quasi-geodesic β witnessing $\eta \notin U(\zeta, r)$ must be an (L, A)–quasi-geodesic with $L^2 < r/3$ and $A < r/3$.

The proof of Proposition 5.5 depends on two lemmas. The first is a recombination result for quasi-geodesics. Its key feature is that the quasi-geodesic constants of the result depend only on the quasi-geodesic constants of the input, not on the contraction function.

Lemma 5.8 (Tail wagging)**.** *Let ρ be a sublinear function. Let $L \geq 1$ and $A \geq 0$. Let $T \geq 11\kappa'(\rho, L, A)$ and $S \geq T + 6\kappa'(\rho, L, A) + 6\kappa'(\rho, 1, 0)$. Suppose α is a ρ–contracting geodesic ray based at o, γ is a continuous (L, A)–quasi-geodesic ray* *based at o such that $d(\gamma, \alpha_{[T,\infty)}) \le \kappa(\rho, L, A)$, and β is a geodesic ray based at o such that $d(\beta, \alpha_{[S,\infty)}) \le \kappa(\rho, 1, 0)$. Then there are continuous $(2L+1, A)$–quasi-geodesic rays that agree with γ until a point within distance $11\kappa'(\rho, L, A)$ of α_T and share tails with α and β, respectively.*

Proof. We construct the quasi-geodesic ray that shares a tail with β. The construction for the α–tail is similar, but with easier estimates.

Let $T'' := T - 3\kappa'(\rho, 1, 0) - \kappa'(\rho, L, A)$. Let T' be the first time at which γ comes within distance $\kappa'(\rho, L, A)$ of $\alpha_{[T'',\infty)}$. Let S' be such that $d(\beta_{S'}, \alpha_{[S,\infty)}) \le \kappa(\rho, 1, 0)$. Let $t_0 \le T'$ and $r_0 \ge S'$ be times such that $d(\gamma_{t_0}, \beta_{r_0}) = d(\gamma_{[0,T']}, \beta_{[S',\infty)})$, and let δ be a geodesic from γ_{t_0} to β_{r_0}. There are times b, c, b', and c' such that $d(\gamma_{t_0}, \alpha_b)$, $d(\gamma_{T'}, \alpha_c) \le \kappa'(\rho, L, A)$, $d(\beta_{b'}, \alpha_b)$, $d(\beta_{c'}, \alpha_c) \le \kappa'(\rho, 1, 0)$. For any $t \le t_0$ there exist a and a' such that $d(\gamma_t, \alpha_a) \le \kappa'(\rho, L, A)$ and $d(\alpha_a, \beta_{a'}) \le \kappa(\rho, 1, 0)$. See Figure 2.

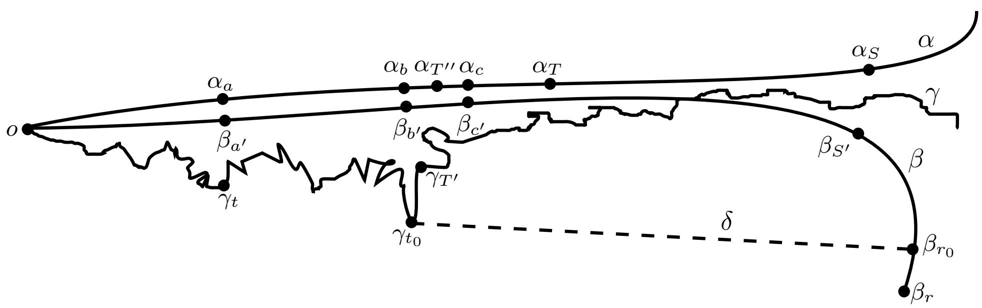

FIGURE 2. Wagging the tail of γ.

The desired quasi-geodesic ray is $\gamma_{[0,t_0]} + \delta + \beta_{[r_0,\infty)}$.

First, we verify $d(\gamma_{t_0}, \alpha_T) \le 11\kappa'(\rho, L, A)$. The definitions of t_0 and r_0 demand $d(\gamma_{t_0}, \beta_{r_0}) \le d(\gamma_{T'}, \beta_{S'})$. The left-hand side is at least $S' - b' - (\kappa'(\rho, L, A) + \kappa'(\rho, 1, 0))$, while the right-hand side is no more than $S' - c' + (\kappa'(\rho, L, A) + \kappa'(\rho, 1, 0))$, so $c' - b' \le 2(\kappa'(\rho, L, A) + \kappa'(\rho, 1, 0))$. Since $T'' \le c \le T'' + 2\kappa'(\rho, L, A)$ we have $d(\alpha_c, \alpha_T) \le \kappa'(\rho, L, A) + 3\kappa'(\rho, 1, 0)$. Together, these allow us to estimate:

$$\begin{aligned} d(\gamma_{t_0}, \alpha_T) &\le d(\gamma_{t_0}, \beta_{b'}) + d(\beta_{b'}, \beta_{c'}) + d(\beta_{c'}, \alpha_c) + d(\alpha_c, \alpha_T) \\ &\le (\kappa'(\rho, L, A) + \kappa'(\rho, 1, 0)) + c' - b' + \kappa'(\rho, 1, 0) \\ &\qquad + (\kappa'(\rho, L, A) + 3\kappa'(\rho, 1, 0)) \\ &\le 7\kappa'(\rho, 1, 0) + 4\kappa'(\rho, L, A) \le 11\kappa'(\rho, L, A) \end{aligned}$$

Next we verify the quasi-geodesic constants. Since we have a concatenation of quasi-geodesics, we only need to check that points on different pieces are not closer than they ought to be with respect to the parameterization.

First we claim $\gamma_{[0,t_0]} + \delta$ is an (L', A)–quasi-geodesic for $L' := 2L + 1$. This is true for $\gamma_{[0,t_0]}$ and δ individually. Suppose there are $0 \le t < t_0$ and $0 < u \le |\delta|$ such that $d(\gamma_t, \delta_u) < \frac{t_0 - t + u}{L'} - A$. Now, $d(\delta_u, \gamma_t) \ge d(\delta_u, \gamma_{t_0}) = u$, which implies

$u < \frac{L'}{L'-1}(\frac{t_0-t}{L'} - A)$. But then:

$$\begin{aligned}\frac{t_0-t}{L} - A \leq d(\gamma_t, \gamma_{t_0}) &\leq d(\gamma_t, \delta_u) + d(\delta_u, \gamma_{t_0})\\ &\leq \left(\frac{t_0-t+u}{L'} - A\right) + u\end{aligned}$$

Plugging in the value for L' and the bound for u yields a contradiction.

The same argument shows $\delta + \beta_{[r_0,\infty)}$ is a $(3,0)$–quasi-geodesic.

Now consider points γ_t and β_r for $t \leq t_0$ and $r \geq r_0$.

$$\begin{aligned}d(\gamma_t, \beta_r) &\geq r - a' - (\kappa'(\rho, L, A) + \kappa'(\rho, 1, 0))\\ &= r - r_0 + r_0 - b' + b' - a' - (\kappa'(\rho, L, A) + \kappa'(\rho, 1, 0))\\ &\geq r - r_0 + d(\gamma_{t_0}, \beta_{r_0}) + d(\gamma_t, \gamma_{t_0}) - 4(\kappa'(\rho, L, A) + \kappa'(\rho, 1, 0))\\ &\geq \frac{t_0 - t + r - r_0 + |\delta|}{2L+1} - A + |\delta|\frac{2L}{2L+1} - 4(\kappa'(\rho, L, A) + \kappa'(\rho, 1, 0))\end{aligned}$$

Thus, the ray we have constructed is a $(2L+1, A)$–quasi-geodesic, since $|\delta| \geq 6(\kappa'(\rho, L, A) + \kappa'(\rho, 1, 0))$, as we now verify:

$$\begin{aligned}|\delta| &\geq r_0 - b' - (\kappa'(\rho, L, A) + \kappa'(\rho, 1, 0))\\ &\geq r_0 - b - (\kappa'(\rho, L, A) + 2\kappa'(\rho, 1, 0))\\ &\geq S' - T'' - (\kappa'(\rho, L, A) + 2\kappa'(\rho, 1, 0))\\ &\geq S - T'' - (\kappa'(\rho, L, A) + 3\kappa'(\rho, 1, 0))\\ &\geq 6(\kappa'(\rho, L, A) + \kappa'(\rho, 1, 0))\end{aligned}$$

□

Lemma 5.9. *For every sublinear function ρ and $r \geq 1$ there exists a number $\psi(\rho, r) > r$ such that for every $R \geq \psi(\rho, r)$ and every $\zeta \in \partial_c X$ such that $\rho_\zeta \leq \rho$ we have that for every $\eta \in U(\zeta, R)$ there exists an R' such that $U(\eta, R') \subset U(\zeta, r)$.*

Proof. It suffices to take $\psi(\rho, r) := r + M\kappa(\rho, 2\sqrt{r/3}+1, r/3) + \lambda(\rho, \sqrt{r/3}, r/3)$, where M and λ are as in Lemma 4.6.

Suppose $R \geq \psi(\rho, r)$ and ζ is a point in $\partial_c X$ such that $\rho_\zeta \leq \rho$. Suppose that $\eta \in U(\zeta, R)$ with $\eta \neq \zeta$. Let T_0 be the last time such that $d(\alpha^\eta_{T_0}, \alpha^\zeta) = \kappa(\rho_\zeta, 1, 0)$. Set:

$$R' := T_0 + 2\kappa'(\rho_\zeta, 2\sqrt{r/3}+1, r/3) + 4\kappa(\rho_\zeta, 1, 0) + 28\kappa'(\rho_\eta, \sqrt{r/3}, r/3) + 6\kappa'(\rho_\eta, 1, 0)$$

Suppose that there exists a point $\xi \in U(\eta, R')$ such that $\xi \notin U(\zeta, r)$. The latter implies there exists an $L \geq 1$ and $A \geq 0$ and a continuous (L, A)–quasi-geodesic $\gamma \in \xi$ such that $d(\gamma, N^c_r o \cap \alpha^\zeta) > \kappa(\rho_\zeta, L, A)$. By Observation 5.7, we have L^2, $A < r/3$. Set $\alpha := \alpha^\eta$, $\beta := \alpha^\xi$, $T := T_0 + 4\kappa(\rho_\zeta, 1, 0) + 22\kappa'(\rho_\eta, \sqrt{r/3}, r/3) + 2\kappa'(\rho_\zeta, 2\sqrt{r/3}+1, r/3)$, and $S := R' \geq T + 6\kappa'(\rho_\eta, L, A) + 6\kappa'(\rho_\eta, 1, 0)$. Apply Lemma 5.8 to α, β, γ, T, and S to produce a continuous $(2L+1, A)$–quasi-geodesic $\delta \in \eta$ that agrees with γ at least until a point z in the ball of radius $11\kappa'(\rho_\eta, L, A)$ about α^η_T.

By Corollary 4.3 ($\star$) we have $d(\alpha_T, \alpha^\zeta) \geq (T - T_0)/2 - 2\kappa(\rho_\zeta, 1, 0)$, which implies $d(z, \alpha^\zeta) \geq \kappa'(\rho_\zeta, 2L+1, A)$, so by point z the ray δ has already escaped α^ζ and can never return to its $\kappa(\rho_\zeta, 2L+1, A)$–neighborhood. Therefore, the only points of δ in the $\kappa(\rho_\zeta, 2L+1, A)$–neighborhood of α^ζ are those that were contributed by γ.

By construction, γ does not come $\kappa(\rho_\zeta, L, A)$–close to α^ζ outside the ball of radius r. By applying Lemma 4.6, we see that δ is a witness to $\eta \notin U(\zeta, R)$. This is a contradiction, so $U(\eta, R') \subset U(\zeta, r)$. □

Proof of Proposition 5.5. For every $\zeta \in \partial_c X$ and $1 \leq r < r'$ we have $\zeta \in U(\zeta, r') \subset U(\zeta, r)$. The nesting is immediate from Definition 5.3, and $\zeta \in U(\zeta, r)$ by Corollary 4.3. Now it is easy to see that $\mathcal{FQ}$ is a topology. That a set of the form $U(\zeta, r)$ is a neighborhood of ζ in this topology follows from showing that the set

$$U := \{\eta \in U(\zeta, r) \mid \exists R_\eta,\, U(\eta, R_\eta) \subset U(\zeta, r)\}$$

is open, since then $\zeta \in U \subset U(\zeta, r)$. Now if $\eta \in U$ then there exists R_η so that $U(\eta, R_\eta) \subset U(\zeta, r)$. Lemma 5.9 says that for all $\xi \in U(\eta, \psi(\rho_\eta, R_\eta))$ there exists R' with $U(\xi, R') \subset U(\eta, R_\eta) \subset U(\zeta, r)$. Therefore $U(\eta, \psi(\rho_\eta, R_\eta)) \subset U$ and so U is open. □

From this proof we observe the following consequence.

Corollary 5.10. *For every $\zeta \in \partial_c X$ and $r \geq 1$ there exists an open set U such that $U(\zeta, \psi(\rho_\zeta, r)) \subset U \subset U(\zeta, r)$.*

Proposition 5.11. *The topology $\mathcal{FQ}$ does not depend on the choice of basepoint or on the choices of the representative geodesic rays for each point in $\partial_c X$.*

Proof. Let $\mathcal{C}$ be the set of contracting quasi-geodesic rays based at o and let $\mathcal{C}'$ be the set of contracting quasi-geodesic rays based at o'. There is a map $\phi\colon \mathcal{C} \to \mathcal{C}'$ by prefixing $\gamma \in \mathcal{C}$ with a chosen geodesic segment from o' to o. The map ϕ clearly induces a bijection $\partial_c\phi$ between contracting boundaries of X with respect to different basepoints, and the inverse map can be achieved by simply prefixing quasi-geodesic rays by a geodesic from o to o'. We check that $\partial_c\phi$ is an open map. For $\zeta \in \partial_c^{\mathcal{FQ}} X$ and $r \geq 1$ we show for sufficiently large R that $U'(\partial_c\phi(\zeta), R) \subset \partial_c\phi(U(\zeta, r))$, where $U'(\partial_c\phi(\zeta), R)$ denotes the appropriate neighborhood of $\partial_c\phi(\zeta)$ defined with o' as basepoint.

Let $\alpha := \alpha^\zeta$ be the reference geodesic for ζ based at o, and let α' be the reference geodesic for $\partial_c\phi(\zeta)$ based at o'. Then α' is bounded Hausdorff distance from α. Suppose α is ρ–contracting and α' is ρ'–contracting. Theorem 3.4 implies that α' eventually comes within distance $\kappa(\rho, 1, 0)$ of α, and Theorem 4.2 implies that this first happens at some time no later than $d(o', \alpha) + 3\kappa(\rho, 1, 0) + 2\rho(d(o', \alpha))$. After that time α' remains in the $\kappa'(\rho, 1, 0)$–neighborhood of α. Assume $R > d(o', \alpha) + 3\kappa(\rho, 1, 0) + 2\rho(d(o', \alpha))$.

Assume further that $R > r + 2d(o, o')$ and suppose $\eta \in U'(\partial_c\phi(\zeta), R)$. Let $\gamma \in \partial_c\phi^{-1}(\eta)$ be an arbitrary continuous (L, A)–quasi-geodesic. Our goal is to show that if R is chosen sufficiently large with respect to ρ, ρ', and r, then such a γ must come within distance $\kappa(\rho, L, A)$ of α outside $N_r o$. We then conclude $\partial_c\phi^{-1}(U'(\partial_c\phi(\zeta), R)) \subset U(\zeta, r)$. By Observation 5.7, it suffices to consider the case L^2, $A < r/3$.

Now, $\gamma' := \phi(\gamma) \in \eta$ is a continuous $(L, A + 2d(o, o'))$–quasi-geodesic. Since $\eta \in U'(\partial_c\phi(\zeta), R)$ there exists a point $x' \in \alpha'$ such that $d(\gamma', x') \leq \kappa(\rho', L, A + 2d(o, o'))$ and $d(x', o') \geq R$. The first restriction on R implies there is a point $x \in \alpha$ such that $d(x, x') \leq \kappa'(\rho, 1, 0)$, so $d(\gamma', x) \leq \kappa'(\rho, 1, 0) + \kappa(\rho', L, A + 2d(o, o'))$. We also have $d(x, o) \geq R - \kappa'(\rho, 1, 0) - d(o, o')$. Assuming further that $R > 2d(o, o') + 2\kappa'(\rho, 1, 0) + \kappa(\rho', L, A + 2d(o, o'))$, we have that the point of γ' close to x is actually a point of

γ. Let y be the last point of α at distance $\kappa(\rho, L, A)$ from γ (see Figure 3), and apply Lemma 4.6 to find:

$$\begin{aligned} d(o, y) \geq R - \kappa'(\rho, 1, 0) - d(o, o') \\ - M(\kappa'(\rho, 1, 0) + \kappa(\rho', \sqrt{r/3}, r/3 + 2d(o, o'))) - \lambda(\rho, \sqrt{r/3}, r/3) \end{aligned}$$

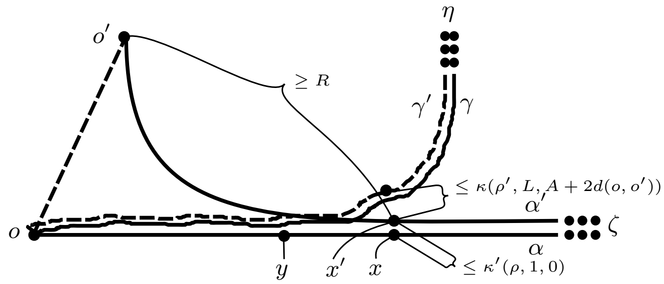

FIGURE 3. Change of basepoint

Assuming that R was chosen large enough to guarantee the right-hand side is at least r, we have that γ comes within distance $\kappa(\rho, L, A)$ of α outside $N_r o$. $\square$

Proposition 5.12. *$\partial_c^{\mathcal{FQ}} X$ is Hausdorff.*

Proof. Let ζ and η be distinct points in $\partial_c X$. Let $\alpha := \alpha^\zeta$ and $\beta := \alpha^\eta$ be representative geodesic rays. Let R be large enough that the $\kappa'(\rho_\zeta, 1, 0)$–neighborhood of $\alpha_{[R,\infty)}$ is disjoint from the $\kappa'(\rho_\eta, 1, 0)$–neighborhood of $\beta_{[R,\infty)}$. Such an R exists by Corollary 4.3.

Choose $\xi \in U(\zeta, R)$. Let $\gamma \in \xi$ be a geodesic ray. Since $\xi \in U(\zeta, R)$ there exists a point $x \in \alpha$ and $y \in \gamma$ with $d(x, o) \geq R$ and $d(x, y) \leq \kappa(\rho_\zeta, 1, 0)$. By construction $d(y, \beta) > \kappa'(\rho_\eta, 1, 0)$, so, by Corollary 4.3, the final visit of γ to the $\kappa(\rho_\eta, 1, 0)$–neighborhood of β must have occurred inside the ball of radius R about o. Thus, $\xi \notin U(\eta, R)$. $\square$

Proposition 5.13. *$\partial_c^{\mathcal{FQ}} X$ is regular.*

Proof. Suppose $C \subset \partial_c^{\mathcal{FQ}} X$ is closed and $\zeta \in C^c$. Then C^c is a neighborhood of ζ, so there exists r' such that for all $r \geq r'$ we have $U(\zeta, r) \subset C^c$. Suppose:

$$\forall \zeta \in \partial_c^{\mathcal{FQ}} X,\ \exists r' \geq 1,\ \forall r \geq r',\ \exists R > r,\ \overline{U(\zeta, R)} \subset U(\zeta, r) \tag{6}$$

Then there exists an $R > r$ such that $\overline{U(\zeta, R)} \subset U(\zeta, r) \subset C^c$, so C is contained in an open set $\overline{U(\zeta, R)}^c$ that is disjoint from $U(\zeta, R)$. By Proposition 5.5, $U(\zeta, R)$ is a neighborhood of ζ, so it contains an open set U that contains ζ. The disjoint open sets U and $\overline{U(\zeta, R)}^c$ separate ζ and C, so (6) implies regularity.

The proof of (6) is similar to the proof of Lemma 5.9: suppose given r and ζ there is no R satisfying the claim. Then there exists a point $\eta \in \overline{U(\zeta, R)} \cap U(\zeta, r)^c$. Now $\eta \in \overline{U(\zeta, R)}$ implies that for all $n \in \mathbb{N}$ there exists $\xi_n \in U(\zeta, R) \cap U(\eta, n)$, while $\eta \notin U(\zeta, r)$ implies there exist L^2, $A < r/3$ and a continuous (L, A)–quasi-geodesic $\gamma \in \eta$ such that $d(\gamma, N_r^c o \cap \alpha^\zeta) > \kappa(\rho_\zeta, L, A)$. For sufficiently large n we wag the tail of γ by Lemma 5.8 to produce a continuous $(2L + 1, A)$–quasi-geodesic $\delta \in \xi_n$ that agrees with γ on a long initial segment. If R is large enough this sets up a contradiction between the fact that $\xi_n \in U(\zeta, R)$ and the fact that γ witnesses $\eta \notin U(\zeta, r)$, so for large enough R we have $\overline{U(\zeta, R)} \subset U(\zeta, r)$, as desired. $\square$

Generally in this paper we will work directly with the topology on the contracting boundary. However, it is worth mentioning that this object that we have called a 'boundary' really is a topological boundary.

Definition 5.14. A *bordification* of a Hausdorff topological space X is a Hausdorff space $\hat{X}$ containing X as an open, dense subset.

The contracting boundary of a proper geodesic metric space provides a bordification of X by $\hat{X} := X \cup \partial_c X$ as follows. For $x \in X$ take a neighborhood basis for x to be metric balls about x. For $\zeta \in \partial_c X$ take a neighborhood basis for ζ to be sets $\hat{U}(\zeta, r)$ consisting of $U(\zeta, r)$ and points $x \in X$ such that we have $d(\gamma, N_r^c o \cap \alpha^\zeta) \leq \kappa(\rho_\zeta, L, A)$ for every $L \geq 1$, $A \geq 0$, and continuous (L, A)–quasi-geodesic segment γ with endpoints o and x.

Proposition 5.15. *$\hat{X} := X \cup \partial_c X$ topologized as above defines a first countable bordification of X such that the induced topology on $\partial_c X$ is the topology of fellow-travelling quasi-geodesics.*

Proof. A similar argument to that of Proposition 5.5 shows we have defined a neighborhood basis in a topology for each point in $\hat{X}$, and the topology agrees with the metric topology on X and topology $\mathcal{FQ}$ on $\partial_c X$ by construction. That $\hat{X}$ is Hausdorff follows from Proposition 5.12. X is clearly open in $\hat{X}$. To see that X is dense, consider $\zeta \in \partial_c X$, which, by definition, is an equivalence class of contracting quasi-geodesic rays. For any quasi-geodesic ray $\gamma \in \zeta$ we have that $(\gamma_n)_{n \in \mathbb{N}}$ is a sequence in X converging to γ, because the subsegments $\gamma_{[0,n]}$ are uniformly contracting. $\square$

Definition 5.16. If G is a finitely generated group acting properly discontinuously on a proper geodesic metric space X with basepoint o we define the *limit set* $\Lambda(G) := \overline{Go} \setminus Go$ of G to be the topological frontier in $\hat{X}$ of the orbit Go of the basepoint.

6. Quasi-isometry invariance

In this section we prove quasi-isometry invariance of the topology of fellow-travelling quasi-geodesics.

Theorem 6.1. *Suppose $\phi\colon X \to X'$ is a quasi-isometric embedding between proper geodesic metric spaces. If ϕ takes contracting quasi-geodesics to contracting quasi-geodesics then it induces an injection $\partial_c\phi\colon \partial_c^{\mathcal{FQ}} X \to \partial_c^{\mathcal{FQ}} X'$ that is an open mapping onto its image with the subspace topology. If $\phi(X)$ is a contracting subset of X' then $\partial_c\phi$ is continuous.*

We will see in Lemma 6.5 that if $\phi(X)$ is contracting then ϕ does indeed take contracting quasi-geodesics to contracting quasi-geodesics, so we get the following corollary of Theorem 6.1.

Corollary 6.2. *If $\phi\colon X \to X'$ is a quasi-isometric embedding between proper geodesic metric spaces and $\phi(X)$ is contracting in X' then $\partial_c\phi$ is an embedding. In particular, if ϕ is a quasi-isometry then $\partial_c\phi$ is a homeomorphism.*

Remark. Cordes [15] proves a version of Theorem 6.1 and Corollary 6.2 for the Morse boundary. The construction of the injective map is exactly the same. For continuity, he defines a map between contracting boundaries to be *Morse-preserving* if for each μ there is a μ' such that the map takes boundary points with a μ–Morse representative to boundary points with a μ'–Morse representative, and shows that if ϕ is a quasi-isometric embedding that induces a Morse-preserving map $\partial_c\phi$ on the contracting boundary then $\partial_c\phi$ is continuous in the direct limit topology.

Similarly, let us say that ϕ is *Morse-controlled* if for each μ there exists μ' such that ϕ takes μ–Morse geodesics to μ'–Morse geodesics. A Morse-controlled quasi-isometric embedding induces a Morse-preserving boundary map. We will see in Lemma 6.5 that the hypothesis that $\phi(X)$ is a contracting set implies that ϕ is Morse-controlled.

Cayley graphs of a fixed group with respect to different finite generating sets are quasi-isometric, so Corollary 6.2 allows us to define the contracting boundary of a finitely generated group, independent of a choice of generating set.

Definition 6.3. If G is a finitely generated group define $\partial_c^{\mathcal{FQ}}G$ to be $\partial_c^{\mathcal{FQ}}X$ where X is any Cayley graph of G with respect to a finite generating set.

The hypothesis in Theorem 6.1 that $\phi(X)$ is contracting already implies that it is undistorted, so in fact we do not need to explicitly require ϕ to be a quasi-isometric embedding. We can relax the hypotheses by only requiring ϕ to be coarse Lipschitz and uniformly proper. This is illustrated by the following easy lemma.

A map $\phi\colon X \to X'$ between metric spaces is *coarse Lipschitz* if there are constants $L \geq 1$ and $A \geq 0$ such that $d(\phi(x),\phi(x')) \leq Ld(x,x') + A$ for all $x, x' \in X$. It is *uniformly proper* if there exists a non-decreasing function $\chi\colon [0,\infty) \to [0,\infty)$ such that $d(x,x') \leq \chi(d(\phi(x),\phi(x')))$ for all $x, x' \in X$. Note that if X is geodesic and ϕ is coarse Lipschitz and uniformly proper then $\chi(r) > 0$ once $r > A$.

Lemma 6.4. *If $\phi\colon X \to X'$ is a coarse Lipschitz, uniformly proper map between geodesic metric spaces and $Z \subset X$ has quasi-convex image in X' then $\phi|_Z\colon Z \to X'$ is a quasi-isometric embedding.*

We will prove a stronger statement than this in Lemma 6.6.

Lemma 6.5. *Suppose $\phi\colon X \to X'$ is a coarse Lipschitz, uniformly proper map between geodesic metric spaces and $Z \subset X$. If $\phi(X)$ is Morse and Z is Morse then $\phi(Z)$ is Morse. If $\phi(Z)$ is Morse then Z is Morse. Moreover, the Morse function of Z determines the Morse function of $\phi(Z)$, and vice versa, up to functions depending on ϕ.*

Before proving Lemma 6.5 let us consider some examples to motivate the hypotheses. If X is a Euclidean plane, X' is a line, Z is a geodesic in X, and ϕ is the composition of projection of X onto Z and an isometry between Z and X' then ϕ is Lipschitz and $\phi(Z)$ is Morse, but ϕ is not proper and Z is not Morse. If X is a line, X' is a plane, $Z = X$, and ϕ is an isometric embedding then ϕ is Lipschitz and uniformly proper and Z is Morse, but $\phi(X) = \phi(Z)$ is not Morse.

In this paper we will only use the lemma in the case that $\phi(X)$ is Morse, and in this case it is easy to prove that Z is Morse when $\phi(Z)$ is. However, the more general statement might be of independent interest, and requires only mild generalizations of known results. The proof of the first claim uses essentially the same argument as the well-known result that quasi-convex subspaces are quasi-isometrically embedded. The key technical point for this direction is made in Lemma 6.6 (in a more general form than needed for Lemma 6.5, for later use). The second claim is proved using the same strategy as used by Drutu, Mozes, and Sapir [22, Lemma 3.25], who proved it in the case that X' is a finitely generated group, $\phi\colon X \to X'$ is inclusion of a finitely generated subgroup, and Z is an infinite cyclic group.

Lemma 6.6. *If $\phi\colon X \to X'$ is a coarse Lipschitz, uniformly proper map between geodesic metric spaces and $Z \subset X$ has Morse image in X' then for every $L \geq 1$ and $A \geq 0$ there exist $L' \geq 1$, $A' \geq 0$, $D' \geq 0$, and $D \geq 0$ such that for every (L, A)–quasi-geodesic γ in X' with endpoints on $\phi(Z)$ there is an (L', A')–quasi-geodesic δ in X with endpoints in Z such that:*

- $\delta \subset N_{D'}(Z)$
- $\gamma \subset N_D(\phi(\delta))$
- *γ and $\phi(\delta)$ have the same endpoints.*

The proof, briefly, is to project γ to $\phi(Z)$ and then pull the image back to X.

Proof. Suppose ϕ is χ–uniformly proper, (L_ϕ, A_ϕ)–coarse-Lipschitz, and $\phi(Z)$ is μ–Morse. Suppose the domain of γ is $[0, T]$. For $z \in \{0, T\}$ choose $\delta_z \in Z$ such that $\phi(\delta_z) = \gamma_z$. For $z \in \mathbb{Z} \cap (0, T)$ choose $\delta_z \in Z$ such that $d(\phi(\delta_z), \gamma_z) \leq \mu(L, A)$. Complete δ to a map on $[0, T]$ by connecting the dots by geodesic interpolation in X. For $D := L/2 + A + \mu(L, A)$ we have $\gamma \subset \bar{N}_D(\phi(\delta))$. Since the reparameterized geodesic segments used to build δ have endpoints on Z and length at most $\chi(L + A + 2\mu(L, A))$, by choosing $D' := \chi(L + A + 2\mu(L, A))/2$ we have $\delta \subset \bar{N}_{D'}(Z)$, and, furthermore, δ is $\chi(L + A + 2\mu(L, A))$–Lipschitz. For any $a \in [0, T]$ we have $d(\phi(\delta_a), \gamma_a) \leq L_\phi D' + A_\phi + D$. Finally, for $a, b \in [0, T]$:

$$\begin{aligned} L_\phi d(\delta_a, \delta_b) + A_\phi &\geq d(\phi(\delta_a), \phi(\delta_b)) \\ &\geq d(\gamma_a, \gamma_b) - 2(L_\phi D' + A_\phi + D) \\ &\geq \frac{|b-a|}{L} - A - 2(L_\phi D' + A_\phi + D) \end{aligned}$$

Thus, δ is an (L', A')–quasi-geodesic for $L' := \max\{L_\phi L, \chi(L + A + 2\mu(L, A))\}$ and $A' := (3A_\phi + A + 2L_\phi D' + 2D)/L_\phi$. ☐

Lemma 6.4 follows by the same argument applied to a geodesic.

Proof of Lemma 6.5. Suppose $\phi(X)$ and Z are Morse. A quasi-geodesic γ in X' with endpoints on $\phi(Z)$ has endpoints on $\phi(X)$, which is Morse. Apply Lemma 6.6 to get a quasi-geodesic δ in X such that $\phi(\delta)$ is coarsely equivalent to γ. We can, and do, choose δ so that it has endpoints on Z. Since Z is Morse, δ stays close to Z, so $\phi(\delta)$ stays close to $\phi(Z)$, so γ is close to $\phi(Z)$. Thus, $\phi(Z)$ is Morse.

Now suppose $\phi(Z)$ is Morse. By Lemma 6.4, ϕ restricted to Z is a quasi-isometric embedding. Suppose that it is an (L, A)–quasi-isometric embedding. (These constants are at least the coarse Lipschitz constants, so we will also assume ϕ is (L, A)–coarse Lipschitz on all of X.) Suppose $\phi(Z)$ is μ–Morse. The Morse property implies that $\phi(Z)$ is $(2\mu(1, 0) + 1)$–coarsely connected, so Z is E–coarsely connected for $E := L(2\mu(1, 0) + 1 + A)$. Let $t := \frac{1}{6L^2}$. If Z has diameter at most $\frac{1+2t}{1-2t}E$ then it is μ'–Morse for μ' the function $\frac{1+2t}{1-2t}E$, which depends only on L, A, and μ. In this case we are done. Otherwise we prove that Z is t–recurrent and apply Theorem 2.2.

We fix $C \geq 1$ and produce the corresponding D from the definition of recurrence.

Since the diameter of Z is bigger than $\frac{1+2t}{1-2t}E$, the fact that Z is E–coarsely connected implies that for every a, $b \in Z$ there exists a point $c \in Z$ such that $td(a, b) + E \geq \min\{d(a, c), d(b, c)\} \geq td(a, b)$: if $d(a, b) \geq \frac{1}{1-2t}E$ then c may be found on a coarse path from a to b, otherwise c may be found on a coarse path joining a to one of two points separated by more than $\frac{1+2t}{1-2t}E$. Such a point c is within distance $td(a,b)+E$ of every path with endpoints a and b, so for any fixed $K \geq 0$ we may restrict our attention to the case $d(a,b) > K$ by assuming D is at least $tK+E$.

Suppose p is path in X with endpoints a and b on Z such that p has length $|p|$ at most $Cd(a,b)$ and $d(a,b) > 8L(A+1)$. Subdivide p into $\lceil |p| \rceil$ many subsegments, all but possibly the last of which has length 1. Denote the endpoints of these subsegments $a = x_0, x_1, \dots, x_{\lceil |p| \rceil} = b$. Let q be a path in X' obtained by connecting each $\phi(x_i)$ to $\phi(x_{i+1})$ by a geodesic. Then q is a path of length at most $(L+A)\lceil |p| \rceil \leq (L+A)\frac{9}{8}|p|$ that coincides with $\phi(p)$ on $\phi(\{x_0, \dots, x_{\lceil |p| \rceil}\})$. Since ϕ is an (L,A)–quasi-geodesic embedding of Z we have that the distance between the endpoints of q is at least $d(a,b)/L - A \geq \frac{7}{8L}d(a,b)$, so that $|q| < C'd(\phi(a),\phi(b))$ for $C' = 9CL(L+A)/7$. Since $\phi(Z)$ is Morse it is recurrent, so given $t' := 1/3$ and C' as above there is a $D' \geq 0$ and $z \in Z$ such that $\min\{d(\phi(z),\phi(a)), d(\phi(z),\phi(b))\} \geq d(\phi(a),\phi(b))/3$ and $d(\phi(z),q) \leq D'$. Thus, there is some i such that $d(\phi(x_i),\phi(z)) \leq D' + (L+A)/2$. If ϕ is χ–uniformly proper then $d(x_i,z) \leq D := \max\{\chi(D' + (L+A)/2), 8tL(A+1)+E\}$. It remains to check that z is sufficiently far from the endpoints of p. This follows easily from our choice of t, the distance bound between $\phi(z)$ and the endpoints of q, and the assumption $d(a,b) > 8LA$, by using the fact that $\phi|_Z$ is an (L,A)–quasi-isometric embedding. □

Corollary 6.7. *If G is a finitely generated group and Z is a subset of a finitely generated subgroup H of G such that Z is Morse in G then Z is Morse in H. If G is a finitely generated group and H is a Morse subgroup of G then every Morse subset Z of H is also Morse in G.*

Proof of Theorem 6.1. Since the topology is basepoint invariant we choose $o \in X$ and let $o' := \phi(o) \in X'$.

Suppose ϕ is an (L,A)–quasi-isometric embedding, and suppose $\bar{\phi}\colon \phi(X) \to X$ is an (L,A)–quasi-isometry inverse to ϕ. We assume $\sup_{x \in X'} d(\phi \circ \bar{\phi}(x), x) \leq A$.

The quasi-isometric embedding ϕ induces an injective map between equivalence classes of quasi-geodesic rays based at o and equivalence classes of quasi-geodesic rays based at o'. The hypothesis that ϕ sends contracting quasi-geodesics to contracting quasi-geodesics implies that it takes equivalence classes of contracting quasi-geodesic rays to equivalence classes of contracting quasi-geodesic rays, so ϕ induces an injection $\partial_c\phi\colon \partial_c X \to \partial_c X'$.

Continuity: Assume $\phi(X)$ is ρ–contracting. By Lemma 6.5, ϕ sends contracting quasi-geodesic rays to contracting quasi-geodesic rays, so we have an injective map $\partial_c\phi$ as above. We claim that:

(7)
$$\forall \zeta \in \partial_c\phi(\partial_c X),\, \forall r > 1,\, \exists R' > 1,\, \forall R \geq R',\, U((\partial_c\phi)^{-1}(\zeta), R) \subset (\partial_c\phi)^{-1}(U(\zeta, r))$$

Given the claim, let U' be an open set in $\partial_c^{\mathcal{FQ}} X'$. For each $\zeta \in U' \cap \partial_c\phi(\partial_c X)$ there exists an r_ζ such that $U(\zeta, r_\zeta) \subset U'$. Apply (7) to get an R_ζ, and choose an open neighborhood of $(\partial_c\phi)^{-1}(\zeta)$ contained in $U((\partial_c\phi)^{-1}(\zeta), R_\zeta)$. Let U be the union of these open sets for all $\zeta \in U' \cap \partial_c\phi(\partial_c X)$. Then U is an open set and (7) implies $U = (\partial_c\phi)^{-1}(U')$.

To prove the claim we play our usual game of supposing the converse, deriving a bound on R, and then choosing R to be larger than that bound. The key point is that all of the constants involved are bounded in terms of ζ, $(\partial_c\phi)^{-1}(\zeta)$, r, and ρ.

Suppose for given $\zeta \in \partial_c\phi(\partial_c X)$ and $r > 1$ there exists an $R > 1$ and a point $\eta \in U((\partial_c\phi)^{-1}(\zeta), R)$ such that $\eta \notin (\partial_c\phi)^{-1}(U(\zeta, r))$. The latter implies there exists a continuous (L', A')–quasi-geodesic $\gamma \in \partial_c\phi(\eta)$ witnessing $\partial_c\phi(\eta) \notin U(\zeta, r)$. By Observation 5.7, the quasi-geodesic constants of γ are bounded in terms of r. We must adjust γ to get it into the domain of $\bar{\phi}$. Since $\partial_c\phi(\eta)$ is in the image of $\partial_c\phi$, the quasi-geodesic γ is asymptotic to a quasi-geodesic contained in $\phi(X)$, so γ is contained in a bounded neighborhood of $\phi(X)$. Since $\phi(X)$ as a whole is ρ–contracting, we can replace γ by a projection γ' of γ to $\phi(X)$ as Lemma 6.6. The Hausdorff distance between γ and γ' is bounded[3] in terms of ρ and the quasi-geodesic constants of γ, hence by r, and the additive quasi-geodesic constant of γ' increases by at most twice the Hausdorff distance.

Tame $\bar{\phi}(\gamma')$ to get a continuous quasi-geodesic $\hat{\gamma} \in \eta$. The Hausdorff distance between them and the quasi-geodesic constants (L'', A'') of $\hat{\gamma}$ are bounded in terms of the quasi-isometry constants of γ' and ϕ.

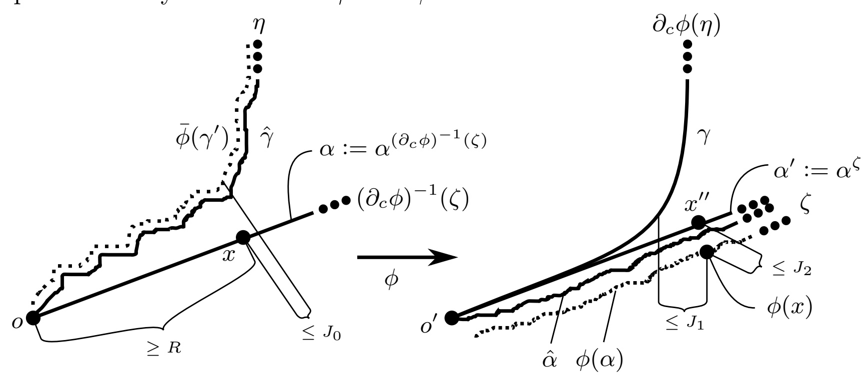

FIGURE 4. Setup for Theorem 6.1

Let $\alpha := \alpha^{(\partial_c\phi)^{-1}(\zeta)}$. Since $\hat{\gamma} \in \eta \in U((\partial_c\phi)^{-1}(\zeta), R)$, there exists $x \in \alpha$ such that $d(o, x) \geq R$ and $d(x, \hat{\gamma}) \leq \kappa(\rho_{(\partial_c\phi)^{-1}(\zeta)}, L'', A'')$. By the argument of the previous paragraph, $\kappa(\rho_{(\partial_c\phi)^{-1}(\zeta)}, L'', A'')$ can be bounded in terms of L, A, r, ρ, ρ_ζ, and $\rho_{(\partial_c\phi)^{-1}(\zeta)}$. This bound plus the Hausdorff distance to $\bar{\phi}(\gamma')$ give a bound $d(\bar{\phi}(\gamma'), x) \leq J_0$. Push forward by ϕ to get $d(\gamma, \phi(x)) \leq J_1 := LJ_0 + 2A + d_{\mathrm{Haus}}(\gamma, \gamma')$. We also know $d(\phi(x), o') \geq R/L - A$.

The quasi-isometric embedding ϕ sends the geodesic α to an (L, A)–quasi-geodesic $\phi(\alpha)$ asymptotic to $\alpha' := \alpha^\zeta$ with $\phi(\alpha)_0 = \phi(o) = o'$. Tame $\phi(\alpha)$ to produce a continuous $(L, 2L + 2A)$–quasi-geodesic $\hat{\alpha}$ at Hausdorff distance at most $L + A$ from $\phi(\alpha)$. Since $\hat{\alpha} \in \zeta$ we have that $\hat{\alpha}$ is contained in the $\kappa'(\rho_\zeta, L, 2L + 2A)$–neighborhood of α', so $\phi(\alpha)$ is contained in the J_2–neighborhood of α' for $J_2 := \kappa'(\rho_\zeta, L, 2L + 2A) + L + A$. In particular, $d(\phi(x), \alpha') \leq J_2$. Let x'' be the closest point of α' to $\phi(x)$, so that $d(\gamma, x'') \leq J_1 + J_2$ and $d(o', x'') \geq R/L - A - J_2$. By Lemma 4.6, since γ is an (L', A')–quasi-geodesic, if y is the last point of α' such

[3]If we had only assumed ϕ to be Morse-controlled this bound would depend on the Morse/contraction function of η, which can be arbitrarily bad, even for η in a small neighborhood of ζ.

that $d(\gamma, y) = \kappa(\rho_\zeta, L', A')$ then:

$$d(o', y) \geq R/L - A - J_2 - M(J_1 + J_2) - \lambda(\rho_\zeta, L', A') \tag{8}$$

Everything except R in (8) can be bounded in terms of L, A, r, ρ, ρ_ζ, and $\rho_{(\partial_c\phi)^{-1}(\zeta)}$, so, given r and ζ we can choose R large enough to guarantee $d(o', y) > r$. For such an R, we have $\partial_c\phi(\eta) \in U(\zeta, r)$ for every $\eta \in U((\partial_c\phi)^{-1}(\zeta), R)$. This finishes the proof of claim (7), so we conclude $\partial_c\phi$ is continuous if $\phi(X)$ is contracting.

Open mapping: The image of $\bar{\phi}$ is coarsely dense in X, so it is contracting. Thus, we can apply the argument of the proof of continuity above to $\bar{\phi}$ to get the following analogue of (7), noting that $\partial_c\bar{\phi} = (\partial_c\phi)^{-1}$:

(9) $\forall \zeta \in \partial_c X, \forall r > 1, \exists R' > 1, \forall R \geq R', \partial_c\phi(U(\zeta, r)) \supset U(\partial_c\phi(\zeta), R) \cap \partial_c\phi(\partial_c X)$

Let U be an open set in $\partial_c^{\mathcal{FQ}} X$. For every $\zeta \in U$ there exists r_ζ such that $U(\zeta, r_\zeta) \subset U$. Apply (9) to get R_ζ, and let U_ζ be an open neighborhood of $\partial_c\phi(\zeta)$ contained in $U(\partial_c\phi(\zeta), R_\zeta)$. Then $U' := \bigcup_{\zeta \in U} U_\zeta$ is an open set in $\partial_c^{\mathcal{FQ}} X'$ containing $\partial_c\phi(U)$. The choices of the R_ζ, by (9), imply that $U' \cap \partial_c\phi(\partial_c^{\mathcal{FQ}} X) = \partial_c\phi(U)$. □

One reason it may be convenient to weaken the stated quasi-isometric embedding hypothesis is that the orbit map of a properly discontinuous group action of a finitely generated group on a proper geodesic metric space is always coarse Lipschitz and uniformly proper, so we get the following consequences of Corollary 6.2.

Proposition 6.8. *Suppose G acts properly discontinuously on a proper geodesic metric space X. Suppose the orbit map $\phi\colon g \mapsto go$ takes contracting quasi-geodesics to contracting quasi-geodesics and has quasi-convex image. Then:*

- *G is finitely generated.*
- *The orbit map $\phi\colon g \mapsto go$ is a quasi-isometric embedding.*
- *The orbit map induces an injection $\partial_c\phi\colon \partial_c^{\mathcal{FQ}} G \to \partial_c^{\mathcal{FQ}} X$ that is an open mapping onto its image, which is $\Lambda(G)$ (recall Definition 5.16). In particular, if $\Lambda(G)$ is compact then so is $\partial_c^{\mathcal{FQ}} G$.*

If $\phi(G)$ is contracting in X then the above are true and $\partial_c\phi$ is an embedding, so $\partial_c^{\mathcal{FQ}} G$ is homeomorphic to $\Lambda(G)$.

Proof. Suppose Go is Q–quasi-convex. A standard argument shows that G is finitely generated and ϕ is a quasi-isometric embedding.

Since ϕ takes contracting quasi-geodesics to contracting quasi-geodesics it induces an open injection $\partial_c\phi$ onto its image by Theorem 6.1.

A point in $\zeta \in \partial_c^{\mathcal{FQ}} G$ is sent to the equivalence class of the contracting quasi-geodesic ray $\phi(\alpha^\zeta)$. The sequence $(\phi(\alpha^\zeta_n))_{n\in\mathbb{N}}$ converges to $\partial_c\phi(\zeta)$ in $\hat{X}$, so the image of $\partial_c\phi$ is contained in $\Lambda(G)$.

Conversely, suppose $\zeta \in \Lambda(G)$. Then there is a sequence $(g_n o)_{n\in\mathbb{N}}$ converging in $\hat{X}$ to $\zeta \in \partial_c^{\mathcal{FQ}} X$. By passing to a subsequence, we may assume $g_n o \in \hat{U}(\zeta, n)$ for all n. The definition of $\hat{U}(\zeta, n)$ implies that for any chosen geodesic γ^n from o to $g_n o$ there are points x_n on α^ζ and y_n on γ^n such that $d(x_n, y_n) \leq \kappa := \kappa(\rho_\zeta, 1, 0)$ and $d(o, x_n) \geq n$. Since Go is quasi-convex, there exists $g_n' \in G$ such that $d(y_n, g_n' o) \leq Q$. Thus, the set $Go \cap N_{\kappa+Q}\alpha^\zeta$ is unbounded. If $go \in N_{\kappa+Q}\alpha^\zeta$ then an application of Lemma 4.6 implies that a geodesic from o to go has an initial segment that is $\kappa'(\rho_\zeta, 1, 0)$–Hausdorff equivalent to an initial segment of α^ζ, and the length of these initial segments is $d(o, go)$ minus a constant depending on ζ and Q, but not g. Since we can take $d(o, go)$ arbitrarily large, and since every geodesic from o to go is contained in the Q–neighborhood of Go, we conclude that α^ζ is contained in a bounded neighborhood of Go. Now project α^ζ to Go and pull back to G to get a contracting quasi-geodesic ray whose ϕ–image is asymptotic to α^ζ, which shows $\zeta \in \partial_c\phi(\partial_c^{\mathcal{FQ}}G)$.

If $\phi(G)$ is contracting then ϕ does indeed take contracting quasi-geodesics to contracting quasi-geodesics, by Lemma 6.5, and have quasi-convex image, so the previous claims are true and Corollary 6.2 says $\partial_c\phi$ is an embedding. □

Corollary 6.9. *If H is a subgroup of a finitely generated group G and H is contracting in G then H is finitely generated and the inclusion $\iota\colon H \to G$ induces an embedding $\partial_c\iota\colon \partial_c^{\mathcal{FQ}}H \to \partial_c^{\mathcal{FQ}}G$.*

Properly speaking, we ought to require that H is a contracting subset of the Cayley graph of G with respect to some specified generating set, but it follows from Theorem 2.2 that the property of being a contracting subset does not depend on the choice of metric within a quasi-isometry class.

Corollary 6.10. *If H is a hyperbolically embedded subgroup (in the sense of* [20]*) in a finitely generated group G then $\partial_c\iota\colon \partial_c^{\mathcal{FQ}}H \to \partial_c^{\mathcal{FQ}}G$ is an embedding. A special case is that of a peripheral subgroup of a relatively hyperbolic group.*

Proof. Sisto [39] shows hyperbolically embedded subgroups are Morse, hence contracting. Peripheral subgroups of relatively hyperbolic groups are a motivating example for the definition of hyperbolically embedded subgroups in [20], but in this special case the fact they are Morse was already shown by Drutu and Sapir [23]. □

Together with Corollary 6.2, Corollary 6.7 implies:

Corollary 6.11. *If G is a finitely generated group and Z is a Morse subset of G then $\partial_c^{\mathcal{FQ}}Z$ embeds into $\partial_c^{\mathcal{FQ}}H$ for every finitely generated subgroup H of G containing Z. In particular, if $\partial_c^{\mathcal{FQ}}Z$ is non-empty then so is $\partial_c^{\mathcal{FQ}}H$, and if $\partial_c^{\mathcal{FQ}}Z$ contains a non-trivial connected component then so does $\partial_c^{\mathcal{FQ}}H$.*

7. Comparison to other topologies

Definition 7.1. Let X be a proper geodesic metric space. Take $\zeta \in \partial_c X$. Fix a geodesic ray $\alpha \in \zeta$. For each $r \geq 1$ define $V(\zeta, r)$ to be the set of points $\eta \in \partial_c X$ such that for every geodesic ray $\beta \in \eta$ we have $d(\beta, \alpha \cap N_r^c o) \leq \kappa(\rho_\zeta, 1, 0)$.

The same argument as Proposition 5.5 shows that $\{V(\zeta, n) \mid n \in \mathbb{N}\}$ gives a neighborhood basis at ζ for a topology $\mathcal{FG}$ on $\partial_c X$. We call $\mathcal{FG}$ the *topology of fellow-travelling geodesics*. It is immediate from the definitions that $\mathcal{FQ}$ is a refinement of $\mathcal{FG}$. The topology $\mathcal{FG}$ need not be preserved by quasi-isometries of X [12]. It is an open question whether $\mathcal{FG}$ is preserved by quasi-isometries when X is the Cayley graph of a finitely generated group.

One might also try to take $V'(\zeta, r)$ to be the set of points $\eta \in \partial_c X$ such that for *some* geodesic ray $\beta \in \eta$ we have $d(\beta, \alpha \cap N_r^c o) \leq \kappa(\rho_\zeta, 1, 0)$. Let $\mathcal{FG}'$ denote the resulting topology. Beware that in general $\{V'(\zeta, r) \mid r \geq 1\}$ is only a filter base converging to ζ, not necessarily a neighborhood base of ζ in $\mathcal{FG}'$; the sets $V'(\zeta, r)$ might not be neighborhoods of ζ.

Lemma 7.2. *Let X be a proper geodesic metric space. Let $\partial_\rho X = \{\zeta \in \partial_c X \mid \rho_\zeta \leq \rho\}$, i.e. $\zeta \in \partial_\rho X$ if all geodesics $\alpha \in \zeta$ are ρ–contracting. The topologies on $\partial_\rho X$ generated by taking, for each $\zeta \in \partial_\rho X$ and $r \geq 1$, the sets $U(\zeta, r) \cap \partial_\rho X$, $V(\zeta, r) \cap \partial_\rho X$, or $V'(\zeta, r) \cap \partial_\rho X$, are equivalent.*

Proof. For each ζ and r we have $U(\zeta, r) \cap \partial_\rho X \subset V(\zeta, r) \cap \partial_\rho X \subset V'(\zeta, r) \cap \partial_\rho X$ by definition.

Given that points in $V'(\zeta, r) \cap \partial_\rho X$ and $U(\zeta, r) \cap \partial_\rho X$ are uniformly contracting, a straightforward application of Lemma 4.6 shows that for all ζ and r, for all sufficiently large R we have $V'(\zeta, R) \cap \partial_\rho X \subset U(\zeta, r) \cap \partial_\rho X$. Also since points of $V'(\zeta, R) \cap \partial_\rho X$ are uniformly contracting, these do, in fact, give a neighborhood basis at ζ for the induced topology, as in Proposition 5.5. □

Proposition 7.3. *Let X be a proper geodesic metric space. If X is hyperbolic then $\partial_c^{\mathcal{FQ}} X \cong \partial_c^{\mathcal{FG}} X \cong \partial_c^{\mathcal{FG}'} X$, and these are homeomorphic to the Gromov boundary. If X is CAT(0) then $\partial_c^{\mathcal{FG}} X \cong \partial_c^{\mathcal{FG}'} X$, and these are homeomorphic to the subset of the visual boundary of X consisting of endpoints of contracting geodesic rays, topologized as a subspace of the visual boundary.*

Proof. For a description of a neighborhood basis for points in the Gromov or visual boundary see [11, III.H.3.6] and [11, II.8.6], respectively. Note that these are equivalent to the neighborhood bases for $\mathcal{FG}'$.

The claim for hyperbolic spaces follows from Lemma 7.2, because geodesics in a hyperbolic space are uniformly contracting.

If X is CAT(0) then $\partial_c^{\mathcal{FG}} X \cong \partial_c^{\mathcal{FG}'} X$ because there is a unique geodesic ray in each asymptotic equivalence class. □

More generally, $\partial_c^{\mathcal{FG}} X \cong \partial_c^{\mathcal{FG}'} X$ if X is a proper geodesic metric space with the property that every geodesic ray in X is either not contracting or has contraction function bounded by a constant. This follows by the same argument as in [12].

Next, we recall the *direct limit topology*, $\mathcal{DL}$, on $\partial_c X$ of Charney and Sultan [14] and Cordes [15].

For a given contraction function ρ consider the set $\partial_\rho X$ of points ζ in $\partial_c X$ such that one can take $\rho_\zeta \leq \rho$, as in Lemma 7.2. The topologies $\mathcal{FQ}$, $\mathcal{FG}$, and $\mathcal{FG}'$ on $\partial_\rho X$ coincide by Lemma 7.2. For $\rho \leq \rho'$ the inclusion $\partial_\rho X \hookrightarrow \partial_{\rho'} X$ is continuous, and $\partial_c X$, as a set, is the direct limit of this system of inclusions over all contraction functions.

Let $\mathcal{DL}$ be the direct limit topology on $\partial_c X$, that is, the finest topology on $\partial_c X$ such that all of the inclusion maps $\partial_\rho X \hookrightarrow \partial_c X$ are continuous.

Proposition 7.4. *$\mathcal{DL}$ is a refinement of $\mathcal{FQ}$.*

Proof. The universal property of the direct limit topology says that a map from the direct limit is continuous if and only if the precomposition with each inclusion map is continuous. Thus, it suffices to show the inclusion $\partial_\rho X \hookrightarrow \partial_c^{\mathcal{FQ}} X$ is continuous. This is clear from Lemma 7.2, since we can take the topology on $\partial_\rho X$ to be the subspace topology induced from $\partial_\rho X \hookrightarrow \partial_c^{\mathcal{FQ}} X$. □

Lemma 7.5. *$\partial_c^{\mathcal{DL}} X$ is homeomorphic to Cordes's Morse boundary.*

Proof. Cordes considers Morse geodesic rays, and defines the Morse boundary to be the set of asymptotic equivalence classes of Morse geodesic rays based at o, topologized by taking the direct limit topology of the system of uniformly Morse subsets. By Theorem 2.2, a collection of uniformly Morse rays is contained in a collection of uniformly contracting rays, and vice versa. It follows as in [15, Remark 3.4] that the direct limit topology over uniformly Morse points and the direct limit topology over uniformly contracting points agree on $\partial_c X$. □

As with the other topologies, if X is hyperbolic then $\partial_c^{\mathcal{DL}} X$ is homeomorphic to the Gromov boundary. Thus, if X is a proper geodesic hyperbolic metric space then all of the above topologies yield a compact contracting boundary. Conversely, Murray [33] showed if X is a complete CAT(0) space admitting a properly discontinuous, cocompact, isometric group action, and if $\partial_c^{\mathcal{DL}} X$ is compact and non-empty, then X is hyperbolic. Work of Cordes and Durham [17] shows that if the contracting boundary, with topology $\mathcal{DL}$, of a finitely generated group is non-empty and compact then the group is hyperbolic. We will prove this for $\mathcal{FQ}$ in Section 10.

We have shown that all of the topologies we consider agree for hyperbolic groups. More generally, we could ask about relatively hyperbolic groups. There are many ways to define relatively hyperbolic groups [26, 10, 35, 27, 21, 23, 29, 38], all of which are equivalent in our setting. Let G be a finitely generated group that is hyperbolic relative to a collection of *peripheral* subgroups $\mathcal{P}$. Fix a finite generating set for G. We again use G to denote the Cayley graph of G with respect to this generating set. Let $\tilde{G}$ be the *cusped space* obtained by gluing a combinatorial horoball onto each left coset of a peripheral subgroup, as in [27]. The cusped space is hyperbolic, and its boundary $\partial\tilde{G}$ is the Bowditch boundary of $(G, \mathcal{P})$. Points in the Bowditch boundary that are fixed by a conjugate of a peripheral subgroup are known as *parabolic points*, and the remaining points are known as *conical points*. As described, G sits as a subgraph in $\tilde{G}$.

Theorem 7.6. *If a finitely generated group G is hyperbolic relative to $\mathcal{P}$, then the inclusion $\iota\colon G \hookrightarrow \tilde{G}$ induces a continuous, G–equivariant map $\iota_*\colon \partial_c^{\mathcal{FG}} G \to \partial\tilde{G}$ that is injective at conical points.*

For $\iota_*\colon \partial_c^{\mathcal{FQ}} G \to \partial\tilde{G}$, the preimage of a parabolic point is the contracting boundary of its stabilizer subgroup embedded in $\partial_c^{\mathcal{FQ}} G$ as in Corollary 6.10.

Let $q\colon \partial_c^{\mathcal{FQ}} G \to \partial_c^{\mathcal{FQ}} G/\iota_*$ be the quotient map from $\partial_c^{\mathcal{FQ}} G$ to its ι_*–decomposition space, that is, the quotient space of $\partial_c^{\mathcal{FQ}} G$ obtained by collapsing to a point the preimage of each point in $\iota_*(\partial_c^{\mathcal{FQ}} G)$. If each peripheral subgroup is hyperbolic or has empty contracting boundary then $\iota_* \circ q^{-1}$ is an embedding.

Theorem 7.6 for $\mathcal{DL}$, without the embedding result, was first observed by Tran [41]. Recall from the introduction that the embedding statement is not true for $\mathcal{DL}$ (cf [41, Remark 8.13]).

Corollary 7.7. *If G is a finitely generated group that is hyperbolic relative to subgroups with empty contracting boundaries then $\partial_c^{\mathcal{FQ}} G = \partial_c^{\mathcal{FG}} G$.*

Since the contracting boundary of a hyperbolic group is the same as the Gromov boundary, we also recover the following well-known result (see [30] and references therein).

Corollary 7.8. *If G is hyperbolic and hyperbolic relative to $\mathcal{P}$ then the Bowditch boundary of $(G, \mathcal{P})$ can be obtained from the Gromov boundary of G by collapsing to a point each embedded Gromov boundary of a peripheral subgroup.*

The following example shows that the embedding statement of Theorem 7.6 can fail when a peripheral subgroup is non-hyperbolic with non-trivial contracting boundary.

Example 7.9. Let $A := \langle a, b \mid [a, b] = 1\rangle$, $H := A * \langle c\rangle$, and $G := H * \langle d\rangle$. Since G is a free product of H and a hyperbolic group, G is hyperbolic relative to H.

A geodesic α in G (or H) is contracting if and only if there is a bound B such that α spends at most time B in any given coset of A.

Consider the sequence $(a^n d^\infty)_{n\in\mathbb{N}}$ in $\partial_c^{\mathcal{FQ}} G$. We have $(\iota_*(a^n d^\infty)) \to \iota_*(\partial_c^{\mathcal{FQ}} H)$, which is a parabolic point in $\partial\tilde{G}$. However, $(q(a^n d^\infty))$ does not converge in $\partial_c^{\mathcal{FQ}} G/\iota_*$. To see this, note that every edge e in the Cayley graph of G with one incident vertex in A determines a clopen subset U_e of $\partial_c^{\mathcal{FQ}} G$ consisting of all $\zeta \in \partial_c^{\mathcal{FQ}} G$ such that α^ζ crosses e. Let U be the union of the U_e for every edge e incident to A and labelled by c or c^{-1}. This is an open set containing $\partial_c^{\mathcal{FQ}} H$ such that $q^{-1}(q(U)) = U$ and $a^n d^\infty \notin U$ for all $n \in \mathbb{N}$. Therefore, $q(U)$ is an open set in $\partial_c^{\mathcal{FQ}} G/\iota_*$ containing the point $q(\partial_c^{\mathcal{FQ}} H)$ but not containing any $q(a^n d^\infty)$.

Before proving the theorem let us recall some of the necessary machinery for relatively hyperbolic groups. Any bounded set meets finitely many cosets of the peripherals, and projections of peripheral sets to one another are uniformly bounded.

Given an (L, A)–quasi-geodesic γ, Drutu and Sapir [23] define the *saturation* $\mathrm{Sat}(\gamma)$ of γ to be the union of γ and all cosets gP of peripheral subgroups $P \in \mathcal{P}$ such that γ comes within distance M of gP, where M is a number depending on L and A. [23, Lemma 4.25] says there exists μ independent of γ such that the saturation of γ is μ–Morse. It follows that the analogous saturation of γ in $\tilde{G}$, that is, the union of γ and all horoballs sufficiently close to γ, is also Morse.

Sisto [38] extends these results, showing, in particular, that peripheral subgroups are strongly contracting.[4]

The other key definition is that of a *transition point* of γ, as defined by Hruska [29]. The idea is that a point of γ is *deep* if it is contained in a long subsegment of γ that is contained in a neighborhood of some gP, and a point is a transition point if it is not deep. A quasi-geodesic in γ is bounded Hausdorff distance from a path of the form $\beta_0 + \alpha_1 + \beta_1 + \alpha_2 + \cdots$ where the β_i are shortest paths connecting some $g_i P_i$ to some $g_{i+1} P_{i+1}$ and the α_i are paths in $g_i P_i$. The transition points are the points close to the β–segments. In $\tilde{G}$ there is an obvious way to shorten such a path by letting the α–segments relax into the corresponding horoballs. If the endpoints of α_i are x and y, this replaces α_i with a segment of length roughly $2\log_2 d_G(x, y)$ in $\tilde{G}$. This is essentially all that happens: if γ is a quasi-geodesic in G then take a geodesic $\hat{\gamma}$ with the same endpoints as γ in the coned-off space $\hat{G}$ obtained by collapsing each coset of a peripheral subgroup. Lift $\hat{g}$ to a nice α-β path in G as above. The β–segments are coarsely well-defined, because the cosets of peripheral subgroups are strongly contracting, and the union of the β segments is Hausdorff equivalent to the set of transition points of γ. Only the endpoints of the α–segments are coarsely well defined, but relaxing the α-segments to geodesics in

[4]Sisto does not use the term ‘strongly contracting’, but observe it is equivalent to the first two conditions of [38, Definition 2.1].

the corresponding horoball yields a uniform quasi-geodesic in $\tilde{G}$ (see [10, Section 7]). Since $\tilde{G}$ is hyperbolic, this is within bounded Hausdorff distance of any $\tilde{G}$ geodesic $\tilde{\gamma}$ with the same endpoints as γ. In particular, $\tilde{\gamma}$ comes boundedly close to the transition points of γ.

Proof of Theorem 7.6. We omit ι from the notation and think of G sitting as a subgraph of $\tilde{G}$. First we show that for $\zeta \in \partial_c G$ the sequence $(\alpha^\zeta_n)_{n\in\mathbb{N}}$ converges to a point of $\partial\tilde{G}$. Distances in G give an upper bound for distances in $\tilde{G}$, so all quasi-geodesics in G asymptotic to α^ζ also converge to this point in $\partial\tilde{G}$, which we define to be $\iota_*(\zeta)$. Let $(x \cdot y) := \frac{1}{2}(d(\mathbb{1}, x) + d(\mathbb{1}, y) - d(x, y))$ denote the Gromov product of $x, y \in \tilde{G}$ with respect to the basepoint $\mathbb{1}$ corresponding to the identity element of G. (See [11, Section III.H.3] for background on boundaries of hyperbolic spaces.)

To see that the sequence $(\alpha^\zeta_n)_{n\in\mathbb{N}}$ does indeed converge, there are two cases. If α^ζ has unbounded projection to gP for some $g \in G$ and $P \in \mathcal{P}$, then a tail of α^ζ is contained in a bounded neighborhood of gP, but leaves every bounded subset of gP. It follows that (α^ζ_n) converges to the parabolic point in $\partial\tilde{G}$ fixed by gPg^{-1} corresponding to the horoball attached to gP. Furthermore, the projection of the tail of α^ζ to gP is a contracting quasi-geodesic ray in gP (by Corollary 6.7), so P has non-trivial contracting boundary.

The other case is that α^ζ has bounded (not necessarily uniformly!) projection to every gP. Now, given any r there are only finitely many horoballs in $\tilde{G}$ that meet the r–neighborhood of $\mathbb{1}$. Since α^ζ has bounded projection to each of these, for sufficiently large s none of these are in $\mathrm{Sat}(\alpha^\zeta_{[s,\infty)})$. Since $\mathrm{Sat}(\alpha^\zeta_{[s,\infty)})$ is μ–Morse in $\tilde{G}$ for some μ independent of α^ζ, for any $m,\, n \geq s$, geodesics connecting α^ζ_m and α^ζ_n in $\tilde{G}$ stay outside the $(r-\mu(1,0))$–ball about $\mathbb{1}$. We conclude $\lim_{m,\,n\to\infty}(\alpha^\zeta_m \cdot \alpha^\zeta_n)_{\tilde{G}} = \infty$, so $(\alpha^\zeta_n)_{n\in\mathbb{N}}$ converges to a point in $\partial\tilde{G}$, which, in this case, is a conical point.

If α^ζ and α^η tend to the same conical point in $\partial\tilde{G}$ then the sets of transition points of α^ζ and α^η are unbounded and at bounded Hausdorff distance from one another in G. Since they are contracting geodesics in G they can only come close on unbounded sets if they are in fact asymptotic, so ι_* is injective at conical points.

Continuity: To show $\iota_*\colon \partial^{\mathcal{FG}}_c G \to \partial\tilde{G}$ is continuous we show that for all $\zeta \in \partial_c G$ and all r there exists an R such that for all $\eta \in V(\zeta, R))$ we have $(\iota_*(\zeta) \cdot \iota_*(\eta))_{\tilde{G}} > r$.

Recall that there is a bound B such that a $\tilde{G}$ geodesic comes B–close to the transition points of a G geodesic with the same endpoints. There exists B' so that $\mathrm{diam}\, \pi_{gP}(x) \leq B'$ for each $x \in G, g \in G, P \in \mathcal{P}$, and so that for any deep point x of a geodesic along gP we have $\mathrm{diam}\{x\} \cup \pi_{gP}(x) \leq B'$. Finally, there exists a constant B'' depending on B so that if $x, y \in G$ satisfy $d(\pi_{gP}(x), \pi_{gP}(y)) \geq B''$ for some $g \in G, P \in \mathcal{P}$, then any geodesic from x to y has a deep component along gP whose transition points at the ends are within B'' of $\pi_{gP}(x)$ and $\pi_{gP}(y)$, respectively.

Suppose ζ and r are given. If $\iota_*(\zeta)$ is conical then given any $r' \geq 0$ there is an R' such that for all $n \geq R'$ we have $d_{\tilde{G}}(\mathbb{1}, \alpha^\zeta_n) > r'$. Choose $R \geq R'$ such that α^ζ_R is a transition point, and moreover that any deep component along α^ζ within $\kappa'(\rho_\zeta, 1, 0) + B' + B''$ of α^ζ_R has distance at least R' from $\mathbb{1}$. If $\eta \in V(\zeta, R)$ then α^η comes $\kappa(\rho_\zeta, 1, 0)$–close to $\alpha^\zeta_{[R,\infty)}$ in G, so there is a point α^η_t that is $\kappa'(\rho_\zeta, 1, 0)$–close to α^ζ_R. If α^η_t is a deep point of α^η, let $g'P'$ be the corresponding coset. If

$d(\pi_{g'P'}(\mathbb{1}),\alpha^\eta_t) > J := B'' + 2B' + 2\kappa'(\rho,1,0))$ then the geodesic α^ζ must also have a deep component along $g'P'$ with one endpoint $(\kappa'(\rho_\zeta,1,0)+B'+B'')$–close to α^ζ_R and the other $z := \alpha^\zeta_s \in \bar{N}_{B''}\pi_{g'P'}(\mathbb{1})$; by assumption, $s \geq R'$. If α^η_t is a transition point of α^η, or if $d(\pi_{g'P'}(\mathbb{1}),\alpha^\eta_t) \leq J$, then $z := \alpha^\zeta_R$ is $(J + B'')$–close to a transition point of α^η. In either case then, z is a transition point of α^ζ which is $(J + B'')$–close to a transition point of α^η, and has $d(z,\mathbb{1}) \geq R'$. Thus, by the choice of B, there are points $x \in [\mathbb{1},\iota_*(\zeta)]$ and $y \in [\mathbb{1},\iota_*(\eta)]$ with $d_{\tilde{G}}(\mathbb{1},x) \geq r' - B$ and $d(x,y) \leq 2B + J + B''$. This allows us to bound $(\iota_*(\zeta)\cdot\iota_*(\eta))$ below in terms of these constants and the hyperbolicity constant for $\tilde{G}$, and by choosing r' large enough we guarantee $(\iota_*(\zeta)\cdot\iota_*(\eta)) > r$.

Now suppose $\iota_*(\zeta)$ is parabolic. Then there is some $R_0 \geq 0$, $M \geq 0$, $g \in G$, and $P \in \mathcal{P}$ such that $\alpha^\zeta_{[R_0,\infty)} \subset N_M gP$. For $R \gg R_0$, if $\eta \in V(\zeta,R)$ then α^η comes within distance $\kappa(\rho_\zeta,1,0)$ of $\alpha^\zeta_{[R,\infty)}$. If $\iota_*(\zeta) \neq \iota_*(\eta)$ then eventually α^η escapes from gP, so it has a transition point at G–distance greater than $R - R_0 - \kappa(\rho_\zeta,1,0)$ from $\alpha^\zeta_{R_0}$. This implies $\operatorname{diam}\pi_{gP}([\mathbb{1},\iota_*(\eta)]) > R - R_0 - C$, where C depends on M, B, the contraction function of gP, and $\kappa(\rho_\zeta,1,0)$. It follows from the geometry of the horoballs that for $\iota_*(\zeta)$ is the parabolic boundary point corresponding to gP and $y \in \partial\tilde{G}$ we have $(\iota_*(\zeta)\cdot y)$ is roughly $d_{\tilde{G}}(\mathbb{1},gP) + \log_2 \operatorname{diam}\pi_{gP}(\mathbb{1}) \cup \pi_{gP}(y)$, so by choosing $R > 2^r + R_0 + C$ we guarantee $(\iota_*(\zeta)\cdot\iota_*(\eta)) > r + d_{\tilde{G}}(\mathbb{1},gP) \geq r$.

Embedding: Suppose that $U' \subset \partial^{\mathcal{FQ}}_c G/\iota_*$ is open. Define $U := q^{-1}(U')$, which is open in $\partial^{\mathcal{FQ}}_c G$. We claim that for each $p \in \iota_*(U)$ there exists an $R_p > 0$ such that for $p' \in \iota_*(U)$, if $(p\cdot p') > R_p$ then $\iota_*^{-1}(p') \subset U$. Given the claim, the proof concludes by choosing, for each $p \in \iota_*(U)$, an open neighborhood V_p of p such that $V_p \subset \{p' \in \partial\tilde{G} \mid (p\cdot p') > R_p\}$, and setting $V = \bigcup_{p\in\iota_*(U)} V_p$. Then V is open and $\iota_*(\partial^{\mathcal{FQ}}_c G) \cap V = \iota_*(U)$, so that $\iota_* \circ q^{-1}(\partial^{\mathcal{FQ}}_c G/\iota_*) \cap V = \iota_* \circ q^{-1}(U')$.

First we prove the claim when p is conical. In this case there is a unique point $\zeta \in \iota_*^{-1}(p)$, and since U is open there exists $r_\zeta > 1$ such that $U(\zeta,r_\zeta) \subset U$. Let x be a transition point of α^ζ and choose R such that $d_{\tilde{G}}(\mathbb{1},x) \leq R$. If the claim is false then there exists an $\eta \in \partial^{\mathcal{FQ}}_c G$ such that $(\iota_*(\zeta)\cdot\iota_*(\eta)) > R$ and $\eta \notin U(\zeta,r_\zeta)$. Since $\eta \notin U(\zeta,r_\zeta)$, there exists an L and A and a continuous (L,A)–quasi-geodesic $\gamma \in \eta$ such that the last point $y \in \alpha^\zeta$ such that $d_G(y,\gamma) = \kappa(\rho_\zeta,L,A)$ satisfies $d(\mathbb{1},y) < r_\zeta$. By Observation 5.7, we can take $L < \sqrt{r_\zeta/3}$ and $A < r_\zeta/3$.

By hyperbolicity, geodesics in $\tilde{G}$ tending to $\iota_*(\zeta)$ and $\iota_*(\eta)$ remain boundedly close together for distance approximately $(\iota_*(\zeta)\cdot\iota_*(\eta)) > R$. Since x is a transition point of α^ζ there is a B such that any geodesic $[\mathbb{1},\iota_*(\zeta)]$ comes B–close to x, so some point z' in a geodesic $[\mathbb{1},\iota_*(\eta)]$ also comes boundedly close to x. If the point z' lies in a horoball along which γ has a deep component, whose transition points at both ends are close to $[\mathbb{1},\iota_*(\zeta)]$, then this deep component must be of bounded size else $x \in \alpha^\zeta$ would not be a transition point. It follows that γ must contain a point z at bounded distance from x. Since x and z are transition points, we also get a bound on $d_G(x,z)$. Then, by applying Lemma 4.6, we get an upper bound on $d_G(\mathbb{1},x)$ depending on r_ζ and ρ_ζ, but independent of γ and η. However, if the set of transition points of α^ζ is bounded in G then it is bounded in $\tilde{G}$, which would imply $\iota_*(\zeta) = p$ is parabolic, contrary to hypothesis.

Now suppose p is parabolic. By hypothesis, its stabilizer G_p is a hyperbolic group conjugate into $\mathcal{P}$. Since the maps are G–equivariant we may assume $G_p \in \mathcal{P}$. We may assume that we have chosen a generating set for G extending one for G_p. Since

G_p is quasi-isometrically embedded, by Lemma 6.4, there exist $L_p \geq 1$, $A_p \geq 0$ with $\frac{1}{L_p} d_{G_p}(x,y) - A_p \leq d(x,y) \leq d_{G_p}(x,y)$ for all $x, y \in G_p$. The contracting boundary $\partial_c^{\mathcal{FQ}} G_p$ embeds into $\partial_c^{\mathcal{FQ}} G$ and is compact—it is homeomorphic to the Gromov boundary of G_p. Geodesic rays in G_p are uniformly contracting, by hyperbolicity, so there exists a contracting function ρ such that for all $\zeta \in \partial_c^{\mathcal{FQ}} G_p \subset \partial_c^{\mathcal{FQ}} G$ we have that ζ is ρ–contracting.

We will verify the following fact at the end of this proof:

$$\exists R'_p > 1 \, \forall \xi \in \partial_c^{\mathcal{FQ}} G_p, \, U(\xi, R'_p) \subset U \tag{10}$$

Assuming (10), let $\eta \in \iota_*^{-1}(p')$ and let $\gamma \in \eta$ be a continuous (L, A)–quasi-geodesic for some $L < \sqrt{R'_p/3}$, $A < R'_p/3$. Since G_p is strongly contracting, there exist C and C' such that the diameter of $\pi_{G_p}(\alpha^\eta \cap N_C^c G_p)$ is at most C'. We define $\pi_{G_p}(\eta)$ to be this finite diameter set. Since γ stays $\kappa'(\rho_\eta, L, A)$–close to α^η, strong contraction implies $\pi_{G_p}(\gamma \cap N^c_{\kappa'(\rho_\eta, L, A)} G_p) \subset N_{C'} \pi_{G_p}(\eta)$. We apply [38, Lemma 1.15] to any sufficiently long[5] initial subsegment of γ to conclude there is a function K, a point $z \in \gamma$, and a point $x \in \pi_{G_p}(\eta)$ such that $d(x, z) \leq K(L, A)$.

Since G_p is a hyperbolic group there exists a constant D such that every point is within D of a geodesic ray based at $\mathbb{1}$. Let $\xi \in \partial_c^{\mathcal{FQ}} G_p$ be a point such that there is a G_p–geodesic $[\mathbb{1}, \xi]$ containing a point w with $d(w, x) \leq D$. Since this G_p–geodesic is a (L_p, A_p)–quasi-geodesic in G, there exists $y' \in \alpha^\xi$ such that $d(y', w) \leq \kappa'(\rho, L_p, A_p)$.

We have $d(z, y') \leq K(L, A) + D + \kappa'(\rho, L_p, A_p)$ and:

$$\begin{aligned} d(y', \mathbb{1}) &\geq d(x, \mathbb{1}) - D - \kappa'(\rho, L_p, A_p) \\ &\geq \frac{d_{G_p}(\mathbb{1}, \pi_{G_p}(\eta))}{L_p} - A_p - D - \kappa'(\rho, L_p, A_p) \end{aligned}$$

Lemma 4.6 implies that, for M and λ as in the lemma, γ comes within distance $\kappa(\rho, L, A)$ of α^ξ outside the ball of radius:

(11)
$$\frac{d_{G_p}(\mathbb{1}, \pi_{G_p}(\eta))}{L_p} - A_p - D - \kappa'(\rho, L_p, A_p) - M(K(L, A) + D + \kappa'(\rho, L_p, A_p)) - \lambda(\rho, L, A)$$

Now, $d_{G_p}(\mathbb{1}, \pi_{G_p}(\eta)) \asymp 2^{(p \cdot p')} > 2^{R_p}$. Since $L < \sqrt{R'_p/3}$, $A < R'_p/3$, all the negative terms are bounded in terms of R'_p, so we can guarantee (11) is greater than R'_p by taking R_p sufficiently large. This means the quasi-geodesic γ does not witness $\eta \notin U(\xi, R'_p)$. Since γ was arbitrary, $\eta \in U(\xi, R'_p)$, which, by (10), is contained in U. Thus, $\iota_*^{-1}(p') \subset U$ when $(p \cdot p') > R_p$ for R_p sufficiently large with respect to R'_p.

It remains to determine R'_p and verify (10). Define:

$$\theta(s) := s + M(\kappa'(\rho, \sqrt{s/3}, s/3) + \kappa(\rho, 1, 0)) + \lambda(\rho, \sqrt{s/3}, s/3)$$

Since U is open, for every $\zeta \in \partial_c^{\mathcal{FQ}} G_p$ there exists r_ζ such that $U(\zeta, r_\zeta) \subset U$. For each $\zeta \in \partial_c^{\mathcal{FQ}} G_p$, let U_ζ be an open neighborhood of ζ such that $U_\zeta \subset U(\zeta, \theta(r_\zeta))$. Then $\{U_\zeta\}_{\zeta \in \partial_c^{\mathcal{FQ}} G_p}$ is an open cover of $\partial_c^{\mathcal{FQ}} G_p$, which is compact, so there exists

[5] Long enough to leave the $\max\{D_0(L, A), \kappa'(\rho_\eta, L, A)\}$–neighborhood of G_p where D_0 is as in [38, Lemma 1.15].

a finite set $F \subset \partial_c^{\mathcal{FQ}} G_p$ such that $\partial_c^{\mathcal{FQ}} G_p \subset \bigcup_{\zeta \in F} U_\zeta \subset \bigcup_{\zeta \in F} U(\zeta, \theta(r_\zeta))$. Define $r := \max_{\zeta \in F} r_\zeta$ and:

$$R'_p := r + \kappa'(\rho, 1, 0) + M(\kappa(\rho, \sqrt{r/3}, r/3) + \kappa'(\rho, 1, 0)) + \lambda(\rho, \sqrt{r/3}, r/3)$$

Suppose that $\xi \in \partial_c^{\mathcal{FQ}} G_p$ and $\eta \in U(\xi, R'_p)$. There exists $\zeta \in F$ such that $\xi \in U_\zeta \subset U(\zeta, \theta(r_\zeta))$. Let $\gamma \in \eta$ be a continuous (L, A)–quasi-geodesic for some $L < \sqrt{r_\zeta/3}$, $A < r_\zeta/3$. Since $\eta \in U(\xi, R'_p)$, there exist $z \in \gamma$ and $x \in \alpha^\xi$ such that $d(x, z) \leq \kappa(\rho, L, A)$ and $d(x, \mathbb{1}) \geq R'_p$.

There are now two cases to consider. First, suppose that there exists $y' \in \alpha^\zeta$ with $d(x, y) \leq \kappa'(\rho, 1, 0)$. Then $d(y', \mathbb{1}) \geq R'_p - \kappa'(\rho, 1, 0)$. By Lemma 4.6, γ comes $\kappa(\rho, L, A)$ close to α^ζ outside the ball of radius:

$$R'_p - \kappa'(\rho, 1, 0) - M(\kappa'(\rho, 1, 0) + \kappa(\rho, L, A)) - \lambda(\rho, L, A)$$

By definition of R'_p and the conditions $L < \sqrt{r_\zeta/3}$, $A < r_\zeta/3$, this radius is at least r, which is at least r_ζ, so γ does not witness $\eta \notin U(\zeta, r_\zeta)$.

The second case, where the above y' does not exist, is the case that x occurs after α^ξ has already escaped α^ζ. In this case there exists x' between $\mathbb{1}$ and x on α^ξ and $y' \in \alpha^\zeta$ such that $d(x', y') \leq \kappa(\rho, 1, 0)$ and $d(\mathbb{1}, y') \geq \theta(r_\zeta)$. Moreover, by Lemma 4.4, there exists $z' \in \gamma$ such that $d(z', x') \leq \kappa'(\rho, L, A)$. By Lemma 4.6, γ comes within distance $\kappa(\rho, L, A)$ of α^ζ outside the ball of radius:

$$\theta(r_\zeta) - M(\kappa(\rho, 1, 0) + \kappa'(\rho, L, A)) - \lambda(\rho, L, A)$$

By definition of θ and the conditions $L < \sqrt{r_\zeta/3}$, $A < r_\zeta/3$, this radius is at least r_ζ, so γ does not witness $\eta \notin U(\zeta, r_\zeta)$. This verifies (10). ☐

8. Metrizability for group boundaries

In this section let G be a finitely generated group with nonempty contracting boundary. Consider the Cayley graph of G with respect to some fixed finite generating set, which is a proper geodesic metric space we again denote G, and take the basepoint to be the vertex $\mathbb{1}$ corresponding to the identity element of the group.

There is a natural action of G on $\partial_c^{\mathcal{FQ}} G$ by homeomorphisms defined by sending $g \in G$ to the map that takes $\zeta \in \partial_c G$ to the equivalence class of the quasi-geodesic that is the concatenation of a geodesic from $\mathbb{1}$ to g and the geodesic $g\alpha^\zeta$.

The following two results generalize results of Murray [33] for the case of $\partial_c^{\mathcal{DL}} X$ when X is CAT(0). See also [28].

Remark. If $\beta \colon I \to G$ is a geodesic and β_m is a vertex for some $m \in \mathbb{Z} \cap I$ then β_n is a vertex for every $n \in \mathbb{Z} \cap I$. Vertices in the Cayley graph are in one-to-one correspondence with group elements. If Z is a subset of the Cayley graph we use $\beta_n Z$ to denote the image of Z under the action by the group element corresponding to the vertex β_n.

We will always parameterize bi-infinite geodesics in G so that integers go to vertices.

Proposition 8.1. *G is virtually (infinite) cyclic if and only if $G \curvearrowright \partial_c^{\mathcal{FQ}} G$ has a finite orbit.*

Proof. If G is virtually cyclic then $|\partial_c G| = 2$ and every orbit is finite.

Conversely, if G has a finite orbit then it has a finite index subgroup that fixes a point in $\partial_c G$. The inclusion of a finite index subgroup is a quasi-isometry, so we may assume that G fixes a point $\zeta \in \partial_c G$.

Let $\alpha \in \zeta$ be geodesic and ρ–contracting. Let β be an arbitrary geodesic ray or segment with $\beta_0 = \mathbb{1}$. Since $G\zeta = \zeta$, for all $n \in \mathbb{N}$ the geodesic rays α and $\beta_n\alpha$ are asymptotic. By Theorem 3.4, α and $\beta_n\alpha$ eventually stay within distance κ'_ρ of one another. Truncate α and $\beta_n\alpha$ when their distance is κ'_ρ. By Lemma 3.6, these segments are contracting, and they form a geodesic almost triangle with $\beta_{[0,n]}$, so, by Lemma 3.8, $\beta_{[0,n]}$ is ρ'–contracting for some $\rho' \asymp \rho$ depending only on ρ. Since this is true uniformly for all n, β is ρ'–contracting. Since β was arbitrary and G is homogeneous, every geodesic in G is uniformly contracting, which means G is hyperbolic and $\partial_c^{\mathcal{FQ}} G$ is the Gromov boundary. If G is hyperbolic and not virtually cyclic then its boundary is uncountable and every orbit is dense, hence infinite. □

Proposition 8.2. *Suppose $|\partial_c^{\mathcal{FQ}} G| > 2$, and fix a point $\eta \in \partial_c G$. For every $\zeta \in \partial_c G$ and every $r \geq 1$ there exists an $R' \geq 1$ such that for all $R_2 \geq R_1 \geq R'$ there exist $g \in G$ such that $\zeta \in U(g\eta, R_2) \subset U(g\eta, R_1) \subset U(\zeta, r)$.*

Corollary 8.3. *$\partial_c^{\mathcal{FQ}} G$ is separable.*

Corollary 8.4. *If G is not virtually cyclic then $G \curvearrowright \partial_c^{\mathcal{FQ}} G$ is minimal, that is, every orbit is dense.*

Remark. For the corollaries we just need to know that we can push η into $U(\zeta, r)$ via the group action. The stronger statement of Proposition 8.2 is used in Proposition 8.5 to upgrade first countable and separable to second countable. The reason for having two parameters R_1 and R_2 is to be able to apply Corollary 5.10 in case $U(g\eta, R_1)$ is not an open set.

Proof of Proposition 8.2. By Proposition 8.1, $G \curvearrowright \partial_c^{\mathcal{FQ}} G$ does not have a finite orbit, so there exists a $g' \in G$ with $\eta' := g'\eta \neq \eta$. Let β be a geodesic joining η' and η. It suffices to assume $\beta_0 = \mathbb{1}$; otherwise, we could consider $\beta' := \beta_0^{-1}\beta$, which is a geodesic with $\beta'_0 = \mathbb{1}$ and endpoints in $G\eta$.

Let $\alpha := \alpha^\zeta$ be the geodesic representative of ζ. Choose ρ so that α, $\beta_{[0,\infty)}$, and $\bar{\beta}_{[0,-\infty)}$ are all ρ–contracting.

For each integral $t \gg 0$, at most one of $\alpha_t\beta_{[0,\infty)}$ and $\alpha_t\bar{\beta}_{[0,-\infty)}$ remains in the closed $\kappa'(\rho, 1, 0)$–neighborhood of $\alpha_{[0,t]}$ for distance greater than $2\kappa'(\rho, 1, 0)$, otherwise we contradict the fact that $\alpha_t\beta$ is a geodesic. Define $g_t := \alpha_t$ if $\alpha_t\beta_{[0,\infty)}$ does not remain in the closed $\kappa'(\rho, 1, 0)$–neighborhood of $\alpha_{[0,t]}$ for distance greater than $2\kappa'(\rho, 1, 0)$. Otherwise, define $g_t := \alpha_t g'$. For each $s \in \mathbb{N}$ consider a geodesic triangle with sides $\alpha_{[0,t]}$, $g_t\beta_{[0,s]}$, and a geodesic $\delta^{s,t}$ joining α_0 to $g_t\beta_s$. By Lemma 3.6, the first two sides are uniformly contracting, so $\delta^{s,t}$ is as well, by Lemma 3.8. Since G is proper, for each fixed t a subsequence of the $\delta^{s,t}$ converges to a contracting geodesic ray $\delta^t \in g_t\eta$. See Figure 5. Moreover, since the $\delta^{s,t}$ are uniformly contracting, the contraction function for δ^t does not depend on t. Now, for any given t it is possible that δ^t does not coincide with the chosen representative $\alpha^{g_t\eta}$ of $g_t\eta$, but they are asymptotic, so Lemma 4.5 tells us that uniform contraction for the δ^t implies uniform contraction for the $\alpha^{g_t\eta}$. Thus, there is a ρ' independent of t such that $\alpha^{g_t\eta}$ is ρ'–contracting. Furthermore, the defining condition for g_t guarantees that there is a C independent of t such that the geodesic representative

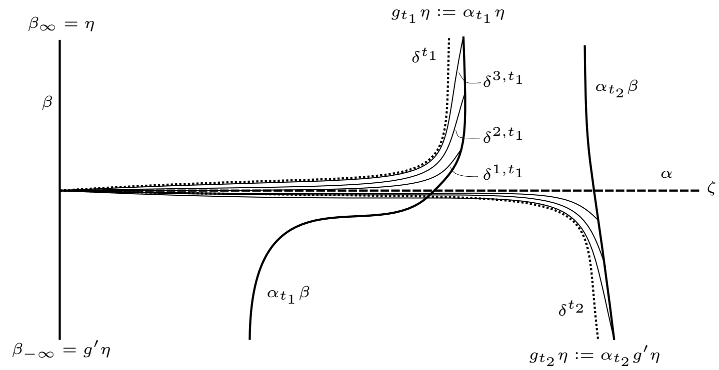

FIGURE 5.

$\alpha^{g_t\eta}$ comes within distance $\kappa(\rho,1,0)$ of α outside of $N_{t-C}\mathbb{1}$, which implies that $\alpha^{g_t\eta}_{[0,t-C]} \subset \bar{N}_{2\kappa'(\rho,1,0)}\alpha_{[0,t-C]}$.

First we give a condition that implies $\zeta \in U(g_t\eta, R)$. Suppose:

$$t \geq R + C + 2M(\kappa'(\rho, \sqrt{R/3}, R/3) + \kappa'(\rho,1,0)) + \lambda(\rho', \sqrt{R/3}, R/3) \tag{12}$$

Suppose that $\gamma \in \zeta$ is a continuous (L, A)–quasi-geodesic. By Observation 5.7 is suffices to consider L^2, $A < R/3$. By Corollary 4.3, $\gamma \subset \bar{N}_{\kappa'(\rho,L,A)}\alpha$, so there is a point γ_a that is $(2\kappa'(\rho,L,A) + 2\kappa'(\rho,1,0))$–close to $\alpha^{g_t\eta}_{t-C}$. By Lemma 4.6, γ comes $\kappa(\rho',L,A)$–close to $\alpha^{g_t\eta}$ outside the ball around $\mathbb{1}$ of radius $t-C-M(2\kappa'(\rho,L,A)+2\kappa'(\rho,1,0)) - \lambda(\rho',L,A)$. By (12), this is at least R. Since $\gamma \in \zeta$ was arbitrary, $\zeta \in U(g_t\eta, R)$.

Next, we give a condition that implies $U(g_t\eta, R) \subset U(\zeta, r)$. Suppose:

$$t - C \geq R \geq r + M(2\kappa'(\rho,1,0) + 2\kappa'(\rho', \sqrt{r/3}, r/3)) + \lambda(\rho, \sqrt{r/3}, r/3) \tag{13}$$

Suppose that γ is a continuous (L, A)–quasi-geodesic such that $[\gamma] \in U(g_t\eta, R)$. By Observation 5.7, it suffices to consider L^2, $A < r/3$. By definition, γ comes $\kappa(\rho',L,A)$ close to $\alpha^{g_t\eta}$ outside $N_R\mathbb{1}$, so some point γ_b is $2\kappa'(\rho',L,A)$–close to $\alpha^{g_t\eta}_R$, which implies that $d(\gamma_b, \alpha_R) \leq (2\kappa'(\rho',L,A) + 2\kappa'(\rho,1,0))$. Now apply Lemma 4.6 to see that γ comes $\kappa(\rho,L,A)$–close to α outside the ball around $\mathbb{1}$ of radius $R - M(2\kappa'(\rho,1,0) + 2\kappa'(\rho',L,A)) - \lambda(\rho,L,A)$, which is at least r, by (13). Thus, $U(g_t\eta, R) \subset U(\zeta, r)$.

Equipped with these two conditions, we finish the proof. The contraction functions ρ and ρ' are determined by ζ and η. Given these and any $r \geq 1$, define R' to be the right hand side of (13). Given any $R_2 \geq R_1 \geq R'$, it suffices to define $g := g_t$ for any t large enough to satisfy both condition (12) for $R = R_2$ and condition (13) for $R = R_1$. □

Proposition 8.5. *$\partial_c^{\mathcal{FQ}}G$ is second countable.*

Proof. If G is virtually cyclic then $\partial_c^{\mathcal{FQ}}G$ is the discrete space with two points, and we are done. Otherwise, fix any $\eta \in \partial_c G$. For each $g \in G$ and $n \in \mathbb{N}$ choose an open set $U_{g,n}$ such that $U(g\eta, \psi(\rho_{g\eta}, n)) \subset U_{g,n} \subset U(g\eta, n)$ as in Corollary 5.10.

Let U be a non-empty open set and let ζ be a point in U. By definition of $\mathcal{FQ}$, there exists an $r \geq 1$ such that $U(\zeta, r) \subset U$. Let R' be the constant of Proposition 8.2 for ζ and r, and let $R' \leq R_1 \in \mathbb{N}$. As noted there, there exists a sublinear ρ' such that the points $g_t\eta$ in the proof of Proposition 8.2 are all ρ'–contracting. Define $R_2 := \psi(\rho', R_1) \geq \psi(\rho_{g_t\eta}, R_1)$. Combining Proposition 8.2 and the definition of the sets $U_{g,n}$, there exists $g \in G$ such that:

$$\zeta \in U(g\eta, R_2) \subset U_{g,R_1} \subset U(g\eta, R_1) \subset U(\zeta, r) \subset U$$

Thus, $\mathcal{U} := \{U_{g,n} \mid g \in G,\, n \in \mathbb{N}\}$ is a countable basis for $\partial_c^{\mathcal{FQ}} G$. □

Corollary 8.6. *$\partial_c^{\mathcal{FQ}} G$ is metrizable.*

Proof. $\partial_c^{\mathcal{FQ}} G$ is second countable by Proposition 8.5, regular by Proposition 5.13, and Hausdorff by Proposition 5.12. The Urysohn Metrization Theorem says every second countable, regular, Hausdorff space is metrizable. □

It is an interesting open problem to describe, in terms of the geometry of G, a metric on $\partial_c^{\mathcal{FQ}} G$ that is compatible with $\mathcal{FQ}$.

9. Dynamics

Definition 9.1. An element $g \in G$ is *contracting* if $\mathbb{Z} \to G : n \mapsto g^n$ is a quasi-isometric embedding whose image is a contracting set.

We use g^∞ and $g^{-\infty}$ to denote the equivalence classes of the contracting quasi-geodesic rays based at $\mathbb{1}$ corresponding to the non-negative powers of g and non-positive powers, respectively. These are distinct points in $\partial_c G$.

Lemma 9.2. *Given a contracting element $g \in G$, an $r \geq 1$, and a point $\zeta \in \partial_c^{\mathcal{FQ}} G \setminus \{g^\infty, g^{-\infty}\}$ there exists an $R' \geq 1$ such that for every $R \geq R'$ and every $n \in \mathbb{N}$ we have $U(\zeta, R) \subset g^{-n} U(g^n \zeta, r)$.*

Proof. Since g is contracting there is a sublinear function ρ such that all geodesic segments joining powers of g as well as geodesic rays based at $\mathbb{1}$ going to g^∞, $g^{-\infty}$, or $g^m\zeta$ for any $m \in \mathbb{Z}$ are all ρ–contracting.

Consider a geodesic triangle with sides $g^{-m}\alpha^{g^m\zeta}$, α^ζ, and a geodesic from $\mathbb{1}$ to g^{-m} for arbitrary $m \in \mathbb{Z}$. All sides are ρ–contracting, and such a triangle is B–thin for some B independent of m. Thus, for sufficiently large s', independent of m, the point $\alpha^\zeta_{s'}$ is more than B–far from $\langle g \rangle$, hence B–close to $g^{-m}\alpha^{g^m\zeta}$. Since α^ζ and $g^{-m}\alpha^{g^m\zeta}$ are asymptotic, they eventually come $\kappa(\rho, 1, 0)$–close and then stay $\kappa'(\rho, 1, 0)$–close thereafter. Theorem 3.4 says the first time they come $\kappa(\rho, 1, 0)$ close occurs no later than $s' + \rho'(B)$. Take $R' \geq s' + \rho'(B)$, which guarantees $d(\alpha^\zeta_s, g^{-m}\alpha^{g^m\zeta}) \leq \kappa'(\rho, 1, 0)$ for all $m \in \mathbb{Z}$ and all $s \geq R'$.

Let L' and A' be the constants of Lemma 4.7 for ρ, $L = \sqrt{r/3}$ and $A = r/3$, and let M and λ be as in Lemma 4.6. Take $T := 1 + r + M(\kappa(\rho, L', A') + \kappa'(\rho, 1, 0)) + \lambda(\rho, 1, 0)$. We require further that R' is larger than L' and A' and large enough so that for all $s \geq R'$ we have $d(\alpha^\zeta_s, \langle g \rangle) > T + \kappa'(\rho, 1, 0)$.

Suppose, for a contradiction, that there exist some $n \in \mathbb{N}$ and $R \geq R'$ such that there exists a point $\eta \in U(\zeta, R) \setminus g^{-n} U(g^n\zeta, r)$. Since $\eta \notin g^{-n} U(g^n\zeta, r)$, for some L, A there exists a continuous (L, A)–quasi-geodesic $\gamma \in g^n\eta$ that does not come $\kappa(\rho, L, A)$–close to $\alpha^{g^n\zeta}$ outside $N_r\mathbb{1}$. By Observation 5.7, it suffices to consider the case that $L < \sqrt{r/3}$ and $A < r/3$.

As in Lemma 4.7, construct a continuous (L', A')–quasi-geodesic δ that first follows a geodesic from $\mathbb{1}$ towards g^{-n}, then a geodesic segment of length $\kappa(\rho, L, A)$, and then follows a tail of $g^{-n}\gamma$. Since it shares a tail with $g^{-n}\gamma$, we have $\delta \in \eta$. Since $\eta \in U(\zeta, R)$, there is some $s \geq R$ such that δ comes within distance $\kappa(\rho, L', A')$ of α^ζ_s. Our choice of R' guarantees that $d(\alpha^\zeta_s, g^{-n}\alpha^{g^n\zeta}) \leq \kappa'(\rho, 1, 0)$

and $d(\alpha^\zeta_s, \langle g\rangle) > T + \kappa'(\rho, 1, 0)$. The latter implies that the point of δ close to α^ζ_s is a point of $g^{-n}\gamma$, so there is a point of $g^{-n}\gamma$ that comes within distance $\kappa(\rho, L', A') + \kappa'(\rho, 1, 0)$ of a point x of $g^{-n}\alpha^{g^n\zeta}$ such that $d(x, \langle g\rangle) \geq T$. Lemma 4.6 says that there is a point y on $g^{-n}\alpha^{g^n\zeta}$ such that $d(y, g^{-n}\gamma) = \kappa(\rho, L, A)$ and $d(x, y) \leq M(\kappa(\rho, L', A') + \kappa'(\rho, 1, 0)) + \lambda(\rho, L, A)$. The definition of T implies $d(y, g^{-n}) \geq d(y, \langle g\rangle) \geq d(x, \langle g\rangle) - d(x, y) > r$. But then $g^n y$ is a point of $\alpha^{g^n\zeta}$ with $d(g^n y, \mathbb{1}) > r$ and $d(g^n y, \gamma) = \kappa(\rho, L, A)$, contradicting the definition of γ. □

Lemma 9.3. *Given a contracting element $g \in G$, an $r \geq 1$, and a point $\zeta \in \partial^{\mathcal{FQ}}_c G \setminus \{g^{-\infty}\}$ there exist constants $R' \geq 1$ and N such that for all $R \geq R'$ and $n \geq N$ we have $g^n U(\zeta, R) \subset U(g^\infty, r)$.*

Proof. The lemma is easy if $\zeta = g^\infty$, so assume not. Since g is contracting the geodesics $\alpha^{g^n\zeta}$ are uniformly contracting. Let ρ be a sublinear function such that α^{g^∞}, $\alpha^{g^{-\infty}}$, and all $\alpha^{g^n\zeta}$ are ρ–contracting. Since these geodesics are uniformly contracting, ideal geodesic triangles with vertices g^∞, $g^{-\infty}$, and $g^n\zeta$ are uniformly thin, independent of n. Thus, for N sufficiently large and for all $n \geq N$ we have that $\alpha^{g^n\zeta}$ stays $\kappa'(\rho, 1, 0)$ close to α^{g^∞} for distance greater than $S' := 1 + r + \lambda(\rho, \sqrt{r/3}, r/3) + M(\kappa'(\rho, \sqrt{r/3}, r/3) + \kappa'(\rho, 1, 0))$, where M and λ are as in Lemma 4.6.

Suppose that $\eta \in U(g^n\zeta, S)$ for some $n \geq N$ and $S \geq S'$. Let $\gamma \in \eta$ be a continuous (L, A)–quasi-geodesic for some L^2, $A \leq r/3$. By hypothesis, γ comes $\kappa(\rho, L, A)$–close to $\alpha^{g^n\zeta}$ outside $N_S\mathbb{1}$. Therefore, γ stays $\kappa'(\rho, L, A)$–close to $\alpha^{g^n\zeta}$ in $N_S\mathbb{1}$. By our choice of N, this implies γ stays $(\kappa'(\rho, L, A) + \kappa'(\rho, 1, 0))$–close to α^{g^∞} in $N_S\mathbb{1}$. By our choice of S and Lemma 4.6, γ comes $\kappa(\rho, L, A)$–close to α^{g^∞} outside the neighborhood of $\mathbb{1}$ of radius:

$$\begin{aligned} &S - M(\kappa'(\rho, L, A) + \kappa'(\rho, 1, 0)) - \lambda(\rho, L, A) \\ &\qquad \geq S' - M(\kappa'(\rho, \sqrt{r/3}, r/3) + \kappa'(\rho, 1, 0)) - \lambda(\rho, \sqrt{r/3}, r/3) > r \end{aligned}$$

Since γ was arbitrary, $\eta \in U(g^\infty, r)$, thus $U(g^n\zeta, S) \subset U(g^\infty, r)$.

By Lemma 9.2, given g, S', and ζ there exists an R' such that for all $R \geq R'$ and every $n \in \mathbb{N}$ we have $U(\zeta, R) \subset g^{-n}U(g^n\zeta, S')$. Thus, for this R' and N as above we have, for all $n \geq N$ and $R \geq R'$, that $g^n U(\zeta, R) \subset U(g^n\zeta, S') \subset U(g^\infty, r)$. □

Theorem 9.4 (Weak North-South dynamics for contracting elements)**.** *Let $g \in G$ be a contracting element. For every open set V containing g^∞ and every compact set $C \subset \partial^{\mathcal{FQ}}_c G \setminus \{g^{-\infty}\}$ there exists an N such that for all $n \geq N$ we have $g^n C \subset V$.*

We remark that if Theorem 9.4 were true for *closed* sets and G contained contracting elements without common powers then we could play ping-pong to produce a free subgroup of G. Such a result cannot be true in this generality because there are Tarski Monsters, non-cyclic groups such that every proper subgroup is cyclic, such that every non-trivial element is Morse, hence, contracting [34, Theorem 1.12].

Proof of Theorem 9.4. Since V is an open set containing g^∞ there exists some $r > 0$ such that $U(g^\infty, r) \subset V$. For this r and for each $\zeta \in C$ there exist R_ζ and N_ζ as in Lemma 9.3 such that for all $n \geq N_\zeta$ we have $g^n U(\zeta, R_\zeta) \subset U(g^\infty, r)$. By Proposition 5.5, $U(\zeta, R_\zeta)$ is a neighborhood of ζ, so there exists an open set U'_ζ such that $\zeta \in U'_\zeta \subset U(\zeta, R_\zeta)$. The collection $\{U'_\zeta \mid \zeta \in C\}$ is an open cover of C, which is compact, so there exists a finite subset C' of C such that $\{U'_\zeta \mid \zeta \in C'\}$ covers C. Define $N := \max_{\zeta\in C'} N_\zeta$. For every $n \geq N$ we then have:

$$\begin{aligned} g^n C &\subset g^n(\bigcup_{\zeta\in C'} U'_\zeta) \subset g^n(\bigcup_{\zeta\in C'} U(\zeta, R_\zeta)) \\ &= \bigcup_{\zeta\in C'} g^n U(\zeta, R_\zeta) \subset U(g^\infty, r) \subset V \end{aligned}$$ □

10. Compactness

In this section we characterize when the contracting boundary of a group is compact, Theorem 10.1, and give a partial characterization of when the limit set of a group is compact, Proposition 10.2.

Theorem 10.1. *Let G be an infinite, finitely generated group. Consider the Cayley graph of G, which we again denote G, with respect to a fixed finite generating set. The following are equivalent:*

1. *Geodesic rays in G are uniformly contracting.*
2. *Geodesic segments in G are uniformly contracting.*
3. *G is hyperbolic.*
4. *$\partial_c^{\mathcal{DL}} G$ is non-empty and compact.*
5. *$\partial_c^{\mathcal{FQ}} G$ is non-empty and compact.*
6. *Every geodesic ray in G is contracting.*

Remark. Work of Cordes and Durham [17] implies '(4) implies (3)'. Roughly the same argument we use for '(1) implies (2)' is contained in the proof of [17, Proposition 4.2]. More interestingly, they prove [17, Lemma 4.1] that compact subsets of the Morse boundary (of a space) are uniformly Morse, which is a more general version of '(4) implies (1)'. We specifically designed the topology of fellow-travelling quasi-geodesics to allow sequences with decaying contraction/Morse functions to converge when geometrically appropriate, so the corresponding statement cannot be true in our setting. In particular, the equivalence of (5) and (6) with (1)-(4) does not follow from their result.

Remark. Previous attempts have been made to prove results similar to '(6) implies (1)'. We point out a difficulty in the obvious approach. Suppose that $(\gamma^n)_{n\in\mathbb{N}}$ is a sequence of geodesics with decaying Morse functions. Let δ^n be paths witnessing the decaying Morse functions, by which we mean that there exist L and A such that for each n the path δ^n is an (L, A)–quasi-geodesic with endpoints on γ^n, and that δ^n is not contained in $N_n\gamma^n$. Let β^n denote the subsegment of γ^n between the endpoints of δ^n. We may assume by translation that for all n the basepoint $\mathbb{1}$ is approximately the midpoint of β^n. By properness, there is a subsequence of (γ^n) that converges to a geodesic γ through $\mathbb{1}$. One would guess that γ is not Morse, but this is not true in general; explicit counterexamples can be constructed. The problem is that the convergence to γ can be much slower than the growth of $|\beta^n|$ so that the subsegment of γ^n that agrees with γ can be a vanishingly small fraction of β^n. In this case the segments δ^n may not have endpoints close to γ, so no conclusion can be drawn.

It seems difficult to fix this argument. Instead, our strategy, roughly, will be to construct for each i a translate $g_i\delta^i$ of δ^i and for each n a geodesic ray that passes suitably close to both endpoints of $g_i\delta^i$ for all $i \leq n$. We argue that a subsequence of these rays converges to a non-Morse geodesic ray.

Proof. Assume (1). Since G is infinite and finitely generated, there exists a geodesic ray α based at $\mathbb{1}$. Recall that for $n \in \mathbb{N}$ the point α_n is a vertex of the Cayley graph, so it corresponds to a unique element of the group G. Thus, $\alpha_n^{-1}\alpha$ is simply the translate of α by the isometry of G defined by left multiplication by the element α_n^{-1}. Since $\mathbb{1} = \alpha_n^{-1}\alpha_n \in \alpha_n^{-1}\alpha$, the sequence $(\alpha_n^{-1}\alpha)_{n\in\mathbb{N}}$ has a subsequence that converges to a bi-infinite geodesic β containing $\mathbb{1}$. By construction, subsegments of β are close to subsegments of translates of α, so by Lemma 3.6 and Lemma 3.7, β is contracting, with contraction function determined by the uniform bound for rays. Let β^+ and β^- denote the two rays based at $\mathbb{1}$ such that $\beta = \beta^+ \cup \beta^-$.

Let g be an arbitrary non-trivial element of G. Consider the ideal geodesic triangle with one side $g\beta$ and whose other two sides are geodesic rays based at $\mathbb{1}$ with endpoints $g\beta^+_\infty$ and $g\beta^-_\infty$. The sides of this triangle are uniformly contracting, so it is uniformly thin, so there is some constant C such that every point on $g\beta$ is C–close to one of the other two sides. In particular, g is C–close to one of the other sides. Since the constant C is independent of g, we have that that for every $g \in G$ there exists a geodesic ray γ^g based at $\mathbb{1}$ and passing within distance C of g.

Let δ be a geodesic segment with endpoints h and hg for some $g, h \in G$. The contraction function of the geodesic $h^{-1}\delta$ from $\mathbb{1}$ to g can be bounded in terms of C and the contraction function of γ^g, but since rays have uniform contraction this gives us a bound for the contraction function for $h^{-1}\delta$, hence for δ. Since every geodesic segment is at Hausdorff distance at most $1/2$ from a geodesic segment with endpoints at vertices, Lemma 3.3 tells us that geodesic segments are uniformly contracting. Thus, (1) implies (2).

If geodesic segments in G are uniformly contracting then geodesic bigons are uniformly thin, so G is hyperbolic by a theorem of Papasoglu [37]. Thus, (2) implies (3).

By [15, Theorem 3.10], if G is hyperbolic then $\partial_c^{\mathcal{DL}}G$ agrees with the Gromov boundary, which is compact, so (3) implies (4).

$\mathcal{DL}$ is a refinement of $\mathcal{FQ}$ by Proposition 7.4, so (4) implies (5).

If G is virtually cyclic then (6) is true. If G is not virtually cyclic then, by Proposition 8.1, if $\partial_c^{\mathcal{FQ}}G$ is non-empty then it is infinite. In particular, there are distinct points in $\partial_c^{\mathcal{FQ}}G$. Choose two of them and connect them by a geodesic β, which is necessarily contracting. By translating β we may assume that $\beta_0 = \mathbb{1}$. Suppose that α is an arbitrary geodesic ray based at $\mathbb{1}$. As in the proof of Proposition 8.2, after possibly exchanging β with $\bar{\beta}$ there is increasing $\sigma' \colon \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have that $\alpha_{\sigma'(n)}\beta_{[0,\infty)}$ does not backtrack far along $\alpha_{[0,\sigma'(n)]}$. This means there are L and A, independent of n, such that the concatenation of $\alpha_{[0,\sigma'(n)]}$ and $\alpha_{\sigma'(n)}\beta_{[0,\infty)}$ is a continuous (L, A)–quasi-geodesic.

If $\partial_c^{\mathcal{FQ}}G$ is compact then the sequence $(\alpha_{\sigma'(n)}\beta_\infty)_{n\in\mathbb{N}}$ has a convergent subsequence, so there is an increasing $\sigma'' \colon \mathbb{N} \to \mathbb{N}$ such that for $\sigma := \sigma' \circ \sigma''$ we have $(\alpha_{\sigma(n)}\beta_\infty)_{n\in\mathbb{N}}$ converges to a point $\zeta \in \partial_c^{\mathcal{FQ}}G$. Then for every $r > 1$ there exists an N such that for all $n \geq N$ we have $\alpha_{\sigma(n)}\beta_\infty \in U(\zeta, r)$. For $r > 3L^2$, $3A$ we then have that the continuous (L, A)–quasi-geodesic $\alpha_{[0,\sigma(n)]} + \alpha_{\sigma(n)}\beta_{[0,\infty)} \in \alpha_{\sigma(n)}\beta_\infty$

comes $\kappa(\rho_\zeta, L, A)$–close to α^ζ outside the ball of radius r about $\mathbb{1}$, for all sufficiently large n, and therefore has initial segment of length at least r contained in the $\kappa'(\rho_\zeta, L, A)$–neighborhood of α^ζ. Since this is true for all sufficiently large n and since longer and longer initial segments of the $\alpha_{[0,\sigma(n)]} + \alpha_{\sigma(n)}\beta_{[0,\infty)}$ are initial segments of α, we conclude that α is asymptotic to α^ζ, which implies that α is contracting. Thus, (5) implies (6).

Finally, we prove (6) implies (1). We do so by assuming (6) is true and (1) is false, and deriving a contradiction. The strategy is as follows. The fact that $\partial_c^{\mathcal{FQ}}G$ is not uniformly contracting implies that no non-empty open subset of $\partial_c^{\mathcal{FQ}}G$ is uniformly contracting. We construct a nested decreasing sequence of neighborhoods focused on points with successively worse contraction behavior. We use properness of G to conclude that a subsequence of representative geodesics of these focal points converges to a geodesic ray. The assumption (6) implies the limiting ray is contracting, so it represents a point in $\partial_c^{\mathcal{FQ}}G$, and we claim that this point is in the intersection of the nested sequence of neighborhoods. Furthermore, the details of the construction ensure that the limiting ray actually experiences the successively worse contraction behavior of the construction, with the conclusion that it is not a contracting ray, contradicting (6).

If rays in G fail to be uniformly contracting then so do bi-infinite geodesics in G. To see this, fix a ray α. Any other ray has a translate β with the same basepoint as α. Since rays are contracting there is a contracting bi-infinite geodesic γ with endpoints α_∞ and β_∞. Now α, β, and γ make a geodesic triangle, so the contraction function of β can be bounded in terms of those of α and γ. If bi-infinite geodesics are all ρ–contracting then this would mean the contraction function for β can be bounded in terms of only ρ and ρ_α, so rays would be uniformly contracting. Combining this with Theorem 2.2, we have that $\neg(1)$ implies bi-infinite geodesics in G are not uniformly Morse. They are all Morse, as each bi-infinite geodesic can be written as a union of two rays, which are contracting, by (6). For each $L \geq 1, A \geq 0$, and bi-infinite geodesic γ in G, define $D(\gamma, L, A)$ to be the supremum of the set:

$$\{d(z,\gamma) \mid z \text{ is a point on a continuous } (L,A)\text{–quasi-geodesic with endpoints on } \gamma\}$$

Since γ is Morse, $D(\gamma, L, A)$ exists for each L and A. If $\sup_\gamma D(\gamma, L, A)$ exists for every L and A then we can define $\mu(L, A) := \sup_\gamma D(\gamma, L, A)$ and we have that all bi-infinite geodesics are μ–Morse, contrary to hypothesis, so there exist some $L \geq 1$ and $A \geq 0$ such that for all $n \in \mathbb{N}$ there exists a bi-infinite geodesic γ^n and a continuous (L, A)–quasi-geodesic δ^n with endpoints on γ^n such that δ^n is not contained in $N_n\gamma^n$. By translating and shifting the parameterization of γ^n we may assume that $\gamma_0^n = \mathbb{1}$ and that the distances from $\mathbb{1}$ to the two endpoints of δ^n differ by at most 1.

Now we make a claim and use it to finish the proof:

(14) Let γ be a bi-infinite ρ_γ–contracting geodesic. Given $\zeta \in \partial_c^{\mathcal{FQ}}G$, $R > 1$, $r \geq 0$ there exists $\eta \in U(\zeta, R)$ and $g \in G$ such that α^η passes within distance $\kappa(\rho_\gamma, 1, 0)$ of both endpoints of a segment of $g\gamma$ containing $g\gamma_{[-r,r]}$.

See Figure 6 and Figure 7 for illustrations of (14). Assuming (14), we construct a decreasing nested sequence of neighborhoods in $\partial_c^{\mathcal{FQ}}G$ focusing on points with successively worse contraction behavior. The key trick is to build extra padding into our constants to give the contraction function of the eventual limiting geodesic time to dominate. Let M and λ be as in Lemma 4.6; recall that $\lambda(\phi, 1, 0) = 8\kappa(\phi, 1, 0)$. Let ψ be as in Corollary 5.10. Let $\zeta^0 \in \partial_c^{\mathcal{FQ}} G$ and $R_0 > 1$ be arbitrary. Now, supposing ζ^i and R_i have been defined, consider γ^{i+1}. Let r_{i+1} be the least integer greater than half the distance between endpoints of δ^i plus the quantity $(M+1)\kappa(\rho_{\gamma^i}, 1, 0) + (6M+15)(i+1)$. Apply (14) to γ^{i+1}, ζ^i, $\psi(\zeta^i, R_i)$, r_{i+1} and get output $\zeta^{i+1} \in U(\zeta^i, \psi(\zeta^i, R_i)) \subset U(\zeta^i, R_i)$ and $g_{i+1} \in G$. Let R''_{i+1} be $\kappa(\rho_{\gamma^i}, 1, 0)$ plus the larger of the distances to $\mathbb{1}$ of the endpoints of the subsegment of $g_{i+1}\gamma^{i+1}$ given by (14). Define $R'_{i+1} := R''_{i+1} + (M+1)\kappa(\rho_{\zeta^{i+1}}, 1, 0) + 9(i+1)$. By Corollary 5.10, there is an open set U_i such that $U(\zeta^i, \psi(\zeta^i, R_i)) \subset U_i \subset U(\zeta^i, R_i)$, so since $\zeta^{i+1} \in U(\zeta^i, \psi(\zeta^i, R_i))$ we can choose $R_{i+1} \geq R'_{i+1}$ large enough to guarantee $U(\zeta^{i+1}, R_{i+1}) \subset U \subset U(\zeta^i, R_i)$.

Consider the sequence of geodesic rays $(\alpha^{\zeta^n})_{n\in\mathbb{N}}$. Some subsequence converges to a geodesic ray α. By hypothesis, all rays are contracting, so there exists some ρ_α such that α is ρ_α–contracting.

Pick any $i \geq \kappa'(\rho_\alpha, 1, 0) > \kappa(\rho_\alpha, 1, 0)$. There is some $n \gg i$ such that α agrees with α^{ζ^n} for distance $R_i + \kappa(\rho_{\zeta^i}, 1, 0)$. Since the neighborhoods are nested, by construction, $\zeta^n \in U(\zeta^i, R_i)$, which implies that α comes $\kappa(\rho_{\zeta^i}, 1, 0)$–close to α^{ζ^i} outside the ball of radius $R_i \geq R'_i$ about $\mathbb{1}$. The definition of R'_i and the fact that $i > \kappa(\rho_\alpha, 1, 0)$ gives us, by Lemma 4.6, that α^{ζ^i} comes $\kappa(\rho_\alpha, 1, 0)$–close to α outside the ball of radius R''_i about $\mathbb{1}$. In particular, by Lemma 4.4, α passes $(\kappa'(\rho_\alpha, 1, 0) + \kappa(\rho_{\zeta^i}, 1, 0))$-close to both endpoints of a subsegment of $g_i\gamma^i$ containing $g_i\gamma^i_{[-r_i, r_i]}$. The definition of r_i and the fact that $i \geq \kappa(\rho_\alpha, 1, 0)$ give us, by a second application of Lemma 4.6 and Lemma 4.4, that α comes within distance $\kappa'(\rho_\alpha, 1, 0)$ of both endpoints of $g_i\delta^i$. Connect the endpoints of $g_i\delta^i$ to α by shortest geodesic segments. For $A' := A + 2\kappa'(\rho_\alpha, 1, 0)$, the resulting path δ'_i is a continuous (L, A')–quasi-geodesic that is contained in $\bar{N}_{\kappa'(\rho_\alpha, L, A')}(\alpha)$ but leaves the $(i - \kappa'(\rho_\alpha, 1, 0))$–neighborhood of the subsegment of α between its endpoints. For sufficiently large i this contradicts the fact that δ'_i is (L, A')–quasi-geodesic.

We now prove (14). The idea is to take an element g that pushes γ far out along α^ζ and take η to be one of the endpoints of $g\gamma$. Then α^η forms a geodesic triangle with a subsegment of α^ζ and a subray of $g\gamma$. Additionally, we arrange for $g\gamma_{[-r,r]}$ to be suitably far from the quasi-center of this triangle so that it is in one of the thin legs of the triangle, parallel to a segment of α^η.

Let $\alpha := \alpha^\zeta$. Let $r' > r$ represent a number to be determined, and choose any $s > r' + 2\kappa'(\rho_\gamma, 1, 0)$. First, suppose that for arbitrarily large t there exists $g \in N_{\kappa'(\rho_\gamma, 1, 0)}\alpha_t$ such that $d(g\gamma_s^{-1}\gamma_{r'}, \alpha_{[0,t]}) \geq \kappa'(\rho_\gamma, 1, 0)$. We claim that for any sufficiently large t we can take such a g and $\eta := g\gamma_s^{-1}\gamma_{-\infty}$ as the output of (14). To see this, define a continuous quasi-geodesic by following α until we reach the first of either α_t or a point of $\pi_\alpha(g\gamma_s^{-1}\gamma_{r'})$, then follow a geodesic to $g\gamma_s^{-1}\gamma_{r'}$, then follow $g\gamma_s^{-1}\bar{\gamma}$ towards η. By an argument similar to Lemma 4.7, β is an (L, A)–quasi-geodesic, for some L and A not depending on g, r, or s. Now, $\alpha_{[0,t]}$, $g\gamma_s^{-1}\gamma_{(-\infty,s]}$, and α^η form a $\kappa'(\rho_\gamma, 1, 0)$–almost geodesic triangle, so the contraction function of α^η is bounded in terms of the contraction functions of α and γ. Thus, there is some E such that β and α^η are bounded Hausdorff distance E from one another, independent of our choices. By the hypothesis on g and Corollary 4.3 the two sides $\alpha_{[0,t]}$ and $g\gamma_s^{-1}\gamma_{(-\infty,s]}$ of the almost geodesic triangle are diverging at a linear rate, and so $g\gamma_s^{-1}\gamma_{r'}$ is H–close to some point of α^η for some H, again independent of our choices. Assume that we chose $r' \geq r + (M+1)H + 9\kappa(\rho_\gamma, 1, 0)$. Then, by Lemma 4.6, we have that α^η passes within distance $\kappa(\rho_\gamma, 1, 0)$ of some point of $g\gamma_s^{-1}\gamma_{[r,r']}$, and also of some point in $g\gamma_s^{-1}\gamma_{[-r',-r]}$.

We also need to show $\eta \in U(\zeta, R)$. Any continuous (L', A')–quasi-geodesic in η stays bounded Hausdorff distance H' from β, with bound depending on L' and A', but not g, s, or t. We only need to consider $L' < \sqrt{R/3}$ and $A' < R/3$, so we can bound H' in terms of R (and ρ_α and ρ_γ). We therefore have that such a quasi-geodesic passes $(H'+H)$–close to $g\gamma_s^{-1}\gamma_{r'}$, which is $(s-r'+\kappa'(\rho_\gamma,1,0))$–close to α_t. Applying Lemma 4.6 we see that such a geodesic passes $\kappa(\rho_\alpha, L', A')$–close to α outside the ball of radius R provided that t is chosen sufficiently large with respect to R, s, and the contraction functions for α and γ. By hypothesis, we can choose t as large as we like, so in this case we are done.

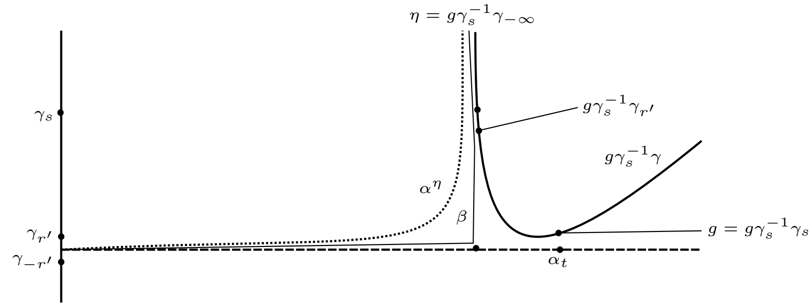

FIGURE 6. First case for (14).

The other case is that there exists T such that for every $g \in N_{\kappa'(\rho_\gamma,1,0)}\alpha_{[T,\infty]}$ we have $d(g\gamma_s^{-1}\gamma_{r'}, \alpha_{t'}) < \kappa'(\rho_\gamma, 1, 0)$ for some t' with $t > T$ and $t - t' = s - r' \pm 2\kappa'(\rho_\gamma, 1, 0) > 0$. Let w be the word in the generators for G read along the path $\gamma_{[r',s]}$. Let $t_0 > T$ be arbitrary, and let $g_0 := \alpha_{t_0}$. Let $g_1 := g_0\gamma_s^{-1}\gamma_{r'}$. By hypothesis there is a $t_1 < t_0$ such that $d(g_1, \alpha_{t_1}) \leq \kappa'(\rho_\gamma, 1, 0)$. The segment $g_0\gamma_s^{-1}\gamma_{[r',s]}$ has edge label w and, by Theorem 3.4, is contained in $\bar{N}_K(\alpha)$ for some K depending only on $\kappa'(\rho_\gamma, 1, 0)$ and ρ_γ. If $t_1 > T$ we can repeat, setting $g_2 := g_1\gamma_s^{-1}\gamma_{r'}$, so that the initial vertex g_1 of $g_1\gamma_s^{-1}\gamma_{[r',s]}$ agrees with the terminal vertex of $g_0\gamma_s^{-1}\gamma_{[r',s]}$. Repeating this construction until $t_i \leq T$, we construct a path from α_t to $\bar{N}_{\kappa'(\rho_\gamma,1,0)}\alpha_{[0,T]}$ that is contained in the K–neighborhood of α and whose edge label is a power of w^{-1}. Since this is true for arbitrarily large t, we conclude that w is a contracting element in G and $\zeta = hw^\infty$ for some $h \in G$ that is $(s-r'+\kappa'(\rho_\gamma, 1, 0))$–close to α_T. Furthermore, we can also take $s' > s$ arbitrarily large and run the same argument to conclude that either we find the g and η we are looking for from the first case, or else arbitrarily long segments $\gamma_{[r',s']}$ can be sent into $\bar{N}_K\alpha_{[T,\infty)}$. We already know this tube contains an infinite path labelled by powers of w. Therefore, there is f which is $(s - r' + 2\kappa'(\rho_\gamma, 1, 0))$–close to γ_s such that $\gamma_\infty = fw^\infty$.

If $fw^{-\infty} = \gamma_{-\infty}$ then we can take $\eta := \zeta$ and $g := hw^a f^{-1}$ for a sufficiently large. Otherwise, for any sufficiently large t and a we can take $g := \alpha_t\gamma_s^{-1}fw^{-a}$ and $\eta := g\gamma_{-\infty}$, see Figure 7. □

Proposition 10.2. *Let G be a group acting properly discontinuously on a proper geodesic metric space X. Suppose that the orbit map $\phi\colon g \mapsto go$ takes contracting quasi-geodesics to contracting quasi-geodesics and has quasi-convex image. If Go*

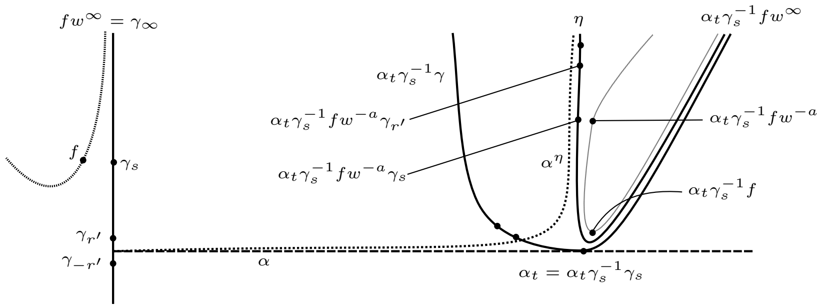

FIGURE 7. Second case for (14).

has compact closure in $\hat{X}$ then G is hyperbolic. If G is infinite and hyperbolic and ϕ is Morse-controlled then $\Lambda(G)$ is non-empty and compact and Go has compact closure in $\hat{X}$.

Proof. If G is finite then Go is compact and G is hyperbolic, so assume G is infinite. Since it has a quasi-convex orbit, G is finitely generated and ϕ is a quasi-isometric embedding. Since G is infinite, Go is unbounded, so if it has compact closure then $\Lambda(G)$ is non-empty and compact. By Proposition 6.8, $\partial_c^{\mathcal{FQ}}G$ is compact. By Theorem 10.1, G is hyperbolic.

Conversely, if G is hyperbolic then, with respect to any finite generating set, geodesics in G are uniformly Morse. Therefore, there is a μ such that for any two points $g, h \in G$ there is a μ–Morse geodesic $\gamma^{g,h}$ in G from g to h. If ϕ is Morse-controlled, there exists a μ' depending only on μ such that $\phi(\gamma^{g,h})$ is μ'–Morse. Any (L, A)–quasi-geodesic with endpoints go and ho therefore stays $\mu'(L, A)$–close to $\phi(\gamma^{g,h})$, but $\phi(\gamma^{g,h})$ is a quasi-geodesic with integral points on Go, so it remains close to Go. Therefore, Go is a Morse subset of X. By Proposition 6.8, $\partial_c\phi\colon \partial_c^{\mathcal{FQ}}G \to \Lambda(G)$ is a homeomorphism. Since G is an infinite hyperbolic group, $\partial_c^{\mathcal{FQ}}G$ is non-empty and compact, so $\Lambda(G)$ is as well.

It remains to show $\overline{Go}$ is compact. Suppose $\mathcal{U}$ is an open cover of $\overline{Go}$. Only finitely many elements of $\mathcal{U}$ are required to cover $\Lambda(G)$. We claim that the part of Go not covered by these finitely many sets is bounded, hence, finite, so only finitely many more elements of $\mathcal{U}$ are required to cover all of $\overline{Go}$. To see this, suppose $(g_no)_{n\in\mathbb{N}}$ is an unbounded sequence in Go that does not enter the chosen finite cover of $\Lambda(G)$. By passing to a subsequence, we may assume $d(o, g_no) \geq n$, in which case $(g_no)_{n\in\mathbb{N}}$ is a sequence with no convergent subsequence in $\hat{X}$. For each n pick a geodesic γ^n from o to g_no. A subsequence $(\gamma^{\sigma(n)})_{n\in\mathbb{N}}$ converges to a geodesic ray γ in X based at o, but since the geodesics γ^n were uniformly contracting, γ is contracting. Moreover, by uniform contraction the endpoints $g_{\sigma(n)}o$ converge to γ_∞ in $\hat{X}$, which is a contradiction. □

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