Quantum ergodicity on large regular graphs

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Abstract. We propose a version of the Quantum Ergodicity theorem on large regular graphs of fixed valency. This is a property of delocalization of "most" eigenfunctions. We consider expander graphs with few short cycles (for instance random large regular graphs). Our method mimics the proof of Quantum Ergodicity on manifolds : it uses microlocal analysis on regular trees, as introduced in [**25**].

1. Introduction and main results

It has been suggested by Kottos and Smilansky that graphs are a good ground of exploration of the ideas of "quantum chaos" [**22, 23**]. This means that the spectrum of the laplacian, as well as its eigenfunctions, should exhibit universal features that depend only on qualitative geometric properties of the graph. Whereas spectral statistics have been extensively studied, both numerically and analytically, the localization of eigenfunctions have (to our knowledge) only been investigated in a few models : the *star* graphs (both metric and discrete) [**3, 20**], the large *regular* discrete graphs [**33, 6, 9**], and a family of metric graphs arising from measure preserving 1-dimensional dynamical systems [**2**]. For the latter, a version of the "Quantum Ergodicity theorem" (also known as Shnirelman theorem) has been established. For star graphs, the paper [**3**] shows on the opposite that "Quantum Ergodicity" holds neither in the high frequency limit nor in the large graph limit. Furthermore it shows there are eigenfunctions that localise on two bonds of the graph. Spectral properties of large regular *discrete* graphs have been studied in [**30, 24, 19, 35, 32**] but eigenfunctions have attracted attention only recently. A statistical study of the auto-correlations and the level sets of eigenvectors appeared in the papers [**10, 11**] that introduce a random wave model (see also [**18**] for a random wave model on metric graphs). The paper [**6**] has pioneered the study of quantum ergodicity on large regular graphs – that is to say, the study of the spatial

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distribution of eigenfunctions of the laplacian. The result of [**6**] shows some form of delocalization of eigenfunctions :

THEOREM 1.1. [**6**] *Let (G_n) be a sequence of $(q+1)$-regular graphs (with q fixed), $G_n = (V_n, E_n)$ with $V_n = \{1, \ldots, n\}$. Assume that*[1] *there exists $c > 0, \delta > 0$ such that, for any $k \leq c \ln n$, for any pair of vertices $x, y \in V_n$,*

$$|\{\text{paths of length } k \text{ in } G_n \text{ from } x \text{ to } y\}| \leq q^{k\left(\frac{1-\delta}{2}\right)}.$$

Fix $\epsilon > 0$. Then, if ϕ is an eigenfunction of the discrete laplacian on G_n and if $A \subset V_n$ is a set such that

$$\sum_{x \in A} |\phi(x)|^2 \geq \epsilon \sum_{x \in V_n} |\phi(x)|^2,$$

then $|A| \geq n^\alpha$ — where $\alpha > 0$ is given as an explicit function of ϵ, δ and c.

A similar form of delocalization (but on weaker scales) is established when the degree $q = q_n$ goes to infinity in [**9, 36**]. See also [**17**] and Remark 1.7 below for fixed q. We also point out the papers [**14, 15, 13, 12, 5**] where various forms of delocalization have been established for eigenvectors of random Wigner matrices and random band matrices.

In this paper, our aim is to establish for large regular graphs a result which reads like an analogue of the "quantum ergodicity theorem" on manifolds. Compared to Theorem 1.1 it pertains to a different definition of delocalization : delocalization is now tested by averaging an observable and comparing with the average along the uniform measure. As a motivation, let us recall the Quantum Ergodicity theorem in its original form.

Quantum Ergodicity Theorem (Shnirelman theorem, [34, 7, 38]). Let (M, g) be a compact Riemannian manifold, with the metric normalized so that $\mathrm{Vol}(M) = 1$. Call Δ the Laplace-Beltrami operator on M. Assume that the geodesic flow of M is *ergodic* with respect to the Liouville measure. Let $(\psi_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $L^2(M, g)$ made of eigenfunctions of the laplacian

$$\Delta \psi_n = -\lambda_n \psi_n, \qquad \lambda_n \leq \lambda_{n+1} \longrightarrow +\infty.$$

Let a be a continuous function on M such that $\int_M a(x) d\,\mathrm{Vol}(x) = 0$. Then

$$\frac{1}{N(\lambda)} \sum_{n, \lambda_n \leq \lambda} |\langle \psi_n, a\psi_n \rangle_{L^2(M)}|^2 \underset{\lambda \longrightarrow +\infty}{\longrightarrow} 0 \tag{1.1}$$

where the normalizing factor is $N(\lambda) = |\{n, \lambda_n \leq \lambda\}|$. Here $\langle \psi_n, a\psi_n \rangle_{L^2(M)} = \int_M a(x)|\psi_n(x)|^2 d\,\mathrm{Vol}(x)$.

REMARK 1.2. Equation (1.1) implies that there exists a subset $S \subset \mathbb{N}$ of density 1 such that

$$\langle \psi_n, a\psi_n \rangle \underset{n \longrightarrow +\infty, n \in S}{\longrightarrow} \int_M a(x) d\,\mathrm{Vol}(x). \tag{1.2}$$

(Note that (1.2) is true even for a function a with non-zero mean.)

In addition, since the space of continuous functions is separable, one can actually find $S \subset \mathbb{N}$ of density 1 such that (1.2) holds for *all* $a \in C^0(M)$. In other

[1] This assumption holds in particular if the injectivity radius is $\geq c \ln n$. The interest of the weaker assumption is that it holds for typical random regular graphs [**29**].

words, the sequence of measures $(|\psi_n(x)|^2 d\,\mathrm{Vol}(x))_{n\in S}$ converges weakly to the uniform measure $d\,\mathrm{Vol}(x)$.

Actually, the full statement of the theorem says that there exists a subset $S \subset \mathbb{N}$ of density 1 such that

$$\langle \psi_n, A\psi_n\rangle \underset{n\longrightarrow +\infty, n\in S}{\longrightarrow} \int_{S^*M} \sigma^0(A) dL \tag{1.3}$$

for every pseudodifferential operator A of order 0 on M. On the right-hand side, $\sigma^0(A)$ is the principal symbol of A, that is a function on the unit cotangent bundle S^*M, and L is the normalized Liouville measure (uniform measure), arising naturally from the symplectic structure of the cotangent bundle.

In this paper we consider a sequence of $(q+1)$-regular connected graphs $(G_n)_{n\in\mathbb{N}}$, $G_n = (V_n, E_n)$ with vertices $V_n = \{1, \ldots, n\}$ and edge set E_n. The valency $(q+1)$ is fixed. We denote by $\mathfrak{X}$ the $(q+1)$-regular tree and identify it with its set of vertices, equipped with the geodesic distance $d_{\mathfrak{X}}$, that we will simply denote by d most of the time. We will denote by d_{G_n} the geodesic distance on the graph G_n. Each G_n is seen as a quotient of $\mathfrak{X}$ by a group of automorphisms Γ_n : $G_n = \Gamma_n\backslash\mathfrak{X}$, where we assume that the elements of Γ_n act without fixed points. Accordingly, a function $f : V_n \longrightarrow \mathbb{C}$ may be seen as a function $f : \mathfrak{X} \longrightarrow \mathbb{C}$ satisfying $f(\gamma\cdot x) = f(x)$ for each $x \in \mathfrak{X}, \gamma \in \Gamma_n$. We will denote by D_n a subtree of $\mathfrak{X}$ which is a fundamental domain for the action of Γ_n on vertices.

We consider the stochastic operator acting on Γ_n-invariant functions,

$$Mf(x) = \frac{1}{q+1}\sum_{x\sim y} f(y) \tag{1.4}$$

where $x \sim y$ means that x and y are neighbours in the tree[2]. It is related to the discrete laplacian by $M - I = \Delta$.

Whereas the Shnirelman theorem deals with the *high frequency* asymptotics ($\lambda_n \longrightarrow +\infty$), there is no such asymptotic régime for discrete graphs since the laplacian is a bounded operator. We will instead work (like in [**6**]) in the *large spatial scale* régime $n \longrightarrow +\infty$.

We will assume the following conditions on our sequence of graphs :

(EXP) The sequence of graphs is a family of expanders. More precisely, there exists $\beta > 0$ such that the spectrum of M on $L^2(G_n)$ is contained in $\{1\} \cup [-1 + \beta, 1 - \beta]$ for all n.

(BST) For all R, $\frac{|\{x\in V_n, \rho(x)<R\}|}{n} \longrightarrow 0$ where $\rho(x)$ is the injectivity radius at x (meaning the largest ρ such that the ball $B(x,\rho)$ in G_n is a tree).

The condition (BST) means that our sequence of graphs converges to a tree in the sense of Benjamini-Schramm. It is equivalent to saying that there exists $R_n \longrightarrow +\infty$ and $\alpha_n \longrightarrow 0$ such that

$$\frac{|\{x \in V_n, \rho(x) < R_n\}|}{n} \leq \alpha_n.$$

In particular, it is satisfied if the injectivity radius goes to infinity (with R_n taken to be the minimal injectivity radius and $\alpha_n = 0$).

[2]This is also the (normalized) adjacency matrix of the graph G_n, but note that this definition allows G_n to have loops and multiple edges.

Condition (EXP) replaces the ergodicity assumption in the usual quantum ergodicity theorem.

EXAMPLE 1. The graph G_n can be chosen uniformly at random among the $(q+1)$-regular graphs with n vertices (see [**4**] section 2.4 for an introduction to this model). We can then take $R_n = k$ and $n\alpha_n = 40Akq^k$ for any $k = k(n)$ such that $kq^k n^{-1} \underset{n\to+\infty}{\longrightarrow} 0$, and $A = A(n)$ such that $A \geq c > 1$ (see [**29**], Theorem 4). For instance, we can take $k = (1-\delta)\log_q(n)$, with $0 < \delta < 1$, and $A = 2$. In this case we have $R_n = (1-\delta)\log_q(n)$ and $\alpha_n = 80(1-\delta)\log_q(n)n^{-\delta}$. For this choice of parameters, (BST) is satisfied with a probability tending to 1 when $n \longrightarrow +\infty$. More precisely, this probability is greater than $1-e^{-Cn^{1-\delta}}$, for some constant $C > 0$ independent of n.

Condition (EXP) is also satisfied by these sequences of random graphs : [**1**] proves an equivalence between having a uniform spectral gap and having a uniform Cheeger constant. The latter condition was shown to hold generically in [**31**]. In [**16**], a spectral gap estimate is established that is close to optimal.

EXAMPLE 2. An explicit example of sequence of $(q+1)$-regular graphs to which our results apply is given by the construction of Ramanujan graphs of [**26**] for prime q. The sequence obtained satisfies conditions (EXP) and (BST) even more strongly than the sequences of random graphs of Example 1. A method for obtaining bipartite Ramanujan graphs of arbitrary degrees has appeared recently in [**27**].

Eigenvalues of M on a $(q+1)$-regular graph may be parameterized by their "spectral parameter" s thanks to the relation

$$\lambda = \lambda(s) = \frac{2\sqrt{q}}{q+1}\cos(s\ln q). \tag{1.5}$$

The case $s \in \mathbb{R}$ corresponds to $\lambda \in \left[-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}\right]$, which is the tempered spectrum. In that case we will usually choose $s \in [0,\tau]$ ($\tau = \frac{\pi}{\ln(q)}$). The case $s \in i(-1/2,1/2)+ ik\frac{\pi}{\ln(q)}$ ($k \in \mathbb{Z}$) corresponds to $\lambda \in [-1,1] \setminus \left(-\frac{2\sqrt{q}}{q+1}, \frac{2\sqrt{q}}{q+1}\right)$, which is the untempered spectrum. The result of this paper will only be of interest in the tempered part of the spectrum.

In what follows, (r_n) will be a sequence of integers such that $r_n \longrightarrow +\infty$, satisfying $r_n + 2 \leq R_n$ and $q^{r_n}\alpha_n \longrightarrow 0$. The sequence (δ_n) will be assumed to satisfy $\delta_n^K r_n^{K-4} \longrightarrow +\infty$, for some integer K. We also assume that $\delta_n \longrightarrow 0$, although it is not necessary for the general proof of section 4.

Our aim is to prove the following :

THEOREM 1. *Let (G_n) be a sequence of $(q+1)$-regular graphs, $G_n = (V_n, E_n)$ with $V_n = \{1,\dots,n\}$. Assume that (G_n) satisfies (BST) and (EXP). Fix $s_0 \in (0,\tau)$ and let $I_n = [s_0-\delta_n, s_0+\delta_n]$. Call $(s_1^{(n)},\dots,s_n^{(n)})$ the spectral parameters associated with the eigenvalues of M on G_n, and $(\psi_1^{(n)},\dots,\psi_n^{(n)})$ a corresponding orthonormal eigenbasis.*

Let $N(I_n, G_n) = \left|\{j \in \{1, \dots, n\}, s_j^{(n)} \in I_n\}\right|$ *be the number of eigenvalues in* I_n.[3] *Finally, let* $a_n : V_n \longrightarrow \mathbb{C}$ *be a sequence of functions such that*

$$\sum_{x \in V_n} a_n(x) = 0, \qquad \sup_x |a_n(x)| \leq 1.$$

Then

$$\frac{1}{N(I_n, G_n)} \sum_{s_j^{(n)} \in I_n} \left|\langle \psi_j^{(n)}, a_n \psi_j^{(n)} \rangle\right|^2 \underset{n \longrightarrow +\infty}{\longrightarrow} 0.$$

Remark 1.3. If a_n does not have zero mean, then by applying the theorem to $a_n - \overline{a_n}$ (where $\overline{a_n} = \frac{1}{n}\sum_{x \in V_n} a_n(x)$) we obtain

$$\frac{1}{N(I_n, G_n)} \sum_{s_j^{(n)} \in I_n} \left|\langle \psi_j^{(n)}, a_n \psi_j^{(n)} \rangle - \overline{a_n}\right|^2 \underset{n \longrightarrow +\infty}{\longrightarrow} 0.$$

Remark 1.4. If we exclude the case $s_0 = \tau/2$ in Theorem 1, a careful inspection of the proof shows that we can assume, instead of (EXP), the following weaker condition : there exists $\beta > 0$ such that the spectrum of A on $L^2(G_n)$ is contained in $\{1\} \cup [-1, 1-\beta]$ for all n. In particular, the theorem applies for bipartite regular graphs in this case.

We can also say something in the case $s_0 = \tau/2$ for bipartite expander graphs, that is if there exists $\beta > 0$ such that the spectrum of A on $L^2(G_n)$ is contained in $\{-1, 1\} \cup [-1+\beta, 1-\beta]$ for all n. We need to strengthen the condition on the functions $a_n(x)$ in the theorem for the conclusion to apply : if $V_n = V_n^1 \sqcup V_n^2$ is the bi-partition of V_n, then we need that

$$\sum_{x \in V_n^1} a_n(x) = \sum_{x \in V_n^2} a_n(x) = 0.$$

The theorem then tells us that we have equidistribution of most eigenfunctions with eigenvalue near $\tau/2$ on each set V_n^1 and V_n^2, without providing information on the relative weight of these two sets.

Remark 1.5. After proving a "quantum ergodicity result", one may ask about "quantum unique ergodicity". A first difficulty is to define the question. A reasonable formulation of "quantum unique ergodicity" would be to ask whether, for any sequence of integers $j_n \in \{1, \dots, n\}$ and for any observable a_n as in Theorem 1 we have $\langle \psi_{j_n}^{(n)}, a_n \psi_{j_n}^{(n)} \rangle \underset{n \longrightarrow +\infty}{\longrightarrow} 0$. Or we could ask for the stronger property $\sup_{j=1,\dots,n} \left|\langle \psi_j^{(n)}, a_n \psi_j^{(n)} \rangle\right| \underset{n \longrightarrow +\infty}{\longrightarrow} 0$. For regular graphs we have no clue about the proof of such a statement. Note that the recent paper [**5**] proves the first form of quantum unique ergodicity for eigenvectors of random Wigner matrices.

Since random $(q+1)$-regular graphs satisfy both (EXP) [**31, 1, 16**] and (BST) [**29**], our theorem applies to them with the values of R_n, α_n given in Example 1.

Corollary 1. *Choose* (G_n) *uniformly at random amongst the* $(q+1)$*-regular graphs* $G_n = (V_n, E_n)$ *such that* $V_n = \{1, \dots, n\}$*. Choose* j *uniformly at random in* $\{1, \dots, N(I_n, G_n)\}$.

[3]Note that with the assumptions on δ_n, we know that $N(I_n, G_n) \longrightarrow +\infty$ when $n \to +\infty$ (see Corollary 5.2).

Let $a_n : V_n = \{1, \ldots, n\} \longrightarrow \mathbb{C}$ *be a sequence of functions such that*

$$\sum_{x \in V_n} a_n(x) = 0, \qquad \sup_x |a_n(x)| \leq 1.$$

Then for any fixed $\epsilon > 0$,

$$P\left(\left| \langle \psi_j^{(n)}, a_n \psi_j^{(n)} \rangle \right| \geq \epsilon \right) \longrightarrow 0$$

when $n \longrightarrow +\infty$.

The statement of Theorem 1 is the exact analogue of the Shnirelman theorem in its form (1.1). However, we do not have a statement analogous to the convergence of measures (1.2), because our sequence of measures does not live on a single space; instead, it is defined on the sequence of graphs G_n. We do not know of a notion that would be adapted to describe the limit of the family (G_n) endowed with the probability measure $(|\psi_j^{(n)}(x)|^2)_{x \in V_n}$.

We can generalize Theorem 1 by replacing the function a with any finite range operator :

Theorem 2. *Let* (G_n) *be a sequence of* $(q+1)$*-regular graphs,* $G_n = (V_n, E_n)$ *with* $V_n = \{1, \ldots, n\}$*. Assume that* (G_n) *satisfies (BST) and (EXP). Fix* $s_0 \in (0, \tau)$ *and let* $I_n = [s_0 - \delta_n, s_0 + \delta_n]$*. Call* $(s_1^{(n)}, \ldots, s_n^{(n)})$ *the spectral parameters associated with the eigenvalues of* M *on* G_n*, and* $(\psi_1^{(n)}, \ldots, \psi_n^{(n)})$ *a corresponding orthonormal eigenbasis.*

Let $N(I_n, G_n) = \left| \{ j \in \{1, \ldots, n\}, s_j^{(n)} \in I_n \} \right|$.

Fix $D \in \mathbb{N}$*. Let* A_n *be a sequence of operators on* $L^2(G_n)$ *whose kernels* $K_n : V_n \times V_n \longrightarrow \mathbb{C}$ *are such that* $K_n(x, y) = 0$ *for* $d(x, y) > D$.

Assume that $\sup_{x, y \in V_n} |K_n(x, y)| \leq 1$.

Then there exists a number $\overline{A_n}(s_0)$ *such that*

$$\frac{1}{N(I_n, G_n)} \sum_{s_j^{(n)} \in I_n} \left| \langle \psi_j^{(n)}, A_n \psi_j^{(n)} \rangle - \overline{A_n}(s_0) \right|^2 \underset{n \longrightarrow +\infty}{\longrightarrow} 0.$$

With the notation of §3.2, we can write $A_n = \mathrm{Op}(a_n)$*,* $a_n \in S_o^D$*; and we have the expression* $\overline{A_n}(s_0) = \frac{1}{n} \sum_{x \in D_n} \int_\Omega a_n(x, \omega, s_0) d\nu_x(\omega)$.

Remark 1.6. Quantitative statements (i.e. rates of convergence) will be given in Section 6. For instance, in the context of Theorem 1, what we prove is that, for any n,

$$\frac{1}{N(I_n, G_n)} \sum_{s_j^{(n)} \in I_n} \left| \langle \psi_j^{(n)}, a_n \psi_j^{(n)} \rangle \right|^2 \leq O\left(r_n^{-1/3} \beta^{-1} \|a\|_2^2 + r_n^{8/3} q^{r_n} \alpha_n \|a\|_\infty^2 \right),$$

where we can take $r_n = \min\left\{ R_n - 2, -(1 - \epsilon') \log_q(\alpha_n) \right\}$, for any $0 < \epsilon' < 1$, and δ_n in the definition of I_n is $\delta_n = r_n^{-1+\epsilon}$ for any $0 < \epsilon < 1$.

Here we denoted $\|a_n\|_2^2 = \frac{1}{n} \sum_{x \in V_n} |a_n(x)|^2$ and $\|a_n\|_\infty = \sup_x |a_n(x)|$.

We see in particular that we can weaken condition (EXP), by allowing the spectral gap β to decay with n "not too fast" ($\frac{r_n^{-1/3}}{\beta} \longrightarrow 0$ is enough).

If the length of the shortest loop goes to infinity (in other word, if one can take $\alpha_n = 0$ in the quantitative version of assumption (BST)) then we see that

$$\frac{1}{N(I_n, G_n)} \sum_{s_j^{(n)} \in I_n} \left| \langle \psi_j^{(n)}, a_n \psi_j^{(n)} \rangle \right|^2 \leq O\left(r_n^{-1/3} \beta^{-1} \|a\|_2^2 \right).$$

In other word the assumption that $\|a\|_\infty \leq 1$ may be replaced by $\|a_n\|_2 \leq 1$.

Remark 1.7. For random regular graphs, and in the case where the "observable" $a_n(x)$ in Theorem 1 is deterministic (or independent of the graph), it was pointed to us by Charles Bordenave and Alice Guionnet that one can use the results of [**17**] or [**9, 36**] and the fact that the $\psi_j^{(n)}(x)$ are identically distributed for $j \in \{1, \dots, n\}$ and $x \in V_n$, to prove that

$$\mathbb{E}\left(\frac{1}{n} \sum_{j=1}^{n} \left| \langle \psi_j^{(n)}, a_n \psi_j^{(n)} \rangle \right|^2 \right) \underset{n \longrightarrow +\infty}{\longrightarrow} 0.$$

The strength of Theorem 1 is that it is a pointwise, deterministic statement; if we apply it to random regular graphs, we can very well allow a_n to be correlated with the graph. In addition, we do not know how to prove the version for general operators (Theorem 2) for random regular graphs directly from [**9, 17, 36**].

Another look at the proof. For the reader's convenience, let us sketch the proof of Theorem 2 without using the quantization procedure Op.

Let us assume, without loss of generality, that A_n is an operator on $L^2(\mathfrak{X})$ that preserves the Γ_n-invariant functions. For fixed n, the quantity $\overline{A_n}(s_0)$ appearing in Theorem 2 turns out to be

$$\overline{A_n}(s_0) = \lim_{\varepsilon \to 0} \frac{\mathrm{Tr}(\chi_\varepsilon(M - \lambda(s_0))1_{G_n} A_n)}{\mathrm{Tr}(\chi_\varepsilon(M - \lambda(s_0))1_{G_n})}$$

where χ_ε is a sequence of smooth functions converging to a Dirac mass as $\varepsilon \to 0$, 1_{G_n} is the characteristic function of a set in the tree $\mathfrak{X}$ that projects bijectively to G_n under the map $\mathfrak{X} \longrightarrow G_n = \Gamma_n \backslash \mathfrak{X}$. Above we are taking the traces in $L^2(\mathfrak{X})$. Note that if $A_n = f(M)$ is a (continuous) function of the adjacency matrix M, we have $\overline{A_n}(s_0) = f(\lambda(s_0))$.

To estimate the size of operators we use a normalized Hilbert-Schmidt norm, such that the norm of the identity is 1. For the operator A_n it is $\frac{1}{\sqrt{n}}\|A_n\|_{HS}$, where $\|A_n\|_{HS}$ is the usual Hilbert-Schmidt norm of A_n.

The proof can be schematically seen as follows : without loss of generality we can assume that $\overline{A_n}(s) = 0$ for every s (replace A_n by $A_n - f_n(M)$ where f_n is the function such that $\overline{A_n}(s) = f_n(\lambda(s))$, show that $f_n(M)$ can be approximated by operators of finite range in the normalized Hilbert-Schmidt norm). Then we are able to write

$$A_n = [\Delta, B_n] + R_n \tag{1.6}$$

where R_n is a family of operators that go to 0 in the normalized Hilbert-Schmidt norm and B_n is some operator on $L^2(G_n)$. We prove (1.6) by providing explicit expressions for B_n and R_n. Once we have (1.6) it is obvious that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} |\langle \psi_j^{(n)}, A_n \psi_j^{(n)} \rangle|^2 = 0.$$

A slight refinement in this argument (replacing A_n by $A_n\chi_n(M)$ where χ_n shrinks around an interval) allows averaging over shrinking spectral intervals

We see that this sketch of proof could, in principle, work for wide classes of graphs. However, our explicit construction of B_n and R_n is specific to graphs satisfying (BST), and the proof that R_n goes to 0 in the normalized Hilbert-Schmidt norm uses the (EXP) condition.

2. Theorem 1 : outline of the proof in the case $s_0 = \tau/2$

We first give a proof in the special case $s_0 = \tau/2$. The reason for treating this case separately is that one can give a proof which is exactly parallel to that of the Shnirelman theorem on manifolds [**34, 7, 38, 39**]. The case of arbitrary s_0 requires additional arguments and will be treated in Section 4.

2.1. Upper bound on the variance. Fix an integer $T > 0$. Let χ be a smooth cut-off function supported in $[-1, 1]$ and taking the constant value 1 on $[-1/2, 1/2]$. We write

$$\chi_n(s) = \chi\left(\frac{s - s_0}{2\delta_n}\right) \tag{2.1}$$

so that $\chi_n \equiv 1$ on I_n. We use the pseudodifferential calculus and the notation defined in Section 3, taking the cut-off parameter r equal to r_n (from condition (BST), as explained before the statement of Theorem 1).

To simplify the notation, we will write $\psi_j = \psi_j^{(n)}$, $s_j = s_j^{(n)}$, and $a = a_n$. The observable a is a function on G_n, in other words a Γ_n-invariant function on $\mathfrak{X}$. Let Ω be the boundary of $\mathfrak{X}$ (see section 3), then a extends to a function on $\mathfrak{X} \times \Omega \times [0, \tau]$ that does not depend on the last two coordinates. The notation $\mathrm{Op}(a)$ is then defined in section 3.

Let T be a positive integer. Let $\sigma : \mathfrak{X} \times \Omega \longrightarrow \mathfrak{X} \times \Omega$ be the shift (see §3.4). We apply the "Egorov property" Corollary 3.8 to the function $\tilde{a}_T(x, \omega)\chi_n(s)$ where

$$\tilde{a}_T = \frac{1}{T}\sum_{k=0}^{T-2}(T - 1 - k)a \circ \sigma^{2k},$$

which satisfies $\tilde{a}_T \circ \sigma^2 - \tilde{a}_T = a^T - a$, where

$$a^T := \frac{1}{T}\sum_{k=0}^{T-1} a \circ \sigma^{2k}$$

.

Combining Lemma 3.9 and Corollary 3.8, we have[4]

$$
\begin{aligned}
\sum_{j=0}^{n} \chi_n(s_j)^2 \left|\langle \psi_j, a\psi_j\rangle\right|^2 &= \sum_{j=0}^{n} \left|\langle \psi_j, \mathrm{Op}_{G_n}(a\chi_n)\psi_j\rangle\right|^2 + nr^3 O\left(\frac{1}{(r\delta_n)^\infty}\right) \\
&= \sum_{j=0}^{n} \left|\langle \psi_j, \mathrm{Op}_{G_n}(a^T\chi_n)\psi_j\rangle\right|^2 + nr^3 O\left(\frac{1}{(r\delta_n)^\infty}\right) \\
&\quad + O\left(\frac{1}{r^2}\right) \sum_{x\in D_n} \int |\tilde{a}_T(x,\omega,s)|^2 \chi_n^2(s) d\nu_x(\omega) d\mu(s) \\
&\quad + O\left(q^{r+2}\right) |\{x\in D_n, \rho(x)\le r+2\}| \int \|\tilde{a}_T(\cdot,\cdot,s)\|_\infty^2 \chi_n^2(s) d\mu(s) \\
&\quad + \sum_{x\in D_n} \int O(|s-s_0|^2) |\tilde{a}_T(x,\omega,s)|^2 \chi_n^2(s) d\nu_x(\omega) d\mu(s) \\
&= \sum_{j=0}^{n} \left|\langle \psi_j, \mathrm{Op}_{G_n}(a_T\chi_n)\psi_j\rangle\right|^2 + nr^3 O\left(\frac{1}{(r\delta_n)^\infty}\right) \\
&\quad + O\left(\frac{T^2}{r^2}\right) n \int \chi_n^2(s) d\mu(s) \\
&\quad + O(T^2 q^{r+2}) |\{x\in D_n, \rho(x)\le r+2\}| \int \chi_n^2(s) d\mu(s) \\
&\quad + O(T^2) n \int O(|s-s_0|^2) \chi_n^2(s) d\mu(s)
\end{aligned}
$$

Next, we use Lemma 3.3 to write

$$
\begin{aligned}
\sum_{j=0}^{n} \left|\langle \psi_j, \mathrm{Op}_{G_n}(a^T\chi_n)\psi_j\rangle\right|^2 &\le \|\mathrm{Op}_{G_n}(a^T\chi_n)\|_{HS}^2 \\
&\le \sum_{x\in D_n} \int |a^T(x,\omega)|^2 d\nu_x(\omega) \chi_n(s)^2 d\mu(s) \\
&\quad + q^r \sum_{x\in D_n, \rho(x)\le r} \int |a^T(x,\omega)|^2 d\nu_x(\omega) \chi_n(s)^2 d\mu(s) \\
&\le \sum_{x\in D_n} \int |a^T(x,\omega)|^2 d\nu_x(\omega) \int \chi_n(s)^2 d\mu(s) \\
&\quad + q^r |\{x\in D_n, \rho(x)\le r\}| \int \chi_n(s)^2 d\mu(s).
\end{aligned}
$$

We also know from the Kesten-McKay law (Section 5, Corollary 5.2) that

$$
\frac{n}{N(I_n,G_n)} \int \chi_n(s)^2 d\mu(s) = O(1).
$$

[4]To prove the extended Theorem 2, we also need Lemma 3.10.

We thus have

$$\begin{aligned}\frac{1}{N(I_n,G_n)}\sum_{s_j\in I_n}|\langle\psi_j,a\psi_j\rangle|^2 &= O(1)\frac{1}{n}\sum_{x\in D_n}\int|a^T(x,\omega)|^2d\nu_x(\omega)\\ &\quad+O(T^2q^{r+2})\frac{|\{x\in D_n,\rho(x)\le r+2\}|}{n}\\ &\quad+\frac{n}{N(I_n,G_n)}r^3O\left(\frac{1}{(r\delta_n)^\infty}\right)+O\left(\frac{T^2}{r^2}\right)+O(T^2\delta_n^2)\\ &= O(1)\frac{1}{n}\sum_{x\in D_n}\int|a^T(x,\omega)|^2d\nu_x(\omega)\\ &\quad+O(T^2q^r)\alpha_n+O(1)\frac{r^3}{\delta_n}O\left(\frac{1}{(r\delta_n)^\infty}\right)\\ &\quad+O\left(\frac{T^2}{r^2}\right)+O(T^2\delta_n^2).\end{aligned}$$

Our choices of $r=r_n$ and δ_n imply that the last four terms vanish as n goes to infinity while T is fixed.

2.2. Expansion and ergodicity. We write, using (3.3)

$$\frac{1}{n}\sum_{x\in D_n}\int|a^T(x,\omega)|^2d\nu_x(\omega)=\frac{1}{n}\frac{1}{T^2}\sum_{k=0}^{T-1}\sum_{j=0}^{T-1}\sum_{x\in D_n}\int a\circ\sigma^{2|k-j|}(x,\omega)a(x)d\nu_x(\omega)$$

For every $x\in\mathfrak{X}$ and $k\in\mathbb{N}$, define the partition of Ω in $(q+1)q^{k-1}$ sets[5]

$$\Omega(x,y):=\{\omega\in\Omega\mid[x,\omega)=(x,x_1,x_2,\ldots)\text{ and }x_k=y\},$$

where $y\in\mathfrak{X}$ and $d(x,y)=k$. Then $\omega\mapsto a\circ\sigma^k(x,\omega)$ is constant on $\Omega(x,y)$ for every $y\in\mathfrak{X}$ such that $d(x,y)=k$, and $\nu_x(\Omega(x,y))=\frac{1}{(q+1)q^{k-1}}$. We have

$$\begin{aligned}\sum_{x\in D_n}\int_\Omega a\circ\sigma^k(x,\omega)a(x)d\nu_x(\omega) &= \sum_{x\in D_n}\sum_{\substack{y\in\mathfrak{X}\\ d_{\mathfrak{X}}(x,y)=k}}\int_{\Omega(x,y)}a\circ\sigma^k(x,\omega)a(x)d\nu_x(\omega)\\ &=\sum_{x\in D_n}S_ka(x)a(x),\end{aligned}$$

where $S_0=\mathrm{Id}$ and for all $k\ge1$, S_k is the stochastic operator defined as follows by its kernel on the tree $\mathfrak{X}$:

$$S_kf(x)=\frac{1}{(q+1)q^{k-1}}\sum_{d_{\mathfrak{X}}(x,y)=k}f(y). \tag{2.2}$$

We thus have

$$\frac{1}{n}\sum_{x\in D_n}\int|a^T(x,\omega)|^2d\nu_x(\omega)=\frac{1}{n}\frac{1}{T^2}\sum_{k=0}^{T-1}\sum_{j=0}^{T-1}\sum_{x\in D_n}S_{2|k-j|}a(x)a(x).$$

On the quotient G_n, the spectrum of S_k is the set $\{\phi_{s_j}(k),j=1,\ldots,n\}$ where ϕ_s is the spherical function,

$$\phi_s(k)=q^{-k/2}\left(\frac{2}{q+1}\cos(ks\ln(q))+\frac{q-1}{q+1}\frac{\sin(k+1)s\ln(q)}{\sin s\ln(q)}\right) \tag{2.3}$$

[5]See section 3 for a definition of the boundary Ω and of the notation $[x,\omega)$.

and $\{s_j, j = 1, \dots, n\}$ are the spectral parameters for the operator A defined by (1.4).

Using the parameterization (1.5) of the spectrum, the eigenvalue $\lambda = 1$ corresponds to

$$is = \frac{1}{2}.$$

Because of the (EXP) condition, the other untempered eigenvalues satisfy $is \in (0, \frac{1}{2} - \beta)$ or $is + i\frac{\pi}{\ln q} \in (0, \frac{1}{2} - \beta)$, for some $\beta > 0$ independent of n. It follows that $|\phi_{s_j}(k)| \le Cq^{-\beta k}$ with C, β independent of n. The eigenvalues of the self-adjoint stochastic operator $\frac{1}{T^2}\sum_{k=0}^{T-1}\sum_{j=0}^{T-1} S_{2|k-j|}$ are therefore bounded by

$$\frac{C}{T^2}\sum_{k=0}^{T-1}\sum_{j=0}^{T-1} q^{-2\beta|k-j|} \le \frac{C}{2T(1-q^{-2\beta})}$$

in modulus. They are contained in $\{1\} \cup \left[-\frac{C}{T\beta}, \frac{C}{T\beta}\right]$ for some C, independent of n, T, β (the eigenvalue 1 has multiplicity 1, corresponding to the constant function).

Thus, if a satisfies $\sum_{x\in V_n} a(x) = 0$ and $\sup_x |a(x)| \le 1$, we have

$$\frac{1}{n}\frac{1}{T^2}\sum_{k=0}^{T-1}\sum_{j=0}^{T-1}\sum_{x\in D_n} S_{2|k-j|}a(x)a(x) = O\left(\frac{1}{T\beta}\right).$$

2.3. Conclusion. We obtain, using the results of the previous sections and the Kesten-McKay law (Corollary 5.2),

$$\frac{1}{N(I_n, G_n)}\sum_{s_j\in I_n} |\langle\psi_j, a\psi_j\rangle|^2 \le O\left(\frac{r^3}{\delta_n(r\delta_n)^\infty}\right) + O\left(\frac{T^2}{r^2}\right) + O(T^2\delta_n^2) \tag{2.4}$$

$$+ O(T^2 q^r \alpha_n) + O\left(\frac{1}{T\beta}\right). \tag{2.5}$$

If we choose the sequences $r = r_n$ and δ_n satisfying $q^r\alpha_n \longrightarrow 0$ and $\frac{r^3}{\delta_n(r\delta_n)^K} \longrightarrow 0$ for some integer K, we finally have

$$\limsup_{n\to\infty}\frac{1}{N(I_n, G_n)}\sum_{s_j\in I_n} |\langle\psi_j, a\psi_j\rangle|^2 = O\left(\frac{1}{T\beta}\right).$$

As the left-hand side of the equality does not depend on T, we take the limit $T \longrightarrow \infty$ to obtain

$$\lim_{n\to\infty}\frac{1}{N(I_n, G_n)}\sum_{s_j\in I_n} |\langle\psi_j, a\psi_j\rangle|^2 = 0.$$

Remark 2.1. If we had wanted to optimize all the estimates, we could have used the property (EXP) to replace the crude bound

$$\sum_{x\in D_n}\int |\tilde{a}_T(x,\omega,s)|^2 d\nu_x(\omega) \le nO(T^2)\|a\|_\infty^2$$

in §2.1 with

$$\sum_{x\in D_n}\int |\tilde{a}_T(x,\omega,s)|^2 d\nu_x(\omega) \le nO\left(\frac{T}{\beta}\right)\|a\|_2^2,$$

which would lead to replace the term $O\left(\frac{T^2}{r^2}\right)$ in (2.4) by $O\left(\frac{T}{\beta r^2}\right)$, and $O(T^2\delta_n^2)$ by $O\left(\frac{T\delta_n^2}{\beta}\right)$.

3. Elements of pseudodifferential calculus

In §3.1–3.2 we recall some of the tools of pseudodifferential calculus that were introduced in [**25**]. However, the following important remark has to be made : in order for Theorems 1 and 2 to have full strength, we should not impose regularity conditions on the symbols a that are too restrictive, which would have the effect of making the theorems trivial consequences of the (EXP) condition. Thus, we pay attention to only use from [**25**] the properties that do not require regularity of a with respect to the x-variable.

In the following sections we try to construct a pseudodifferential calculus on the quotient.

3.1. Definition of $\mathrm{Op}(a)$ on the infinite $(q+1)$-regular tree. Let Ω be the boundary of the tree. It is the set of equivalence classes of infinite half-geodesics of $\mathfrak{X}$ for the relation : two half-geodesics $(x_1, x_2, x_3, \ldots)$ and $(y_1, y_2, y_3, \ldots)$ are equivalent iff there exist $k, N \in \mathbb{N}$ such that for all $n \geq N$, $x_{n+k} = y_n$. For every $\omega \in \Omega$, we will denote by $[x, \omega)$ the unique half-geodesic starting at x and equivalent to ω.

Let $a : \mathfrak{X} \times \Omega \times [0, \tau] \longrightarrow \mathbb{C}$ be a bounded measurable function. In [**25**], the operator $\mathrm{Op}_{\mathfrak{X}}(a)$ was defined by

$$\mathrm{Op}_{\mathfrak{X}}(a)u(x) = \sum_{y\in\mathfrak{X}} \int_\Omega \int_0^\tau q^{\left(\frac{1}{2}+is\right)(h_\omega(y)-h_\omega(x))} a(x,\omega,s) u(y) d\nu_x(\omega) d\mu(s)$$

for every $u : \mathfrak{X} \to \mathbb{C}$ with finite support. Here $h_\omega(x)$ is the height function (or Busemann function), $d\mu(s)$ is the Plancherel measure associated to the $(q+1)$-regular tree[6], and ν_x is the harmonic measure on Ω, seen from the point x. We refer to [**8**] for more background. We will denote by

$$K_{\mathfrak{X}}(x, y; a) = \int_\Omega \int_0^\tau q^{\left(\frac{1}{2}+is\right)(h_\omega(y)-h_\omega(x))} a(x,\omega,s) d\nu_x(\omega) d\mu(s)$$

the kernel of $\mathrm{Op}_{\mathfrak{X}}(a)$.

3.2. Class of symbols. From [**8**], we know that the fact that $K_{\mathfrak{X}}(x, y; a) = 0$ for $d_{\mathfrak{X}}(x, y) > D$ is equivalent to the four following conditions on a :

- a is a continuous function;
- $a(x, \omega, \cdot)$ extends to a 2τ-periodic entire function of exponential type D uniformly in ω; i.e. for all x there exists $C(x) > 0$ such that
$$|a(x,\omega,z)| \leq C(x) q^{D|\Im m(z)|} \qquad \forall\omega \in \Omega, \forall x \in \mathfrak{X};$$
- a satisfies the symmetry condition
$$\int_\Omega q^{\left(\frac{1}{2}-is\right)(h_\omega(y)-h_\omega(x))} a(x,\omega,s) d\nu_x(\omega) = \int_\Omega q^{\left(\frac{1}{2}+is\right)(h_\omega(y)-h_\omega(x))} a(x,\omega,-s) d\nu_x(\omega)$$
for all $s \in \mathbb{C}$, $x \in \mathfrak{X}$.

---

[6] $d\mu(s) = |c(s)|^{-2} ds$ where c is the Harish-Chandra function.

- a is a D-cylindrical function, that is : if the two half-geodesics $[x, \omega) = (x_0, x_1, x_2, \ldots)$ and $[x', \omega') = (x'_0, x'_1, x'_2, \ldots)$ satisfy $x_j = x'_j$ for $0 \le j \le D$, then $a(x, \omega, s) = a(x', \omega', s)$.

We shall denote by $S_o^D(\mathfrak{X})$ the class of such functions. In [**25**], another class of symbols was considered :

Definition 3.1. $S(\mathfrak{X})$ is the class of bounded functions $a : \mathfrak{X} \times \Omega \times [0, \tau] \longrightarrow \mathbb{C}$ such that

- For every $k \in \mathbb{N}$, and every $x \in \mathfrak{X}$, $\partial_s^k a(x, \cdot, \cdot)$ is continuous on $\Omega \times [0, \tau]$, and for every $l \in \mathbb{N}$, there exists $C_l > 0$ such that, for all $n \in \mathbb{N}$, for every (x, ω, s),
$$|(a - \mathcal{E}_n^x a)(x, \omega, s)| \le \frac{C_l}{(1+n)^l}.$$
- For every $k \in \mathbb{N}$, and every (x, ω), $\partial_s^k a(x, \omega, 0) = \partial_s^k a(x, \omega, \tau) = 0$.

Here $\mathcal{E}_n^x a(x, \omega, s)$ is the projection of $a(x, \omega, s)$ on functions depending only on the first n vertices of the half-geodesic $[x, \omega)$ (see [**25**] for a formula). In particular, $\mathcal{E}_n^x a(x, s) = a(x, s)$ if a does not depend on ω.

It is proven in [**25**] that $S(\mathfrak{X})$ endowed with usual addition and multiplication is an algebra. This makes it more suitable for semiclassical analysis than the class $S_o^D(\mathfrak{X})$. It also has the property, crucial for us, that
$$a \in S(\mathfrak{X}) \Longrightarrow a \circ \sigma \in S(\mathfrak{X})$$
where σ is the shift, $\sigma(x, \omega) = (x_1, \omega)$ if $[x, \omega) = (x_0, x_1, x_2, \ldots)$. It is proven in [**25**] that
$$|K_{\mathfrak{X}}(x, y; a)| \le C \left(\|a\|_{\Omega, M} + \sum_{k=0}^{M+1} \|\partial_s^k a\|_\infty \right) \frac{q^{-\frac{d(x,y)}{2}}}{(1 + d(x, y))^M} \tag{3.1}$$
for any M, where $\|a\|_{\Omega, M} = \sup_{(x,\omega,s)} \sup_n (1+n)^M |a - \mathcal{E}_n^x a|(x, \omega, s)$, and that as a consequence, $\mathrm{Op}_{\mathfrak{X}}(a)$ extends to a bounded operator on $L^2(\mathfrak{X})$ if $a \in S(\mathfrak{X})$.

If $a(x, \omega, s) = a(x)$ depends only on x, then $\mathrm{Op}_{\mathfrak{X}}(a)$ is the operator of multiplication by a. At several places we will use the remark that
$$\mathrm{Op}_{\mathfrak{X}}(a)\, \mathrm{Op}_{\mathfrak{X}}(\varphi) = \mathrm{Op}_{\mathfrak{X}}(a\varphi) \tag{3.2}$$
if $\varphi = \varphi(s)$ only depends on the last variable and $a(x, \omega, s) \in S(\mathfrak{X})$, say.

In most of what follows, we will actually need very few conditions on the symbols $a(x, \omega, s)$. Essentially it will be required that $a \in L^\infty(\mathfrak{X} \times \Omega \times [0, \tau])$. For convenience, we can assume that $a \in S_o^D(\mathfrak{X})$ for some $D \in \mathbb{N}$, or that $a \in S(\mathfrak{X})$. In Lemma 3.10 we use the condition $a \in S_o^D(\mathfrak{X})$.

3.3. Definition of $\mathrm{Op}_{G_n}(a)$ on a finite graph. Recall that G_n is written as a quotient $\Gamma_n \backslash \mathfrak{X}$, where Γ_n is a group of automorphisms of $\mathfrak{X}$, whose elements act without fixed points.

Let us now assume that a is Γ_n-invariant, meaning that $a(\gamma \cdot x, \gamma \cdot \omega, s) = a(x, \omega, s)$ for all (x, ω, s) and all $\gamma \in \Gamma_n$ (where the action of Γ_n on the boundary Ω is obtained by extending its action on $\mathfrak{X}$). For a Γ_n-invariant symbol, we have
$$K_{\mathfrak{X}}(\gamma \cdot x, \gamma \cdot y; a) = K_{\mathfrak{X}}(x, y; a)$$
for all $x, y \in \mathfrak{X}$ and $\gamma \in \Gamma_n$. The proof of this fact is identical to the proof of Proposition 1.1 in [**37**].

We now define $\mathrm{Op}(a)$ on the quotient.

DEFINITION 3.2. Assume (G_n) satisfies (BST). Fix $n \in \mathbb{N}$ and let $r = r_n$ be a positive integer. In this section, it can be considered to be an arbitrary positive integer, but later on it should be chosen according to the comment preceding the statement of Theorem 1. If a is Γ_n-invariant, we define $\mathrm{Op}_{G_n}(a)$ to be the operator with Γ_n-bi-invariant kernel

$$K_{G_n}(x, y; a) = \sum_{\gamma \in \Gamma_n} K_{\mathfrak{X}}(x, \gamma \cdot y; a) \chi\left(\frac{d(x, \gamma \cdot y)}{r}\right).$$

Here χ is a cut-off function that satisfies the conditions of §2.1 (although it needs not be the same cut-off as in §2.1, we use the same notation).

Compared to the case of manifolds, a difficulty we meet is that we are not able to prove that $\mathrm{Op}_{G_n}(a)$ is bounded on $L^2(V_n)$ independently of n (actually, inspection of simple examples shows that our conditions on a are not sufficient to ensure this). Note however that we are only interested in $\mathrm{Op}_{G_n}(a)\psi_j^{(n)}$ for $\lambda(s_j^{(n)})$ in the tempered spectrum; more precisely, we shall only need to estimate quantities such as $\frac{1}{N(I_n,G_n)} \sum_{s_j^{(n)} \in I_n} \left| \langle \psi_j^{(n)}, \mathrm{Op}_{G_n}(a)\psi_j^{(n)} \rangle \right|^2$. For that purpose it will be sufficient to know that the Hilbert-Schmidt norm of $\mathrm{Op}_{G_n}(a)$ does not grow too fast :

LEMMA 3.3. *If $a \in L^\infty(\mathfrak{X} \times \Omega \times [0, \tau])$ is Γ_n-invariant, we have for every $r \geq 0$*

$$\begin{aligned}
\|\mathrm{Op}_{G_n}(a)\|_{HS}^2 &\leq \sum_{\substack{x \in D_n \\ \rho(x) \geq r}} \sum_{\substack{y \in \mathfrak{X} \\ d(y,x) \leq r}} |K_{\mathfrak{X}}(x, y; a)|^2 + q^r \sum_{\substack{x \in D_n \\ \rho(x) \leq r}} \sum_{\substack{y \in \mathfrak{X} \\ d(y,x) \leq r}} |K_{\mathfrak{X}}(x, y; a)|^2 \\
&\leq \sum_{x \in D_n} \int |a(x, \omega, s)|^2 d\nu_x(\omega) d\mu(s) + q^r \sum_{\substack{x \in D_n \\ \rho(x) \leq r}} \int |a(x, \omega, s)|^2 d\nu_x(\omega) d\mu(s)
\end{aligned}$$

PROOF. By definition we have

$$\begin{aligned}
\|\mathrm{Op}_{G_n}(a)\|_{HS}^2 &= \sum_{x,y \in D_n} |K_{G_n}(x, y; a)|^2 \\
&= \sum_{x,y \in D_n} \left| \sum_{\gamma \in \Gamma_n} K_{\mathfrak{X}}(x, \gamma \cdot y; a) \chi\left(\frac{d(x, \gamma \cdot y)}{r}\right) \right|^2.
\end{aligned}$$

We split the sum into two parts, whether $\rho(x) \geq r$ or not. If $\rho(x) \geq r$, then the sum over $\gamma \in \Gamma_n$ is reduced to only one term, thanks to the cut-off function. If $\rho(x) \leq r$, then there are at most q^r terms in the sum over $\gamma \in \Gamma_n$, and we can use Cauchy-Schwarz inequality to bound it as follows

$$\begin{aligned}
\|\mathrm{Op}_{G_n}(a)\|_{HS}^2 &\leq \sum_{\substack{x \in D_n \\ \rho(x) \geq r}} \sum_{\substack{y \in \mathfrak{X} \\ d(y,x) \leq r}} |K_{\mathfrak{X}}(x, y; a)|^2 + q^r \sum_{\substack{x \in D_n \\ \rho(x) \leq r}} \sum_{\substack{y \in \mathfrak{X} \\ d(y,x) \leq r}} |K_{\mathfrak{X}}(x, y; a)|^2 \\
&\leq \sum_{\substack{x \in D_n \\ \rho(x) \geq r}} \sum_{y \in \mathfrak{X}} |K_{\mathfrak{X}}(x, y; a)|^2 + q^r \sum_{\substack{x \in D_n \\ \rho(x) \leq r}} \sum_{y \in \mathfrak{X}} |K_{\mathfrak{X}}(x, y; a)|^2.
\end{aligned}$$

Plancherel formula for the Fourier-Helgason transform, applied to $y \mapsto K_{\mathfrak{X}}(x, y; a)$ with x fixed, converts this last expression to

$$\begin{aligned}\|\mathrm{Op}_{G_n}(a)\|_{HS}^2 \leq &\sum_{x\in D_n} \int |a(x,\omega,s)|^2 d\nu_x(\omega) d\mu(s) \\ &+ q^r \sum_{\substack{x\in D_n \\ \rho(x)\leq r}} \int |a(x,\omega,s)|^2 d\nu_x(\omega) d\mu(s).\end{aligned}$$

□

3.4. "Egorov"-type properties. For Quantum Ergodicity on manifolds, the Egorov theorem is a statement saying that the matrix elements $\langle \psi_j, \mathrm{Op}(a)\psi_j\rangle$ remain almost invariant when transporting a along the geodesic flow, when ψ_j are the eigenfunctions of the Laplace-Beltrami operator Δ. This is proven by showing that taking the bracket $[\Delta, \mathrm{Op}(a)]$ amounts to differentiating a along the geodesic flow (up to some "negligible" error term). Here we try to perform a similar calculation.

We define a map $\sigma : \mathfrak{X} \times \Omega \longrightarrow \mathfrak{X} \times \Omega$ by $\sigma(x, \omega) = (x', \omega)$ where $x' \in \mathfrak{X}$ is the unique point such that $d_{\mathfrak{X}}(x, x') = 1$ and x' belongs to the half-geodesic $[x, \omega)$. If $[x, \omega) = (x, x_1, x_2, x_3, \ldots)$, then $\sigma(x, \omega) = (x_1, \omega)$, corresponding to the half-geodesic $[x_1, \omega) = (x_1, x_2, x_3, \ldots)$. In symbolic dynamics, σ is the shift, and if we compare with Quantum Ergodicity on manifolds, σ plays the role of the "geodesic flow" on phase space $\mathfrak{X} \times \Omega$. This map is not invertible, actually each point has exactly q pre-images. We shall denote by U the operator $a \mapsto a \circ \sigma$. For $a : \mathfrak{X} \times \Omega \longrightarrow \mathbb{C}$, we define $La : \mathfrak{X} \times \Omega \longrightarrow \mathbb{C}$ by

$$La(x,\omega) = \frac{1}{q} \sum_{y\in\mathfrak{X}, \sigma(y,\omega)=(x,\omega)} a(y,\omega).$$

If a and b are compactly supported functions, we have

$$\sum_{x\in\mathfrak{X}} \int_\Omega a\circ\sigma(x,\omega) b(x,\omega) d\nu_x(\omega) = \sum_{x\in\mathfrak{X}} \int_\Omega a(x,\omega) Lb(x,\omega) d\nu_x(\omega),$$

in other words L is the adjoint of U on the Hilbert space $L^2(\mathfrak{X} \times \Omega, \sum_x \delta_x d\nu_x(\omega))$. In addition, we also have $LU = I$, reflecting the fact that U is an isometry of $L^2(\mathfrak{X} \times \Omega, \sum_x \delta_x d\nu_x(\omega))$. The operators U and L preserve the Γ_n-invariant functions. If a and b are a Γ_n-invariant functions, we still have

$$\sum_{x\in D_n} \int_\Omega a\circ\sigma(x,\omega) b(x,\omega) d\nu_x(\omega) = \sum_{x\in D_n} \int_\Omega a(x,\omega) Lb(x,\omega) d\nu_x(\omega). \tag{3.3}$$

Recall that D_n is a fundamental domain for the action of Γ_n on $\mathfrak{X}$.

In what follows, we extend the definition of U and L to functions on $\mathfrak{X} \times \Omega \times \mathbb{R}$, by a trivial action on the last component. The crucial bracket calculation is the following :

PROPOSITION 3.1. *If $a \in L^\infty(\mathfrak{X} \times \Omega \times [0, \tau])$ is Γ_n-invariant, then*

$$[\Delta, \mathrm{Op}_{G_n}(a)] = \mathrm{Op}_{G_n}(c) + R$$

where $c = c(a)$ is given by

$$c(x,\omega,s) = \frac{q^{1/2}}{q+1} \left(q^{is}(a\circ\sigma - a)(x,\omega,s) + q^{-is}(La - a)(x,\omega,s) \right),$$

and R is a remainder such that

$$\|R\|_{HS}^2 \leq O\left(\frac{1}{r^2}\right)\left(\sum_{x\in D_n}\int |a(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)\right.$$
$$\left. + q^{r+2}\sum_{\substack{x\in D_n\\ \rho(x)\leq r+2}}\int |a(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)\right).$$

Since $\langle \psi_j, [\Delta, \mathrm{Op}_{G_n}(a)]\psi_j\rangle = 0$ for every laplacian eigenfunction ψ_j, this implies

$$\sum_j |\langle \psi_j, \mathrm{Op}_{G_n}(c)\psi_j\rangle|^2 \leq O\left(\frac{1}{r^2}\right)\left(\sum_{x\in D_n}\int |a(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)\right.$$
$$\left. + q^{r+2}\sum_{\substack{x\in D_n\\ \rho(x)\leq r+2}}\int |a(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)\right). \tag{3.4}$$

As indicated in the statement of the proposition, we shall write $c = c(a)$ when we want to emphasize the dependence of c on the original symbol a.

Proof. Let us denote by $K_{G_n}(x,y;[.])$ the kernel of $[\Delta, \mathrm{Op}_{G_n}(a)]$. We know from [**25**] that $K_{\mathfrak{X}}(x,y;c)$ is the kernel of $[\Delta, \mathrm{Op}_{\mathfrak{X}}(a)]$. We are interested in the difference $K_{G_n}(x,y;[.]) - K_{G_n}(x,y;c)$, which is the kernel of the operator R.

We have

$$K_{G_n}(x,y;[.]) = \frac{1}{q+1}\left(\sum_{d(x,z)=1} K_{G_n}(z,y;a) - \sum_{d(z,y)=1} K_{G_n}(x,z;a)\right)$$
$$= \frac{1}{q+1}\sum_{\gamma\in\Gamma_n} K_{\mathfrak{X}}(x,\gamma\cdot y), \tag{3.5}$$

where

$$K_{\mathfrak{X}}(x,y) := \sum_{d(x,z)=1} K_{\mathfrak{X}}(z,y;a)\chi\left(\frac{d(z,y)}{r}\right) - \sum_{d(z,y)=1} K_{\mathfrak{X}}(x,z;a)\chi\left(\frac{d(x,z)}{r}\right) \tag{3.6}$$

Because of the cut-off functions, the sum (3.5) only runs on those $\gamma\in\Gamma_n$ for which $d(x,\gamma\cdot y)\leq r+1$; and in (3.6) we have $K_{\mathfrak{X}}(x,y) = K_{\mathfrak{X}}(x,y)\mathbb{1}_{\{d(x,y)\leq r+1\}}$.

In the first sum of the right-hand side of equality (3.6), $d(z,y) = d(x,y)\pm 1$, because x and z are neighbours. In the second sum $d(x,z) = d(x,y)\pm 1$, because z and y are neighbours. Since χ is a smooth function, both $\chi\left(\frac{d(z,y)}{r}\right)$ and $\chi\left(\frac{d(x,z)}{r}\right)$ are equal to $\chi\left(\frac{d(x,y)}{r}\right) + O\left(\frac{1}{r}\right)$, and we have

$$\begin{aligned}
K_{\mathfrak{X}}(x,y) &= \left(\sum_{d(x,z)=1} K_{\mathfrak{X}}(z,y;a) - \sum_{d(z,y)=1} K_{\mathfrak{X}}(x,z;a)\right)\chi\left(\frac{d(x,y)}{r}\right) \\
&\quad + O\left(\frac{1}{r}\right)\left(\sum_{d(x,z)=1} |K_{\mathfrak{X}}(z,y;a)| + \sum_{d(z,y)=1} |K_{\mathfrak{X}}(x,z;a)|\right)\mathbb{1}_{\{d(x,y)\leq r+1\}} \\
&= (q+1)K_{\mathfrak{X}}(x,y;c)\chi\left(\frac{d(x,y)}{r}\right) \\
&\quad + O\left(\frac{1}{r}\right)\left(\sum_{d(x,z)=1} |K_{\mathfrak{X}}(z,y;a)| + \sum_{d(z,y)=1} |K_{\mathfrak{X}}(x,z;a)|\right)\mathbb{1}_{\{d(x,y)\leq r+1\}}.
\end{aligned}$$

Now if we go back to (3.5) we get $K_{G_n}(x,y;[.]) = K_{G_n}(x,y;c) + K_{G_n}(x,y;R)$, where $K_{G_n}(x,y;R)$ is the kernel of the operator R, given by

$$K_{G_n}(x,y;R) = O\left(\frac{1}{r}\right)\sum_{\gamma\in\Gamma_n}\left(\sum_{\substack{z\in\mathfrak{X}\\ d(x,z)=1}} |K_{\mathfrak{X}}(z,\gamma\cdot y;a)| + \sum_{\substack{z\in\mathfrak{X}\\ d(z,\gamma\cdot y)=1}} |K_{\mathfrak{X}}(x,z;a)|\right)\mathbb{1}_{\{d(x,\gamma\cdot y)\leq r+1\}}.$$

We estimate the Hilbert-Schmidt norm of R by first writing

$$\begin{aligned}
&\sum_{x,y\in D_n}\left|\sum_{\gamma\in\Gamma_n}\left(\sum_{\substack{z\in\mathfrak{X}\\ d_{\mathfrak{X}}(x,z)=1}} |K_{\mathfrak{X}}(z,\gamma\cdot y;a)| + \sum_{\substack{z\in\mathfrak{X}\\ d_{\mathfrak{X}}(z,\gamma\cdot y)=1}} |K_{\mathfrak{X}}(x,z;a)|\right)\mathbb{1}_{\{d_{\mathfrak{X}}(x,\gamma\cdot y)\leq r+1\}}\right|^2 \\
&\leq \sum_{x,y\in D_n}\left|\sum_{\gamma\in\Gamma_n}\left(\sum_{\substack{z\in\mathfrak{X}\\ d_{\mathfrak{X}}(x,z)=1}} |K_{\mathfrak{X}}(z,\gamma\cdot y;a)|\mathbb{1}_{\{d_{\mathfrak{X}}(z,\gamma\cdot y)\leq r+2\}} + \sum_{\substack{z\in\mathfrak{X}\\ d_{\mathfrak{X}}(z,\gamma\cdot y)=1}} |K_{\mathfrak{X}}(x,z;a)|\mathbb{1}_{\{d_{\mathfrak{X}}(x,\gamma\cdot y)\leq r+1\}}\right)\right|^2.
\end{aligned}$$

We then use the Cauchy-Schwarz inequality and reason along the same lines as in lemma 3.3, to bound the latter expression by

$$\begin{aligned}
&\leq 2(q+1)\Bigg(\sum_{\substack{z\in D_n\\ \rho(z)\geq r+2}}\sum_{y\in\mathfrak{X}}\sum_{\substack{x\in D_n\\ d_{G_n}(z,x)=1}} |K_{\mathfrak{X}}(z,y;a)|^2 + \sum_{\substack{x\in D_n\\ \rho(x)\geq r+1}}\sum_{y\in\mathfrak{X}}\sum_{\substack{z\in\mathfrak{X}\\ d_{\mathfrak{X}}(z,y)=1}} |K_{\mathfrak{X}}(x,z;a)|^2 \\
&\quad + q^{r+2}\sum_{\substack{z\in D_n\\ \rho(z)\leq r+2}}\sum_{y\in\mathfrak{X}}\sum_{\substack{x\in D_n\\ d_{G_n}(z,x)=1}} |K_{\mathfrak{X}}(z,y;a)|^2 + q^{r+1}\sum_{\substack{x\in D_n\\ \rho(x)\leq r+1}}\sum_{y\in\mathfrak{X}}\sum_{\substack{z\in\mathfrak{X}\\ d_{\mathfrak{X}}(z,y)=1}} |K_{\mathfrak{X}}(x,z;a)|^2\Bigg) \\
&\leq 2(q+1)^2\Bigg(\sum_{\substack{z\in D_n\\ \rho(z)\geq r+2}}\sum_{y\in\mathfrak{X}} |K_{\mathfrak{X}}(z,y;a)|^2 + \sum_{\substack{x\in D_n\\ \rho(x)\geq r+1}}\sum_{z\in\mathfrak{X}} |K_{\mathfrak{X}}(x,z;a)|^2 \\
&\quad + q^{r+2}\sum_{\substack{z\in D_n\\ \rho(z)\leq r+2}}\sum_{y\in\mathfrak{X}} |K_{\mathfrak{X}}(z,y;a)|^2 + q^{r+1}\sum_{\substack{x\in D_n\\ \rho(x)\leq r+1}}\sum_{z\in\mathfrak{X}} |K_{\mathfrak{X}}(x,z;a)|^2\Bigg)
\end{aligned}$$

$$\leq 4(q+1)^2 \sum_{x\in D_n} \int |a(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s) + q^{r+2} \sum_{\substack{x\in D_n\\ \rho(x)\leq r+2}} \int |a(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s).$$

Finally we have

$$\|R\|_{HS}^2 \leq O\left(\frac{1}{r^2}\right)\Bigg(\sum_{x\in D_n}\int |a(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s) \\ + q^{r+2} \sum_{\substack{x\in D_n\\ \rho(x)\leq r+2}} \int |a(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)\Bigg).$$

□

In what follows, Proposition 3.1 will be translated into an invariance property of the type

$$\frac{1}{N(I_n,G_n)}\sum_{s_j\in I_n} |\langle\psi_j, \mathrm{Op}_{G_n}(a)\psi_j\rangle|^2 \sim \frac{1}{N(I_n,G_n)}\sum_{s_j\in I_n} |\langle\psi_j, \mathrm{Op}_{G_n}(Ta)\psi_j\rangle|^2$$

for some operator T. A key idea is then to take advantage of the spectral properties of T and its iterates T^k for $k\in\mathbb{N}$. In the special case where $s_0=\tau/2$ we can take $T=U^2$ (Corollary 3.8), which makes this spectral value special. For general values of s_0, T is a linear combination with complex coefficients of the non-commuting operators L and U, and its spectral properties are not so nice. The aim of the successive operations done in Corollaries 3.4 to 3.7 is to replace Ta with Ua up to some error term. We first replace a with $q^{is}a\circ\sigma$ in (3.4) to obtain

Corollary 3.4. *If $a\in L^\infty(\mathfrak{X}\times\Omega\times[0,\tau])$ is Γ_n-invariant, then*

$$\sum_j |\langle\psi_j, \mathrm{Op}\left((U-I)(q^{2is}U-I)a\right)\psi_j\rangle|^2 \\ \leq O\left(\frac{1}{r^2}\right)\sum_{x\in D_n}\int |a(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s) \\ + O(q^r)|\{x\in D_n, \rho(x)\leq r+2\}| \int \|a(\cdot,\cdot,s)\|_\infty^2 d\mu(s)$$

where $Ua=a\circ\sigma$

Proof. Recall that the symbol $c=c(a)$ of Proposition 3.1 is given by

$$c(a) = \frac{q^{1/2}}{q+1}(q^{is}(Ua-a)+q^{-is}(La-a)).$$

If we replace a with $q^{is}Ua$ we have

$$c(q^{is}Ua) = \frac{q^{1/2}}{q+1}(q^{2is}(U^2a-Ua)+(a-Ua)) = \frac{q^{1/2}}{q+1}(U-I)(q^{2is}U-I)a.$$

It follows from (3.4) that

$$\sum_j |\langle \psi_j, \mathrm{Op}\left((U-I)(q^{2is}U-I)a\right)\psi_j\rangle|^2$$
$$\leq O\left(\frac{1}{r^2}\right)\sum_{x\in D_n}\int |Ua(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)$$
$$+O\left(\frac{q^r}{r^2}\right)\sum_{x\in D_n,\rho(x)\leq r+2}\int |Ua(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)$$
$$\leq O\left(\frac{1}{r^2}\right)\sum_{x\in D_n}\int |a(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)$$
$$+O\left(\frac{q^r}{r^2}\right)|\{x\in D_n,\rho(x)\leq r+2\}|\int \|a(\cdot,\cdot,s)\|_\infty^2 d\mu(s),$$

where we used the fact that U preserves the L^2 and L^∞ norms. □

The idea is then to invert $(q^{2is}U-I)$. As the series $\sum_k q^{2iks}U^k$ is a formal inverse to $(q^{2is}U-I)$, we apply Corollary 3.4 to $a=\sum_{k=0}^{N-1} q^{2iks}U^k b := b_{N-1}$, where $b\in L^\infty$ and N is an arbitrary integer. We obtain

Corollary 3.5.

$$\sum_j |\langle \psi_j, \mathrm{Op}\left((U-I)(I-q^{2iNs}U^N)b\right)\psi_j\rangle|^2$$
$$\leq O\left(\frac{1}{r^2}\right)\sum_{x\in D_n}\int |b_{N-1}(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)$$
$$+O\left(\frac{N^2q^r}{r^2}\right)|\{x\in D_n,\rho(x)\leq r+2\}|\int \|b(\cdot,\cdot,s)\|_\infty^2 d\mu(s)$$

where $b_{N-1}=\sum_{k=0}^{N-1} q^{2iks}U^k b$.

Proof. We apply Corollary 3.4 to $a=\sum_{k=0}^{N-1} q^{2iks}U^k b := b_{N-1}$ and use the identity

$$(q^{2is}U-I)b_{N-1}=(q^{2iNs}U^N-I)b$$

combined with the fact that U preserves the L^2 and L^∞ norms. □

If we apply Corollary 3.4 with a replaced by $\frac{1}{N}\sum_{k=0}^{N-1} b_k := S_N(b)$, we obtain

Corollary 3.6.

$$\sum_j \left|\langle \psi_j, \mathrm{Op}\left(Ub-b-q^{2is}U(U-I)b^{(s,N)}\right)\psi_j\rangle\right|^2$$
$$\leq O\left(\frac{1}{r^2}\right)\sum_{x\in D_n}\int |S_N(b)(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)$$
$$+O\left(\frac{N^2q^r}{r^2}\right)|\{x\in D_n,\rho(x)\leq r+2\}|\int \|b(\cdot,\cdot,s)\|_\infty^2 d\mu(s)$$

where $b^{(s,N)}=\frac{1}{N}\sum_{k=0}^{N-1} q^{2isk}U^k b$.

Note that the "remainder term" $q^{2is}U(U-I)b^{(s,N)}$ is not small in the symbol norm : in Section 4, the (EXP) assumption will be used to show that it is small in the L^2-norm. This is a major difference with the Egorov theorem on manifolds, where no ergodicity assumption is needed.

PROOF. We know from the proof of the previous corollary that

$$(I-q^{2is}U)b_k = (I-q^{2i(k+1)s}U^{k+1})b.$$

It follows that $(I-q^{2is}U)\frac{1}{N}\sum_{k=0}^{N-1} b_k = b - q^{2is}Ub^{(s,N)}$, and

$$(U-I)(I-q^{2is}U)\frac{1}{N}\sum_{k=0}^{N-1} b_k = Ub - b - q^{2is}U(U-I)b^{(s,N)}.$$

□

Combining with the Hilbert-Schmidt estimate, Lemma 3.3, we get

COROLLARY 3.7. *If $b \in L^\infty(\mathfrak{X}\times\Omega\times[0,\tau])$ is Γ_n-invariant, then we have*

$$\begin{aligned}\sum_j |\langle \psi_j, \mathrm{Op}\,(Ub-b)\,\psi_j\rangle|^2 \leq O\left(\frac{1}{r^2}\right)&\sum_{x\in D_n}\int |S_N(b)(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)\\ &+O(1)\sum_{x\in D_n}\int |b^{(s,N)}(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)\\ &+O(N^2q^r)|\{x\in D_n, \rho(x)\leq r+2\}|\int \|b(\cdot,\cdot,s)\|_\infty^2 d\mu(s)\end{aligned}$$

PROOF. We write

$$\begin{aligned}\sum_j |\langle \psi_j, \mathrm{Op}\,(Ub-b)\,\psi_j\rangle|^2 \leq 2\sum_j &\left|\langle \psi_j, \mathrm{Op}\left(Ub-b-q^{2is}U(U-I)b^{(s,N)}\right)\psi_j\rangle\right|^2\\ &+2\sum_j \left|\langle \psi_j, \mathrm{Op}\left(q^{2is}U(I-U)b^{(s,N)}\right)\psi_j\rangle\right|^2\end{aligned}$$

The first term on the right-hand side is estimated by Corollary 3.6. We estimate the last term thanks to Lemma 3.3, and we use the fact that U preserves the L^2 norm :

$$\begin{aligned}&\sum_j \left|\langle \psi_j, \mathrm{Op}\left(q^{2is}U(I-U)b^{(s,N)}\right)\psi_j\rangle\right|^2\\ &\leq \sum_{x\in D_n}\int |(I-U)b^{(s,N)}(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)\\ &\quad + q^r\sum_{x\in D_n,\rho(x)\leq r}\int |(I-U)b^{(s,N)}(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)\\ &\leq O(1)\sum_{x\in D_n}\int |b^{(s,N)}(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)\\ &\quad + 4N^2q^r|\{x\in D_n, \rho(x)\leq r\}|\int \|b(\cdot,\cdot,s)\|_\infty^2 d\mu(s).\end{aligned}$$

□

As already mentioned, the value $s_0 = \tau/2$ is special and the previous corollaries may be replaced by the following, simpler one. In case the support of a shrinks around $s_0 = \tau/2$, this is a closer analogue of the Egorov theorem on manifolds in the sense that no ergodicity or expanding assumption is needed to show that the remainder term goes to 0.

COROLLARY 3.8. *If $a \in L^\infty(\mathfrak{X} \times \Omega \times [0,\tau])$ is Γ_n-invariant, then we have*

$$\sum_j |\langle \psi_j, \mathrm{Op}(a - a\circ\sigma^2)\psi_j\rangle|^2 \leq O\left(\frac{1}{r^2}\right) \sum_{x\in D_n} \int |a(x,\omega,s)|^2 d\nu_x(\omega) d\mu(s)$$
$$+ O\left(q^{r+2}\right) |\{x \in D_n, \rho(x) \leq r+2\}| \int \|a(\cdot,\cdot,s)\|_\infty^2 d\mu(s)$$
$$+ \sum_{x\in D_n} \int O(|s-s_0|^2) |a(x,\omega,s)|^2 d\nu_x(\omega) d\mu(s)$$

PROOF. We replace the symbol a in Proposition 3.1 with $q^{is} a\circ\sigma$. As $L(a\circ\sigma) = a$, we have $c(q^{is} a\circ\sigma) = a - Ta$, where

$$Ta(x,\omega,s) = -q^{2is} a\circ\sigma^2(x,\omega,s) + 2q^{is}\cos(s\ln q) a\circ\sigma(x,\omega,s)$$

and we have $\sum_j |\langle\psi_j, \mathrm{Op}(a - Ta)\psi_j\rangle|^2 \leq \|R\|_{HS}^2$, where

$$\begin{aligned}
\|R\|_{HS}^2 &\leq O\left(\frac{1}{r^2}\right) \sum_{x\in D_n} \int |a\circ\sigma(x,\omega,s)|^2 d\nu_x(\omega) d\mu(s) \\
&\quad + O\left(q^{r+1}\right) \sum_{x\in D_n, \rho(x)\leq r+1} \int |a\circ\sigma(x,\omega,s)|^2 d\nu_x(\omega) d\mu(s) \\
&\leq O\left(\frac{1}{r^2}\right) \sum_{x\in D_n} \int |a(x,\omega,s)|^2 d\nu_x(\omega) d\mu(s) \\
&\quad + O\left(q^{r+1}\right) |\{x \in D_n, \rho(x) \leq r+1\}| \int \|a(\cdot,\cdot,s)\|_\infty^2 d\mu(s).
\end{aligned}$$

Now write $Ta = a\circ\sigma^2 - b$, with

$$b(x,\omega,s) = (1 + q^{2is}) a\circ\sigma^2(x,\omega,s) + 2q^{is}\cos(s\ln q) a\circ\sigma(x,\omega,s)$$

and we can write

$$\begin{aligned}
\sum_j |\langle\psi_j, \mathrm{Op}(a - a\circ\sigma^2)\psi_j\rangle|^2 &\leq 2\sum_j |\langle\psi_j, \mathrm{Op}(b)\psi_j\rangle|^2 + 2\|R\|_{HS}^2 \\
&\leq 2\|\mathrm{Op}(b)\|_{HS}^2 + 2\|R\|_{HS}^2.
\end{aligned}$$

Recalling that $s_0 = \frac{\pi}{2\ln q}$, we have

$$\begin{aligned}
\|\mathrm{Op}(b)\|_{HS}^2 \leq & \sum_{x\in D_n} \int |(1+q^{2is})a\circ\sigma^2 + 2q^{is}\cos(s\ln q)a\circ\sigma)|^2 d\nu_x(\omega)d\mu(s) \\
& + q^r \sum_{x\in D_n, \rho(x)\leq r} \int |(1+q^{2is})a\circ\sigma^2 + 2q^{is}\cos(s\ln q)a\circ\sigma)|^2 d\nu_x(\omega)d\mu(s) \\
\leq & \sum_{x\in D_n} \int O(|s-s_0|^2)(|a\circ\sigma^2| + |a\circ\sigma|)^2(x,\omega,s)d\nu_x(\omega)d\mu(s) \\
& + q^r \sum_{x\in D_n, \rho(x)\leq r} \int O(|s-s_0|^2)(|a\circ\sigma^2| + |a\circ\sigma|)^2(x,\omega,s)d\nu_x(\omega)d\mu(s) \\
\leq & \sum_{x\in D_n} \int O(|s-s_0|^2)|a(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s) \\
& + q^r |\{x\in D_n, \rho(x)\leq r\}| \int O(|s-s_0|^2) \sup_{x,\omega} |a(x,\omega,s)|^2 d\mu(s)
\end{aligned}$$

□

3.5. Two more formulas about $\mathrm{Op}_{G_n}(\chi_n)$.

LEMMA 3.9.

$$\mathrm{Op}_{G_n}(\chi_n)\psi_j^{(n)} = \lambda_j^{(n)}\psi_j^{(n)}$$

with

$$\lambda_j^{(n)} = \chi_n(s_j) + r^3 O\left(\frac{1}{(r\delta_n)^\infty}\right)$$

if $s_j \in [0,\tau]$ *(tempered eigenfunctions).*

PROOF. First note that $\psi_j^{(n)}$ is associated to a Γ_n-invariant eigenfunction of the laplacian on the tree $\mathfrak{X}$, that we will still denote by $\psi_j^{(n)}$. We have on the tree

$$\mathrm{Op}_{G_n}(\chi_n)\psi_j^{(n)}(x) = \sum_{y\in\mathfrak{X}} K_{\mathfrak{X}}(x,y;\chi_n)\chi\left(\frac{d(x,y)}{r}\right)\psi_j^{(n)}(y)$$

and $K_{\mathfrak{X}}(x,y;\chi_n)\chi\left(\frac{d(x,y)}{r}\right)$ depends only on $d(x,y)$ because χ_n does not depend on (x,ω). We thus have

$$\mathrm{Op}_{G_n}(\chi_n)\psi_j^{(n)}(x) = f(s_j)\psi_j^{(n)}(x)$$

where $f(s_j)$ is given by the spherical transform of the kernel

$$f(s_j) = \sum_{y\in\mathfrak{X}} K_{\mathfrak{X}}(x,y;\chi_n)\chi\left(\frac{d(x,y)}{r}\right)\phi_{s_j}(d(x,y))$$

and ϕ_{s_j} is the spherical function associated with s_j defined in (2.3). Now

$$\begin{aligned} f(s_j) &= \sum_{y\in\mathfrak{X}} K_{\mathfrak{X}}(x,y;\chi_n)\phi_{s_j}(d(x,y)) \\ &\quad - \sum_{y\in\mathfrak{X}} K_{\mathfrak{X}}(x,y;\chi_n)\phi_{s_j}(d(x,y)) \left(1-\chi\left(\frac{d(x,y)}{r}\right)\right) \\ &= \chi_n(s_j) \\ &\quad - \sum_{y\in\mathfrak{X}} K_{\mathfrak{X}}(x,y;\chi_n)\phi_{s_j}(d(x,y)) \left(1-\chi\left(\frac{d(x,y)}{r}\right)\right). \end{aligned}$$

Because $\chi_n \in S(\mathfrak{X})$, according to the rapid decay property of the kernel of pseudodifferential operators (3.1), we have

$$|K_{\mathfrak{X}}(x,y;\chi_n)| \leq C \left(\sum_{k=0}^{M+1} \|\partial_s^k \chi_n\|_\infty\right) \frac{q^{-\frac{d(x,y)}{2}}}{(1+d(x,y))^M}$$

for every $M \in \mathbb{N}$. Moreover, if s_j is a tempered eigenvalue, then

$$|\phi_{s_j}(d(x,y))| \leq C q^{-\frac{d(x,y)}{2}}.$$

So, using also the fact that $\|\partial_s^k \chi_n\|_\infty = O(\delta_n^{-k})$, we obtain that for every $M \in \mathbb{N}$

$$\begin{aligned} |f(s_j)-\chi_n(s_j)| &= O\left(\delta_n^{-(M+1)}\right) \sum_{y\in\mathfrak{X}} \frac{q^{-d(x,y)}}{(1+d(x,y))^M} \left(1-\chi\left(\frac{d(x,y)}{r}\right)\right) \\ &= \frac{O\left(\delta_n^{-(M+1)}\right)}{(1+r)^{M-2}} \sum_{y\in\mathfrak{X}} \frac{q^{-d(x,y)}}{(1+d(x,y))^2} \\ &= \frac{O\left(\delta_n^{-(M+1)}\right)}{(1+r)^{M-2}} \sum_{k\in\mathbb{N}} \sum_{y:d(x,y)=k} \frac{q^{-k}}{(1+k)^2} \\ &= \frac{O\left(\delta_n^{-(M+1)}\right)}{(1+r)^{M-2}} \sum_{k\in\mathbb{N}} \frac{1}{(1+k)^2}. \end{aligned}$$

We thus have

$$|f(s_j)-\chi_n(s_j)| = r^3 O\left(\frac{1}{(r\delta_n)^{M+1}}\right)$$

for any M. □

LEMMA 3.10. *Fix an integer D. Let a be such that $K_{\mathfrak{X}}(x,y;a) = 0$ for $d_{\mathfrak{X}}(x,y) > D$ (in other words, $a \in S_o^D(\mathfrak{X})$) and $\varphi = \varphi(s)$. We have*

$$\mathrm{Op}_{G_n}(a\varphi) = \mathrm{Op}_{G_n}(a)\,\mathrm{Op}_{G_n}(\varphi) + R$$

where

$$\|R\|_{HS}^2 \leq O(r^{-1}) \int \varphi^2(s)d\mu(s) \left(n + |\{x \in V_n, \rho(x) \leq r+D\}| q^{r+D}\right)$$

PROOF. The kernel $K_{G_n}(x,z;a\varphi)$ of $\mathrm{Op}_{G_n}(a\varphi)$ is obtained by the periodization

$$K_{G_n}(x,z;a\varphi) = \sum_{\gamma\in\Gamma_n} K_{\mathfrak{X}}(x,\gamma\cdot z;a\varphi)\chi\left(\frac{d(x,\gamma\cdot z)}{r}\right).$$

Because φ only depends on s, we note that

$$K_{\mathfrak{X}}(x,z;a\varphi)=\sum_{y\in\mathfrak{X}}K_{\mathfrak{X}}(x,y;a)K_{\mathfrak{X}}(y,z;\varphi)$$

(see (3.2)) and

$$\begin{aligned}
K_{\mathfrak{X}}(x,z;a\varphi)\chi\left(\frac{d(x,z)}{r}\right) &= \sum_{y\in\mathfrak{X}}K_{\mathfrak{X}}(x,y;a)K_{\mathfrak{X}}(y,z;\varphi)\chi\left(\frac{d(y,z)}{r}\right)\\
&+\sum_{y\in\mathfrak{X}}K_{\mathfrak{X}}(x,y;a)K_{\mathfrak{X}}(y,z;\varphi)\left(\chi\left(\frac{d(x,z)}{r}\right)-\chi\left(\frac{d(y,z)}{r}\right)\right)\\
&=\sum_{y\in\mathfrak{X}}K_{\mathfrak{X}}(x,y;a)K_{\mathfrak{X}}(y,z;\varphi)\chi\left(\frac{d(y,z)}{r}\right)\\
&+O(r^{-1})\sum_{y\in\mathfrak{X}}|K_{\mathfrak{X}}(x,y;a)||K_{\mathfrak{X}}(y,z;\varphi)|\mathbb{1}_{d_{\mathfrak{X}}(x,z)\leq r+D}
\end{aligned}$$

After Γ_n-periodization, we note that $\sum_{\gamma\in\Gamma_n}\sum_{y\in\mathfrak{X}}K_{\mathfrak{X}}(x,y;a)K_{\mathfrak{X}}(y,\gamma\cdot z;\varphi)\chi\left(\frac{d(y,\gamma\cdot z)}{r}\right)$ is the kernel of $\mathrm{Op}_{G_n}(a)\,\mathrm{Op}_{G_n}(\varphi)$ (as soon as $D<r$). Using Cauchy-Schwarz and the fact that $K_{\mathfrak{X}}(x,y;a)$ is supported near the diagonal, the Hilbert-Schmidt norm of the operator with kernel

$$\sum_{\gamma\in\Gamma_n}\sum_{y\in\mathfrak{X}}|K_{\mathfrak{X}}(x,y;a)||K_{\mathfrak{X}}(y,\gamma\cdot z;\varphi)|\mathbb{1}_{d_{\mathfrak{X}}(x,\gamma\cdot z)\leq r+D}$$

on $L^2(G_n)$ can be bounded by

$$q^{2D}\sup_{x,y}|K(x,y;a)|^2\int\varphi^2(s)d\mu(s)\left(n+|\{x\in V_n,\rho(x)\leq r+D\}|q^{r+D}\right).$$

$\square$

4. The proof for arbitrary s_0

4.1. Upper bound on the variance. As in section 2.1, fix an integer $T>0$, and let χ be a smooth cut-off function supported in $[-1,1]$ and taking the constant value 1 on $[-1/2,1/2]$. We write

$$\chi_n(s)=\chi\left(\frac{s-s_0}{2\delta_n}\right)$$

so that $\chi_n\equiv 1$ on I_n. We use the pseudodifferential calculus and the notation defined in Section 3, taking the cut-off parameter r equal to r_n (from condition (BST), as explained before the statement of Theorem 1). To simplify the notation, we will also write $\psi_j=\psi_j^{(n)}$, $s_j=s_j^{(n)}$, and $a=a_n$.

Let us first iterate the "Egorov" property (Corollary 3.7) to put it in the form that we will use. For any integer $k\geq 1$

$$b-b\circ\sigma^k=\sum_{l=0}^{k-1}(b\circ\sigma^l-b\circ\sigma^{l+1}),$$

and by successively applying Corollary 3.7 to $b, b\circ\sigma, \dots, b\circ\sigma^{k-1}$, we have the generalization

$$\sum_j \left|\langle \psi_j, \mathrm{Op}_{G_n}(b - b\circ\sigma^k)\psi_j\rangle\right|^2 \le k\sum_{l=0}^{k-1} R(b\circ\sigma^l) \le k^2 R(b),$$

where $R(b)$ is the remainder in Corollary 3.7, namely

$$\begin{aligned} R(b) := O\left(\frac{1}{r^2}\right)&\sum_{x\in D_n}\int |S_N(b)(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)\\ &+O(1)\sum_{x\in D_n}\int |b^{(s,N)}(x,\omega,s)|^2 d\nu_x(\omega)d\mu(s)\\ &+O(N^2q^r)|\{x\in D_n, \rho(x)\le r+2\}|\int \|b(\cdot,\cdot,s)\|_\infty^2 d\mu(s)\end{aligned}$$

Note the property that $R(b\circ\sigma) = R(b)$. We then write

$$\begin{aligned}\sum_j \left|\langle \psi_j, \mathrm{Op}_{G_n}(b - b^T)\psi_j\rangle\right|^2 &= \sum_j \left|\langle \psi_j, \mathrm{Op}_{G_n}\left(\frac{1}{T}\sum_{k=0}^{T-1}(b - b\circ\sigma^k)\right)\psi_j\rangle\right|^2\\ &\le \frac{1}{T}\sum_{k=0}^{T-1}\sum_j \left|\langle \psi_j, \mathrm{Op}_{G_n}(b - b\circ\sigma^k)\psi_j\rangle\right|^2\\ &\le T^2 R(b)\end{aligned}$$

where $b^T := \frac{1}{T}\sum_{k=0}^{T-1} b\circ\sigma^k$.

We use Lemma 3.9, apply the previous inequality to $b = a\chi_n$ (writing explicitly the remainder $R(a\chi_n)$), and also use the fact that $\|a\|_\infty \le 1$ to obtain[7]

$$\begin{aligned}\sum_{j=0}^n \chi_n(s_j)^2\left|\langle\psi_j, a\psi_j\rangle\right|^2 &= \sum_{j=0}^n \left|\langle \psi_j, \mathrm{Op}_{G_n}(a\chi_n)\psi_j\rangle\right|^2 + nr^3 O\left(\frac{1}{(r\delta_n)^\infty}\right)\\ &= O(1)\sum_{j=0}^n \left|\langle \psi_j, \mathrm{Op}_{G_n}(a^T\chi_n)\psi_j\rangle\right|^2 + nr^3 O\left(\frac{1}{(r\delta_n)^\infty}\right)\\ &\quad +O\left(\frac{T^2}{r^2}\right)\sum_{x\in D_n}\int |S_N(a)(x,\omega)|^2 d\nu_x(\omega)\chi_n(s)^2 d\mu(s)\\ &\quad +O(T^2N^2q^r)|\{x\in D_n, \rho(x)\le r+2\}|\int \chi_n(s)^2 d\mu(s)\\ &\quad +O(T^2)\sum_{x\in D_n}\int |a^{(s,N)}(x,\omega)|^2 d\nu_x(\omega)\chi_n(s)^2 d\mu(s),\end{aligned}$$

where $\quad a^{(s,N)} = \frac{1}{N}\sum_{k=0}^{N-1} q^{2isk} a\circ\sigma^k$.

Remark 4.1. Up to now, we can summarize what has been done in the proof of the Egorov property and in the lines above, by saying that we have found two operators B_T and R_T such that

$$\mathrm{Op}(a\chi_n) = \mathrm{Op}(a^T\chi_n) + [\Delta, B_T] + R_T$$

[7] To prove the extended Theorem 2, we also need Lemma 3.10.

where, for any fixed T, the operator R_T has a "small" Hilbert-Schmidt norm as $n \longrightarrow +\infty$. What we need to show then is that $\mathrm{Op}(a^T\chi_n)$ has a "small" Hilbert-Schmidt norm if T is chosen large enough (uniformly in n).

We use Lemma 3.3 to write

$$\begin{aligned}\sum_{j=0}^{n} \left|\langle \psi_j, \mathrm{Op}_{G_n}(a^T\chi_n)\psi_j\rangle\right|^2 &\leq \|\mathrm{Op}_{G_n}(a^T\chi_n)\|^2{}_{HS} \\ &\leq \sum_{x\in D_n} \int |a^T(x,\omega)|^2 d\nu_x(\omega)\chi_n(s)^2 d\mu(s) \\ &\quad + q^r \sum_{x\in D_n, \rho(x)\leq r} \int |a^T(x,\omega)|^2 d\nu_x(\omega) \int \chi_n(s)^2 d\mu(s).\end{aligned}$$

We have seen in Section 2.2 that $\sum_{x\in D_n}\int |a^T(x,\omega)|^2 d\nu_x(\omega) = O\left(\frac{n}{T\beta}\right)$. The same proof shows that $\sum_{x\in D_n}\int |a^{(s,N)}(x,\omega)|^2 d\nu_x(\omega) = O\left(\frac{n}{N\beta}\right)$. A major difference here with the usual Quantum Ergodicity (and with the special proof of §2) is that condition (EXP) is used already to show that the "remainder term" $a^{(s,N)}$ of the Egorov theorem is small in the L^2-norm.

The estimate $\sum_{x\in D_n}\int |a^{(s,N)}(x,\omega)|^2 d\nu_x(\omega) = O\left(\frac{n}{N\beta}\right)$ also leads to

$$\sum_{x\in D_n}\int |S_N(a)(x,\omega)|^2 d\nu_x(\omega) = O\left(\frac{nN}{\beta}\right).$$

Remark 4.2. For s staying away from $\tau/2$, a more careful proof would show that we only need to assume here that the spectrum of A is contained in $\{1\}\cup[-1, 1-\beta]$. Hence our Remark 1.4.

Recall also, from the Kesten-McKay law (Section 5, Corollary 5.2) that

$$\frac{n}{N(I_n, G_n)}\int \chi_n(s)^2 d\mu(s) = O(1).$$

We obtain finally

$$\begin{aligned}\frac{1}{N(I_n, G_n)}\sum_{s_j\in I_n} |\langle\psi_j, a\psi_j\rangle|^2 &= r^3 O\left(\frac{1}{\delta_n (r\delta_n)^\infty}\right) + O\left(\frac{NT^2}{\beta r^2}\right) \\ &\quad + O(N^2T^2q^r\alpha_n) + O\left(\frac{T^2}{N\beta}\right) + O\left(\frac{1}{T\beta}\right),\end{aligned}$$

and if we choose the sequences $r = r_n$ and δ_n as explained in section 5,

$$\limsup_{n\to\infty}\frac{1}{N(I_n, G_n)}\sum_{s_j\in I_n} |\langle\psi_j, a\psi_j\rangle|^2 = O\left(\frac{T^2}{N\beta}\right) + O\left(\frac{1}{T\beta}\right).$$

As the left-hand side of the equality does not depend on T and N, we take the limit $N\longrightarrow\infty$ and then $T\longrightarrow\infty$ to get

$$\lim_{n\to\infty}\frac{1}{N(I_n, G_n)}\sum_{s_j\in I_n} |\langle\psi_j, a\psi_j\rangle|^2 = 0.$$

5. Kesten-McKay law for sequences of graphs satisfying (BST)

In this section we give an alternative proof of the Kesten-McKay law [**21, 28**], which gives the spectral density for large regular graphs satisfying (BST) and is analogous to the Weyl law for the spectral density of the laplacian on Riemannian manifolds. Note that we consider the density of eigenvalues in intervals that are allowed to shrink as $n \longrightarrow +\infty$.

In the definition of Op_{G_n}, we take $r = r_n$ such that $r_n + 2 \leq R_n$ and $q^{r_n}\alpha_n \longrightarrow 0$ (where R_n and α_n are the quantities occurring in (BST))[8]. We also assume that there exists an integer M such that

$$\frac{1}{\delta_n^{M+1} r_n^{M-3}} \longrightarrow 0.$$

If χ_n is the function defined in (2.1) with $s_0 \in (0, \tau)$, this ensures that

$$r_n^3(\delta_n r_n)^{-M} = o\Big(\int_0^\tau \chi_n(s)^2 d\mu(s)\Big).$$

THEOREM 5.1. *Assume (BST). Let $\chi = \chi_n$ be a smooth function satisfying*

$$\|\partial_s^k \chi\| \leq C_k \delta_n^{-k},$$

with $\delta_n r_n \longrightarrow +\infty$, such that $\frac{1}{\delta_n^{M+1} r_n^{M-3}} \longrightarrow 0$ for some M.

Then we have

$$\frac{1}{n}\sum_{j=1}^{n} \chi_n(s_j)^2 \sim \int_0^\tau \chi_n(s)^2 d\mu(s)$$

when $n \longrightarrow +\infty$.

PROOF. Using Lemma 3.9, we have

$$\begin{aligned}
\sum_{j=1}^{n} \chi_n(s_j)^2 &= \mathrm{Tr}\left(\mathrm{Op}_{G_n}(\chi_n)^2\right) + O(nr_n^3(\delta_n r_n)^{-\infty}) \\
&= \sum_{x\in D_n}\sum_{y\in D_n} |K_{G_n}(x,y;\chi_n)|^2 + O(nr_n^3(\delta_n r_n)^{-\infty}) \\
&= \sum_{\rho(x)>r_n}\sum_{y\in\mathfrak{X}} |K_{\mathfrak{X}}(x,y;\chi_n)|^2 + O(nr_n^3(\delta_n r_n)^{-\infty}) \\
&\quad + O(q^{r_n}) \sum_{\rho(x)\leq r_n}\sum_{y\in\mathfrak{X}} |K_{\mathfrak{X}}(x,y;\chi_n)|^2 \\
&= \sum_{x\in D_n}\sum_{y\in\mathfrak{X}} |K_{\mathfrak{X}}(x,y;\chi_n)|^2 + O(nr_n^3(\delta_n r_n)^{-\infty}) \\
&\quad + (O(q^{r_n})-1) \sum_{\rho(x)\leq r_n}\sum_{y\in\mathfrak{X}} |K_{\mathfrak{X}}(x,y;\chi_n)|^2 \\
&= n\int_0^\tau \chi_n(s)^2 d\mu(s) + O(nr_n^3(\delta_n r_n)^{-\infty}) \\
&\quad + (O(q^{r_n})-1)\,|\{x\in D_n, \rho(x)\leq r_n\}| \int_0^\tau \chi_n(s)^2 d\mu(s).
\end{aligned}$$

[8]We can take for example $r_n = \min\left\{R_n - 2, -(1-\epsilon)\frac{\log \alpha_n}{\log q}\right\}$, for any $0 < \epsilon < 1$.

Thus, we get the desired result if

$$r_n^3(\delta_n r_n)^{-\infty} = o(\int_0^\tau \chi_n(s)d\mu(s)).$$

$\square$

COROLLARY 5.2. *Under the assumptions of Theorem 1, we have*

$$N(I_n, G_n) \sim n \int_{s_0-\delta_n}^{s_0+\delta_n} d\mu(s) \sim 2n\delta_n |c(s_0)|^{-2}$$

where $|c(s_0)|^{-2}$ is the density of the Plancherel measure at s_0.

6. Quantitative statement

In this section we will give explicit upper bounds on the rate of convergence, first in terms of the parameters R_n and α_n associated with the sequence of graphs (G_n) in condition (BST), then depending only on n for sequences of random graphs.

In the general case $s_0 \in (0, \tau)$, we have

LEMMA 6.1. *If $\delta_n = r_n^{-1+\epsilon}$ for some $0 < \epsilon < 1$, then we have*

$$\frac{1}{N(I_n, G_n)} \sum_{s_j^{(n)} \in I_n} \left| \langle \psi_j^{(n)}, a_n \psi_j^{(n)} \rangle \right|^2 = O\left(r_n^{-1/3} \|a\|_2^2 + r_n^{8/3} q^{r_n} \alpha_n \|a\|_\infty^2 \right),$$

where we can take $r_n = \min\left\{ R_n - 2, -(1-\epsilon') \log_q(\alpha_n) \right\}$, for any $0 < \epsilon' < 1$.

PROOF. According to the proof of section 4 (and keeping track of the dependence on a of the remainder terms), we have

$$\frac{1}{N(I_n, G_n)} \sum_{s_j \in I_n} |\langle \psi_j, a_n \psi_j \rangle|^2 \leq r_n^3 O\left(\frac{1}{\delta_n (r_n \delta_n)^\infty} \right) \|a\|_2^2 + O\left(\frac{NT^2}{\beta r_n^2} \right) \|a\|_2^2$$
$$+ O(N^2 T^2 q^{r_n} \alpha_n) \|a\|_\infty^2 + O\left(\frac{T^2}{\beta N} \right) \|a\|_2^2 + O\left(\frac{1}{T\beta} \right) \|a\|_2^2.$$

Take $N = r_n$ and $T = r_n^{1/3}$ such that $\frac{1}{T} = \frac{T^2}{N} = \frac{NT^2}{r_n^2} = r_n^{-1/3}$. For every $M > 0$, we have $O\left(\frac{r_n^3}{\delta_n (r_n \delta_n)^\infty} \right) = O\left(r_n^{3-M} \delta_n^{-(1+M)} \right) = O\left(r_n^{4-(1+M)\epsilon} \right)$ and this term can be made negligible in comparison with the other terms by taking M sufficiently large. Finally $N^2 T^2 q^{r_n} \alpha_n = r_n^{8/3} q^{r_n} \alpha_n$. $\square$

REMARK 6.2. Here we kept the spectral gap β fixed, but we see that the spectral gap β_n could be allowed to decay provided that $\lim_{n\to+\infty} r_n^{-1/3}/\beta_n = 0$.

For sequences of random graphs, we have

LEMMA 6.3. *Let $\delta > 1/2$, $\epsilon > 0$, and $\delta_n = (\log_q(n^{1-\delta}))^{1-\epsilon}$. If G_n is chosen uniformly at random among the $(q+1)$-regular graphs with n vertices, we have*

$$\frac{1}{N(I_n, G_n)} \sum_{s_j^{(n)} \in I_n} \left| \langle \psi_j^{(n)}, a_n \psi_j^{(n)} \rangle \right|^2 = O\left(\log_q(n)^{-1/3} \right),$$

with overwhelming probability.

PROOF. Take R_n and α_n as in example 1. Let $1/2 < \delta < 1$, then $R_n = (1-\delta)\log_q(n)$ and $\alpha_n = 80(1-\delta)\log_q(n)n^{-\delta}$. In this case, we take

$$r_n = \min\left\{R_n - 2, -(1-\epsilon')\frac{\log\alpha_n}{\log q}\right\} = R_n - 2,$$

and apply Lemma 6.1. □

7. Proof of Theorem 2

Without loss of generality, we assume that the operator A_n in Theorem 2 is an operator on $L^2(\mathfrak{X})$ that commutes with the action of Γ_n, so that A_n defines an operator on $L^2(G_n)$. We write $A_n = \mathrm{Op}(a)$ where $a(x,\omega,s) \in S_o^D(\mathfrak{X})$ is invariant under the action of Γ_n on half-geodesics $[x,\omega)$ and note that most steps of the proof carry over immediately to this case. Actually, all that needs modifying is the treatment of the expression

$$\sum_{x\in D_n}\int\left|\frac{1}{N}\sum_{k=0}^{N-1} q^{2isk} a\circ\sigma^k\right|^2 (x,\omega,s)d\nu_x(\omega)d\mu(s) \tag{7.1}$$

that is used in §2.2 (for $s = \frac{\pi}{2\ln q}$ and k even) and Section 4 (for s close to s_0). Equation (7.1) is also

$$\begin{aligned}\frac{1}{N^2}\sum_{x\in D_n}\int\sum_{k=0}^{N-1}\sum_{j=0}^{k} q^{2isj} a\circ\sigma^j(x,\omega,s)a(x,\omega,s)d\nu_x(\omega)d\mu(s)\\ +\frac{1}{N^2}\sum_{x\in D_n}\int\sum_{k=0}^{N-1}\sum_{j=1}^{k} q^{-2isj} a\circ\sigma^j(x,\omega,s)a(x,\omega,s)d\nu_x(\omega)d\mu(s)\end{aligned} \tag{7.2}$$

In §2.2, the integral

$$\sum_{x\in D_n}\int a\circ\sigma^j(x,\omega)a(x,\omega)d\nu_x(\omega)$$

was rewritten as $\sum_{x\in D_n} S_j a(x)a(x)$ using the fact that a did not depend on ω – thus establishing a link between the shift σ and the laplacian. We need to adapt that argument to the case when $a(x,\omega)$ depends on the first D coordinates of the half geodesic $[x,\omega) = (x, x_1, x_2, \ldots)$.

7.1. Proof when $D = 1$. When $D = 1$, $a(x,\omega) = a(x,x_1)$ if $[x,\omega) = (x,x_1,x_2,\ldots)$. Thus a may be seen as a function on the set B of directed bonds of $G = G_n$. Note that B has cardinality $n(q+1)$ if G has n vertices and is $(q+1)$-regular. We use the notation of [**32**] : if e is an element of B, we shall denote by $o(e) \in V_n$ its origin, $t(e) \in V_n$ its terminus, and $\hat{e} \in B$ the reversed bond.

One sees that

$$\sum_{x\in D_n}\int a\circ\sigma^j(x,\omega)a(x,\omega)d\nu_x(\omega) = \frac{1}{q+1}\sum_{e\in B} M^{\sharp j}a(e)a(e)$$

where $M^\sharp$ is a bistochastic matrix indexed by B, defined by

$$M^\sharp(e,e') = \frac{1}{q}$$

if $o(e') = t(e)$ and $e' \neq \hat{e}$; and $M^\sharp(e,e') = 0$ otherwise. This is (up to normalization) the matrix appearing in §3 of [**32**]. It is the (normalized) adjacency matrix of the

q-regular directed graph, whose vertices are the directed bonds of G, and where we draw an edge between two bonds if they are consecutive without allowing back-tracking. What we need is an explicit relation between the spectrum of $M^\sharp$ and the spectrum of the discrete laplacian on G, in other words, of the matrix A. The relation between the eigenvalues is formula (44) in [**32**], but since we also need relations between the eigenfunctions, we shall be more explicit below. We did not write all the detailed calculations because they are lengthy but basic. We assume these relations must already be known but did not find any reference.

(o) Both A and $M^\sharp$ have 1 in their spectrum, corresponding to the constant eigenfunction. The matrix A has -1 in its spectrum iff the graph G is bi-partite, in which case $M^\sharp$ also trivially has -1 in its spectrum.

(i) each eigenvalue $\lambda \neq \pm 1$ of A gives rise to the two eigenvalues

$$\frac{2}{(q+1)\left(\lambda \pm \sqrt{\lambda^2 - \frac{4q}{(q+1)^2}}\right)}$$

of $M^\sharp$;

(ii) in addition, $M^\sharp$ admits the eigenvalue $1/q$ with multiplicity $b := |E_n| - |V_n| + 1$ (the rank of the fundamental group of G); and the eigenvalue $-1/q$ with multiplicity $b-1$ if -1 is not an eigenvalue of A, or b if -1 is an eigenvalue of A.[9]

In particular, the eigenvalue 1 of $M^\sharp$ has multiplicity 1. The tempered spectrum of A corresponds to eigenvalues of $M^\sharp$ of modulus $1/\sqrt{q}$; the untempered spectrum of A contained in $[-1, 1-\beta]$ gives rise to real eigenvalues of $M^\sharp$ contained in $[-1, 1-\beta']$ with

$$1 - \beta' = \frac{2}{(q+1)\left(1-\beta - \sqrt{(1-\beta)^2 - \frac{4q}{(q+1)^2}}\right)}.$$

Since $M^\sharp$ is not normal, the knowledge of its spectrum is not sufficient to control the growth of $M^{\sharp k}$ in a precise manner (we need a bound that is independent of the size of the matrix, in other words, independent of n). Below, we describe explicitly the eigenvectors of $M^\sharp$ in terms of those of A; these eigenvectors do not form an orthogonal family but this is compensated by the fact that one can compute their scalar products explicitly.

(i) an eigenfunction ϕ of A for the eigenvalue $\lambda \neq \pm 1$ gives rise to the two eigenfunctions of $M^\sharp$,

$$f_1(e) = \phi(t(e)) - \epsilon_1 \phi(o(e)); \qquad f_2(e) = \phi(t(e)) - \epsilon_2 \phi(o(e)),$$

where ϵ_1, ϵ_2 are the two roots of $q\epsilon^2 - (q+1)\lambda\epsilon + 1 = 0$ (in what follows we index them so that $|\epsilon_1| \leq |\epsilon_2|$). Special attention has to be paid to the case $\lambda = \pm\frac{2\sqrt{q}}{q+1}$, for which $\epsilon_1 = \epsilon_2$ (see below).

(ii) the eigenvalues $\pm 1/q$ of $M^\sharp$ correspond, respectively, to odd and even[10] solutions of $\sum_{o(e)=x} f(e) = 0$ (for every vertex x). For the eigenvalue $1/q$, an explicit basis of eigenfunctions is indexed by generators of the fundamental group, $(\gamma_1, \ldots, \gamma_b)$: every closed circuit γ made of consecutive

[9] Or, equivalently, if the graph is bi-partite.

[10] Odd means $f(\hat{e}) = -f(e)$ and even means $f(\hat{e}) = f(e)$, for every bond e.

edges $(e_1, \ldots, e_k)$ gives rise to an odd eigenfunction

$$f_\gamma = \sum_{j=1}^{k} \delta_{e_j} - \delta_{\hat{e}_j}.$$

If G is bi-partite, then all circuits have even length and we have an explicit basis of even eigenfunctions for the eigenvalue $-1/q$, again indexed by generators of the fundamental group :

$$g_\gamma = \sum_{j=1}^{k} (-1)^k (\delta_{e_j} + \delta_{\hat{e}_j})$$

if γ is a closed circuit made of consecutive edges $(e_1, \ldots, e_k)$. If G is not bi-partite, there are closed circuits of odd lengths, in which case g_γ is not an eigenfunction of $M^\sharp$. Nevertheless, if γ, γ' are two circuits of odd lengths, $g_\gamma - g_{\gamma'}$ is now an eigenfunction of $M^\sharp$ for the eigenvalue $-1/q$.

The eigenfunctions of the family (ii) are automatically orthogonal to those of the family (i). In (i), eigenfunctions of $M^\sharp$ stemming from different eigenvalues λ of A are orthogonal; however, the two eigenfunctions f_1, f_2 stemming from the same λ are not orthogonal.

To evaluate the norm of a matrix, it is safer to work in an orthogonal basis, and thus we shall consider, instead of a pair (f_1, f_2), the pair

$$f_1(e) = \phi(t(e)) - \epsilon_1 \phi(o(e)), f_2'(e) = \phi(t(e)) - \mu\phi(o(e))$$

which can be checked to be orthogonal for

$$\mu = \frac{\bar{\epsilon}_1 \lambda - 1}{\bar{\epsilon}_1 - \lambda}.$$

In the plane generated by $\left(\frac{f_1}{\|f_1\|}, \frac{f_2'}{\|f_2'\|}\right)$, $M^\sharp$ has matrix

$$\begin{pmatrix} 1/q\epsilon_1 & \star \\ 0 & 1/q\epsilon_2 \end{pmatrix}$$

where $\star$ is a number that can be calculated explicitly in terms of ϵ_1, ϵ_2 and λ, and which is uniformly bounded (since the norm of $M^\sharp$, anyway, is bounded independently of n).

This discussion is also valid for $\lambda = \pm\frac{2\sqrt{q}}{q+1}$, a special case where $\epsilon_1 = \epsilon_2 = \pm\frac{1}{\sqrt{q}}$.

To summarize, the spectrum of $M^\sharp$ is contained in $[-1, 1-\beta'] \cup \{1\}$ (resp. $[-1+\beta', 1-\beta'] \cup \{1\}$) if the spectrum of A is contained in $[-1, 1-\beta] \cup \{1\}$ (resp. $[-1+\beta, 1-\beta] \cup \{1\}$). We can find an orthonormal basis of $L^2(\mathbb{C}^B)$ in which $M^\sharp$ is block diagonal, each diagonal block being an upper triangular matrix of size ≤ 2, and the non-diagonal coefficients are uniformly bounded.

This implies that the operator

$$\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{k} q^{2isj} M^{\sharp j} = \frac{1}{N^2}(q^{2is}M^\sharp - I)^{-2}\left(q^{2is(N+1)}M^{\sharp(N+1)} - q^{2is}M^\sharp - N(q^{2is}M^\sharp - I)\right)$$

has norm $O\left(\frac{1}{N}\right)$ on the orthogonal of the constant function (for any real s if the spectrum of $M^\sharp$ is contained in $[-1+\beta', 1-\beta'] \cup \{1\}$, or for q^{2is} away from -1 if the spectrum of $M^\sharp$ is contained in $[-1, 1-\beta'] \cup \{1\}$). This tells us that (7.2), and hence (7.1), is $O\left(\frac{1}{N}\right)$.

7.2. Case $D \geq 2$. We indicate how to modify the previous discussion when $D \geq 2$. To keep the notation simple, and because the extension to higher D is quite clear, let us take $D = 2$. When $D = 2$, $a(x, \omega) = a(x, x_1, x_2)$ if $[x, \omega) = (x, x_1, x_2, \ldots)$. Thus a may be seen as a function on the set $B(2)$ of pairs of bonds (e_1, e_2) of $G = G_n$ such that $M^\sharp(e_1, e_2) \neq 0$. Note that $B(2)$ has cardinality $|B|q = n(q+1)q$ if G has n vertices and is $(q+1)$-regular.

Now one sees that

$$\sum_{x \in D_n} \int a \circ \sigma^j(x, \omega) a(x, \omega) d\nu_x(\omega) = \frac{1}{(q+1)q} \sum_{(e_1, e_2) \in B(2)} M_{(2)}^{\sharp j} a(e_1, e_2) a(e_1, e_2)$$

where $M_{(2)}^\sharp$ is a bistochastic matrix indexed by $B(2)$, defined by

$$M_{(2)}^\sharp((e_1, e_2), (e_1', e_2')) = \frac{1}{q}$$

if $e_2 = e_1'$ and $M_{(2)}^\sharp((e_1, e_2), (e_1', e_2')) = 0$ otherwise.

It turns out that $L^2(B)$ can be embedded into a $n(q+1)$-dimensional subspace of $L^2(B(2))$, by the map

$$j : \psi \in L^2(B) \mapsto j_\psi$$

where $j_\psi(e_1, e_2) = \psi(e_2)$. It is easily seen that $M_{(2)}^\sharp$, restricted to the image $j(L^2(B))$, coincides with $M^\sharp$.

Notice now that $M_{(2)}^\sharp$ sends $L^2(B(2))$ to $j(L^2(B))$. The conclusion of the previous section thus holds with $M^\sharp$ replaced by $M_{(2)}^\sharp$, and hence the analysis done for $D = 1$ carries over to $D = 2$ (and similarly for higher D).

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