A Random Set and Prototype Theory Interpretation of Intuitionistic Fuzzy Sets

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Abstract. An interpretation of intuitionistic fuzzy sets is proposed based on random set theory and prototype theory. The extension of fuzzy labels are modelled by lower and upper random set neighbourhoods, identifying those element of the universe within an uncertain distance threshold of a set of prototypical elements. These neighbourhoods are then generalised to compound fuzzy descriptions generated as logical combinations of basic fuzzy labels. The single point coverage functions of these lower and upper random sets are then shown to generate lower and upper membership functions satisfying the min-max combination rules of interval fuzzy set theory, the latter being isomophic to intuitionistic fuzzy set theory.

1 Introduction

Intuitionistic fuzzy sets (IFS) were first proposed by Atanassov [1] as a bipolar model of fuzzy sets where membership and non-membership are considered separately. The basis of IFS are two measures τ and ν where, for x an element of the underlying universe and θ a fuzzy description generated recursively from a set of basic fuzzy labels through application of logical connectives $\wedge, \vee$ and $\neg$, $\tau_\theta(x)$ corresponds to the membership degree of x in the extension of θ [1] and $\nu_\theta(x)$ is the non-membership degree of x in the extension of θ. A duality relationship is then defined between τ and ν such that, $\tau_{\neg\theta}(x) = \nu_\theta(x)$ and $\nu_{\neg\theta}(x) = \tau_\theta(x)$. It is also assumed that $\tau_\theta(x) + \nu_\theta(x) \leq 1$. Furthermore, τ and ν are fully truth-functional satisfying the following combination rules for $\wedge$ and $\vee$: For any fuzzy descriptions θ and φ, and element x,

- $\tau_{\theta\wedge\varphi}(x) = \min(\tau_\theta(x), \tau_\varphi(x))$, $\nu_{\theta\wedge\varphi}(x) = \max(\nu_\theta(x), \nu_\varphi(x))$
- $\tau_{\theta\vee\varphi}(x) = \max(\tau_\theta(x), \tau_\varphi(x))$, $\nu_{\theta\vee\varphi}(x) = \min(\nu_\theta(x), \nu_\varphi(x))$

As shown by Atanassov and Gargov [2] and discussed by Dubois etal. [5], there is an isomorphic relationship between IFS and an older notion of interval fuzzy sets independently introduced by Zadeh [21], Grattan-Guiness [7], Jahn [11] and Sambuc [19]. In this framework lower and upper membership degrees are defined,

[1] The extension of θ is the set of elements to which the description θ can be appropriately applied

where $\underline{\mu}_\theta(x)$ is the lower membership degree of element x in the extension of θ, and $\overline{\mu}_\theta(x)$ is the upper membership degree of x in θ. These lower and upper memberships then satisfy the following properties: For any element x and fuzzy descriptions θ and φ

- $\underline{\mu}_\theta(x) \leq \overline{\mu}_\theta(x)$
- $\underline{\mu}_{\neg\theta}(x) = 1 - \overline{\mu}_\theta(x)$, and $\overline{\mu}_{\neg\theta}(x) = 1 - \underline{\mu}_\theta(x)$
- $\underline{\mu}_{\theta\wedge\varphi}(x) = \min(\underline{\mu}_\theta(x), \underline{\mu}_\varphi(x))$, $\overline{\mu}_{\theta\wedge\varphi}(x) = \min(\overline{\mu}_\theta(x), \overline{\mu}_\varphi(x))$
- $\underline{\mu}_{\theta\vee\varphi}(x) = \max(\underline{\mu}_\theta(x), \underline{\mu}_\varphi(x))$, $\overline{\mu}_{\theta\vee\varphi}(x) = \max(\overline{\mu}_\theta(x), \overline{\mu}_\varphi(x))$

The mapping between interval fuzzy sets and IFS is then obtained by taking $\underline{\mu}_\theta(x) = \tau_\theta(x)$ and $\overline{\mu}_\theta(x) = 1 - \nu_\theta(x)$. In fact it is for this interval valued fuzzy set theory for which we shall propose a direct interpretation based on random set theory and prototype theory.

Prototype theory has been proposed by Rosch [16] [17] as an alternative model of concepts in natural language. The fundamental idea is that concepts, instead of being defined by formal rules or mappings, are represented by a set of prototypical cases. These cases correspond to those elements of the underlying universe Ω, which it is certain satisfy the concept. Categorization of elements from Ω is then based on similarity to the prototypes as quantified by a distance metric defined on Ω (see [10] for an overview). By taking typicallity to be a decreasing function of distance from prototypes, this approach would naturally explain the fact that some instances are seen as being more typical exemplars of a concept than others. For instance, robins are viewed as being a more typical example of the concept *bird* than penguins, since the latter have certain atypical characteristics such as the inability to fly. This notion of typicality is also strongly related to concept vagueness where borderline cases have an intermediate range of typicality values. In other words, such cases are not sufficiently similar to the prototypes to be judged as having certain membership in the category but are also not sufficiently dissimilar to the prototypes to be ruled out as being certainly outside the category.

Random set theory has been proposed by Goodman and Nguyen as a framework for linguistic reasoning in rule based systems [14], [8], [9]. Stated simply, random sets are set-valued variables with an associated probability measure. In Goodman and Nguyen's work they provide a model of vague concepts from the perspective that the extension of such a concept is an uncertain set. This is an implicitly epistemic model of vagueness since by using a random set to model a concept an intelligent agent is assuming that there is a correct extension set about which they are uncertain. Notice that this does not require that there is actually some objectively correct definition of the concept (as suggested by Williamson [20]), but rather that the agent *assumes*, for the purposes of decision making and communication, that such a definition exists (see Lawry [13] for a discussion of this *epistemic stance*).

Dubois et al. [6] have identified both random sets and prototype theory as possible interpretations of fuzzy set membership functions. More specifically, given a random set $\mathcal{R}$ modelling a concept, the fuzzy set membership value of an element x in $\mathcal{R}$ is then taken to be the probability that the value of $\mathcal{R}$ is a set which contains x. This is the single point coverage function of the random set $\mathcal{R}$. For the prototype theory model it is assumed that there exists a similarity measure between the elements of Ω, which takes values in $[0, 1]$. Given a set of prototypical elements, the membership of x in the associated fuzzy set is then taken as corresponding to the similarity between x and these prototypes [18]. In [12] we have proposed a natural combination of prototype theory and random set theory to model linguistic labels and descriptions. The idea behind this approach is that, in order to decide whether the assertion 'x is L_i' is appropriate for element x and label L_i with prototypes P_i, an agent would threshold the distance between x and P_i. In other words, L_i would be deemed an appropriate label for x provided that $d(x, P_i) \leq \epsilon$, for some distance function $d : \Omega^2 \to [0, \infty)$ and threshold $\epsilon \geq 0$. However, the inherent uncertainty about the extension of L_i would naturally result in uncertainty about the value of threshold ϵ. Consequently, the extension of L_i would correspond to a random set neighbourhood of the prototypes of L_i as defined by those elements which lie within the uncertain threshold ϵ of P_i. In the sequel we extend this idea, so as to generate lower and upper neighbourhoods as extensions of a concept by introducing lower and upper thresholds.

2 Lower and Upper Membership Functions

We envisage a population of communicating agents applying a finite set of labels to describe the elements of an underlying universe of discourse. Given an element x an agent must decide which labels and compound descriptions are *appropriate* to describe x, where appropriateness is governed by the linguistic conventions of the population. Agents adopt the *epistemic stance* [13] by assuming that there is an uncertain but crisp division between those labels which are, and those which are not appropriate to describe a given element. Now since an agent's knowledge of these linguistic conventions, obtained through their experience of communication with others, is partial and often conflicting they will have significant uncertainty about the appropriateness of labels. It is assumed, however, that there will be prototypical elements for which they will be certain that a given label can describe. These prototypes will then form the basis of the agent's representation of each label. More formally:

Let Ω denote the underlying universe of discourse and $LA = \{L_1, \ldots, L_n\}$ be a finite set of labels for describing elements of Ω. LE then corresponds to the set of compound express generated by recursive application of the connectives $\wedge$, $\vee$ and $\neg$ to the labels in LA. For example, if LA contains labels *red* and *blue*, then LE contains expressions including *red and blue*, *red or blue*, *not red*, *not blue*, *red and not blue* etc. For each label L_i there is a set of prototypical elements $P_i \subseteq \Omega$, such that L_i is *certainly appropriate* to describe any prototypical elements in P_i. Given a distance function $d : \Omega^2 \to [0, \infty)$ satisfying $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in \Omega$, lower and upper extensions of each label are defined to be those elements of Ω with distance from P_i less than or equal to a lower and an upper threshold value respectively. In other words, the lower extension of L_i is taken to be $\{x \in \Omega : d(x, P_i) \leq \underline{\epsilon}\}$ and the upper extension $\{x \in \Omega : d(x, P_i) \leq \overline{\epsilon}\}$, where $\underline{\epsilon} \leq \overline{\epsilon}$ and $d(x, P_i) = \inf\{d(x, y) : y \in P_i\}$. Here, we further assume that both $\underline{\epsilon}$ and $\overline{\epsilon}$ are functions of a single parameter α taking values in $[0, 1]$. The underlying intuition is that α quantifies an agent's overall level of imprecision in their definition of labels, so that as α increases the difference between the upper extension of a label and its lower extension decreases. In effect this means that there exists an increasing function $\underline{f} : [0, 1] \to [0, \infty)$ and a decreasing function $\overline{f} : [0, 1] \to [0, \infty)$ such that $\underline{f} \leq \overline{f}$ and for which $\underline{\epsilon} = \underline{f}(\alpha)$ and $\overline{\epsilon} = \overline{f}(\alpha)$.

Definition 1. *Lower and Upper Threshold Functions*
$\underline{f} : [0, 1] \to [0, \infty)$ *and* $\overline{f} : [0, 1] \to [0, \infty)$ *where* $\underline{f}$ *is an increasing function and* $\overline{f}$ *is a decreasing function satisfying* $\forall \alpha \in [0, 1]$ $\underline{f}(\alpha) \leq \overline{f}(\alpha)$.

The lower and upper extension of the labels and the compound descriptions in LE are then defined recursively as follows:

Definition 2. *Lower and Upper Random Neighbourhoods*

- $\forall L_i \in LA$ $\underline{\mathcal{N}}^{\alpha}_{L_i} = \{x : d(x, P_i) \leq \underline{f}(\alpha)\}$, $\overline{\mathcal{N}}^{\alpha}_{L_i} = \{x : d(x, P_i) \leq \overline{f}(\alpha)\}$.
- $\forall \theta, \varphi \in LE$ $\underline{\mathcal{N}}^{\alpha}_{\theta \wedge \varphi} = \underline{\mathcal{N}}^{\alpha}_{\theta} \cap \underline{\mathcal{N}}^{\alpha}_{\varphi}$, $\overline{\mathcal{N}}^{\alpha}_{\theta \wedge \varphi} = \overline{\mathcal{N}}^{\alpha}_{\theta} \cap \overline{\mathcal{N}}^{\alpha}_{\varphi}$.
- $\forall \theta, \varphi \in LE$ $\underline{\mathcal{N}}^{\alpha}_{\theta \vee \varphi} = \underline{\mathcal{N}}^{\alpha}_{\theta} \cup \underline{\mathcal{N}}^{\alpha}_{\varphi}$, $\overline{\mathcal{N}}^{\alpha}_{\theta \vee \varphi} = \overline{\mathcal{N}}^{\alpha}_{\theta} \cup \overline{\mathcal{N}}^{\alpha}_{\varphi}$.
- $\forall \theta \in LE$ $\underline{\mathcal{N}}^{\alpha}_{\neg\theta} = (\overline{\mathcal{N}}^{\alpha}_{\theta})^c$, $\overline{\mathcal{N}}^{\alpha}_{\neg\theta} = (\underline{\mathcal{N}}^{\alpha}_{\theta})^c$

Now in view of the distributed manner in which language is learnt through the interaction and communications between a population of agents, it is likely that an individual agent will be uncertain as to which value of α should be adopted in a given context. Here, in keeping with the epistemic stance, we model this uncertainty by a probability density function δ on α. The lower and upper membership functions of expression $\theta \in LE$ for element $x \in \Omega$ are then given by the probability of a value of α such that $x \in \underline{\mathcal{N}}^{\alpha}_{\theta}$ and the probability of an α such that $x \in \overline{\mathcal{N}}^{\alpha}_{\theta}$ respectively.

Definition 3. *Lower and Upper Membership Functions*
Let δ *be a density function on* $[0, 1]$. *Then* $\forall \theta \in LE$, $\forall x \in \Omega$ *we define* $\underline{\mu}_{\theta}(x) = \delta(\{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\theta}\})$ *and* $\overline{\mu}_{\theta}(x) = \delta(\{\alpha : x \in \overline{\mathcal{N}}^{\alpha}_{\theta}\})$

Here $\underline{\mu}_{\theta}(x)$ quantifies the agent's belief that expression θ is *definitely appropriate* to describe x, and $\overline{\mu}_{\theta}(x)$ is the belief that θ is *possibly appropriate* to describe x. These lower and upper measures attempt to capture the intuition that 'appropriateness' or 'assertability' of descriptions is inherently bipolar. This bipolarity manifests itself in the distinction between those descriptions which convention would deem clearly appropriate to describe an element x, and those which convention would not classify as incorrect, or perhaps even dishonest, descriptions. Parikh [15] observes that:

> Certain sentences are assertible in the sense that we might ourselves assert them and other cases of sentences which are non-assertible in the sense that we ourselves (and many others) would reproach someone who used them. But there will also be the intermediate kind of sentences, where we might allow their use.

For example, consider a witness in a court of law describing a suspect as being *tall*. Depending on the actual height of the suspect this statement may be deemed as clearly true or clearly false, in which latter case the witness could be accused of perjury. However, there will also be an intermediate height range for which, while there may be doubt and differing opinions concerning the use of the description *tall*, it would not be deemed as definitely inappropriate and hence the witness would not be viewed as committing perjury.

We now investigate some basic properties of lower and upper neighbourhoods and argue that $\underline{\mu}_\theta$ and $\overline{\mu}_\theta$ can indeed be viewed as lower and upper membership functions according to the random set interpretation of fuzzy sets. The following theorem shows that the lower neighbourhood is, as intended, a subset of the upper neighbourhood for any expression in LE.

Theorem 1. $\forall \Psi \in LE$, $\forall \alpha \in [0,1]$ $\underline{\mathcal{N}}^\alpha_\Psi \subseteq \overline{\mathcal{N}}^\alpha_\Psi$

Proof. Let $LE^{(1)} = LA$ and $LE^{(k)} = LE^{(k-1)} \cup \{\theta \wedge \varphi, \theta \vee \varphi, \neg\theta : \theta, \varphi \in LE^{(k-1)}\}$. We now proceed by induction on k. If $\Psi = L_i$ then $\underline{\mathcal{N}}^\alpha_{L_i} = \{x : d(x,P_i) \leq \underline{f}(\alpha)\} \subseteq \{x : d(x,P_i) \leq \overline{f}(\alpha\} = \overline{\mathcal{N}}^\alpha_{L_i}$. Now assuming the result holds for $\Psi \in LE^{(k)}$ we show that it holds for $\Psi \in LE^{(k+1)}$. If $\Psi \in LE^{(k+1)}$ then either $\Psi \in LE^{(k)}$, in which case the result holds trivially, or $\exists \theta, \varphi \in LE^{(k)}$ for which one of the following holds:

- $\Psi = \theta \wedge \varphi$ in which case $\underline{\mathcal{N}}^\alpha_\Psi = \underline{\mathcal{N}}^\alpha_{\theta\wedge\varphi} = \underline{\mathcal{N}}^\alpha_\theta \cap \underline{\mathcal{N}}^\alpha_\varphi \subseteq \overline{\mathcal{N}}^\alpha_\theta \cap \overline{\mathcal{N}}^\alpha_\varphi$ (by the inductive step) $= \overline{\mathcal{N}}^\alpha_{\theta\wedge\varphi} = \overline{\mathcal{N}}^\alpha_\Psi$.
- $\Psi = \theta \vee \varphi$ in which case $\underline{\mathcal{N}}^\alpha_\Psi = \underline{\mathcal{N}}^\alpha_{\theta\vee\varphi} = \underline{\mathcal{N}}^\alpha_\theta \cup \underline{\mathcal{N}}^\alpha_\varphi \subseteq \overline{\mathcal{N}}^\alpha_\theta \cup \overline{\mathcal{N}}^\alpha_\varphi$ (by the inductive step) $= \overline{\mathcal{N}}^\alpha_{\theta\vee\varphi} = \overline{\mathcal{N}}^\alpha_\Psi$.
- $\Psi = \neg\theta$. Now by induction $\overline{\mathcal{N}}^\alpha_\theta \supseteq \underline{\mathcal{N}}^\alpha_\theta$ and therefore $(\overline{\mathcal{N}}^\alpha_\theta)^c \subseteq (\underline{\mathcal{N}}^\alpha_\theta)^c$. Hence, in this case $\underline{\mathcal{N}}^\alpha_\Psi = \underline{\mathcal{N}}^\alpha_{\neg\theta} = (\overline{\mathcal{N}}^\alpha_\theta)^c \subseteq (\underline{\mathcal{N}}^\alpha_\theta)^c = \overline{\mathcal{N}}^\alpha_{\neg\theta} = \overline{\mathcal{N}}^\alpha_\Psi$.

Corollary 1. $\forall \theta \in LE$, $\forall x \in \Omega$ $\underline{\mu}_\theta(x) \leq \overline{\mu}_\theta(x)$

For any expression θ, $\underline{\mathcal{N}}^\alpha_\theta$ and $\overline{\mathcal{N}}^\alpha_\theta$ are both random sets taking as values subsets of Ω. From this perspective $\underline{\mu}_\theta$ and $\overline{\mu}_\theta$ are the single point coverage functions of $\underline{\mathcal{N}}^\alpha_\theta$ and $\overline{\mathcal{N}}^\alpha_\theta$ respectively. Hence, according to the random set interpretation of fuzzy sets proposed in [8], [9] and [6], $\underline{\mu}_\theta(x)$ and $\overline{\mu}_\theta(x)$ can be viewed as membership values of x in the lower and upper extension of θ respectively.

Theorem 2. $\forall \alpha, \alpha' \in [0,1]$ *where* $\alpha \leq \alpha'$ *it holds that* $\forall \theta \in LE$ $\underline{\mathcal{N}}^\alpha_\theta \subseteq \underline{\mathcal{N}}^{\alpha'}_\theta$ *and* $\overline{\mathcal{N}}^\alpha_\theta \supseteq \overline{\mathcal{N}}^{\alpha'}_\theta$.

Proof. Let $LE^{(1)} = LA$ and $LE^{(k)} = LE^{(k-1)} \cup \{\theta \wedge \varphi, \theta \vee \varphi, \neg\theta : \theta, \varphi \in LE^{(k-1)}\}$. We now proceed by induction on k. If $\Psi = L_i$ then $\underline{\mathcal{N}}^{\alpha}_{L_i} = \{x : d(x, P_i) \leq \underline{f}(\alpha)\} \subseteq \{x : d(x, P_i) \leq \underline{f}(\alpha')\} = \underline{\mathcal{N}}^{\alpha'}_{L_i}$ since $\underline{f}$ is an increasing function. Also $\overline{\mathcal{N}}^{\alpha}_{L_i} = \{x : d(x, P_i) \leq \overline{f}(\alpha)\} \supseteq \{x : d(x, P_i) \leq \overline{f}(\alpha')\} = \overline{\mathcal{N}}^{\alpha'}_{L_i}$ since $\overline{f}$ is a decreasing function. Now assuming the result holds for $\Psi \in LE^{(k)}$ we show that it holds for $\Psi \in LE^{(k+1)}$. If $\Psi \in LE^{(k+1)}$ then either $\Psi \in LE^{(k)}$, in which case the result holds trivially, or $\exists \theta, \varphi \in LE^{(k)}$ for which one of the following holds:

– $\Psi = \theta \wedge \varphi$. In this case $\underline{\mathcal{N}}^{\alpha}_{\Psi} = \underline{\mathcal{N}}^{\alpha}_{\theta\wedge\varphi} = \underline{\mathcal{N}}^{\alpha}_{\theta} \cap \underline{\mathcal{N}}^{\alpha}_{\varphi} \subseteq \underline{\mathcal{N}}^{\alpha'}_{\theta} \cap \underline{\mathcal{N}}^{\alpha'}_{\varphi}$ (by induction) $= \underline{\mathcal{N}}^{\alpha'}_{\theta\wedge\varphi} = \underline{\mathcal{N}}^{\alpha'}_{\Psi}$. Also $\overline{\mathcal{N}}^{\alpha}_{\Psi} = \overline{\mathcal{N}}^{\alpha}_{\theta\wedge\varphi} = \overline{\mathcal{N}}^{\alpha}_{\theta} \cap \overline{\mathcal{N}}^{\alpha}_{\varphi} \supseteq \overline{\mathcal{N}}^{\alpha'}_{\theta} \cap \overline{\mathcal{N}}^{\alpha'}_{\varphi}$ (by induction) $= \overline{\mathcal{N}}^{\alpha'}_{\theta\wedge\varphi} = \overline{\mathcal{N}}^{\alpha'}_{\Psi}$.
– $\Psi = \theta \vee \varphi$. In this case $\underline{\mathcal{N}}^{\alpha}_{\Psi} = \underline{\mathcal{N}}^{\alpha}_{\theta\vee\varphi} = \underline{\mathcal{N}}^{\alpha}_{\theta} \cup \underline{\mathcal{N}}^{\alpha}_{\varphi} \subseteq \underline{\mathcal{N}}^{\alpha'}_{\theta} \cup \underline{\mathcal{N}}^{\alpha'}_{\varphi}$ (by induction) $= \underline{\mathcal{N}}^{\alpha'}_{\theta\vee\varphi} = \underline{\mathcal{N}}^{\alpha'}_{\Psi}$. Also $\overline{\mathcal{N}}^{\alpha}_{\Psi} = \overline{\mathcal{N}}^{\alpha}_{\theta\vee\varphi} = \overline{\mathcal{N}}^{\alpha}_{\theta} \cup \overline{\mathcal{N}}^{\alpha}_{\varphi} \supseteq \overline{\mathcal{N}}^{\alpha'}_{\theta} \cup \overline{\mathcal{N}}^{\alpha'}_{\varphi}$ (by induction) $= \overline{\mathcal{N}}^{\alpha'}_{\theta\vee\varphi} = \overline{\mathcal{N}}^{\alpha'}_{\Psi}$.
– $\Psi = \neg\theta$. In this case $\underline{\mathcal{N}}^{\alpha}_{\Psi} = \underline{\mathcal{N}}^{\alpha}_{\neg\theta} = (\overline{\mathcal{N}}^{\alpha}_{\theta})^c \subseteq (\overline{\mathcal{N}}^{\alpha'}_{\theta})^c$ (by induction) $= \underline{\mathcal{N}}^{\alpha'}_{\neg\theta} = \underline{\mathcal{N}}^{\alpha'}_{\Psi}$. Also $\overline{\mathcal{N}}^{\alpha}_{\Psi} = \overline{\mathcal{N}}^{\alpha}_{\neg\theta} = (\underline{\mathcal{N}}^{\alpha}_{\theta})^c \supseteq (\underline{\mathcal{N}}^{\alpha'}_{\theta})^c$ (by induction) $= \overline{\mathcal{N}}^{\alpha'}_{\neg\theta} = \overline{\mathcal{N}}^{\alpha'}_{\Psi}$.

Corollary 2. $\forall \theta, \varphi \in LE$, $\forall x \in \Omega$

– $\underline{\mu}_{\theta\wedge\varphi}(x) = \min(\underline{\mu}_{\theta}(x), \underline{\mu}_{\varphi}(x))$, $\overline{\mu}_{\theta\wedge\varphi}(x) = \min(\overline{\mu}_{\theta}(x), \overline{\mu}_{\varphi}(x))$
– $\underline{\mu}_{\theta\vee\varphi}(x) = \max(\underline{\mu}_{\theta}(x), \underline{\mu}_{\varphi}(x))$, $\overline{\mu}_{\theta\vee\varphi}(x) = \max(\overline{\mu}_{\theta}(x), \overline{\mu}_{\varphi}(x))$
– $\underline{\mu}_{\neg\theta}(x) = 1 - \overline{\mu}_{\theta}(x)$, $\overline{\mu}_{\neg\theta}(x) = 1 - \underline{\mu}_{\theta}(x)$

Proof. From theorem 2 we have that $\forall \theta, \varphi \in LE$ either $\{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\theta}\} \subseteq \{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\varphi}\}$ or $\{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\theta}\} \supseteq \{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\varphi}\}$ and either $\{\alpha : x \in \overline{\mathcal{N}}^{\alpha}_{\theta}\} \subseteq \{\alpha : x \in \overline{\mathcal{N}}^{\alpha}_{\varphi}\}$ or $\{\alpha : x \in \overline{\mathcal{N}}^{\alpha}_{\theta}\} \supseteq \{\alpha : x \in \overline{\mathcal{N}}^{\alpha}_{\varphi}\}$. Now assume w.l.o.g that $\{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\theta}\} \subseteq \{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\varphi}\}$ then:

$$\underline{\mu}_{\theta\wedge\varphi}(x) = \delta(\{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\theta\wedge\varphi}\}) = \delta(\{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\theta}\} \cap \{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\varphi}\})$$
$$= \delta(\{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\theta}\}) = \underline{\mu}_{\theta}(x) = \min(\underline{\mu}_{\theta}(x), \underline{\mu}_{\varphi}(x)) \text{ and}$$
$$\underline{\mu}_{\theta\vee\varphi}(x) = \delta(\{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\theta\vee\varphi}\}) = \delta(\{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\theta}\} \cup \{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\varphi}\})$$
$$= \delta(\{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\varphi}\}) = \underline{\mu}_{\varphi}(x) = \max(\underline{\mu}_{\theta}(x), \underline{\mu}_{\varphi}(x))$$

The result also follows similarly for $\overline{\mu}_{\theta\wedge\varphi}(x)$ and $\overline{\mu}_{\theta\vee\varphi}(x)$. Furthermore,

$$\underline{\mu}_{\neg\theta}(x) = \delta(\{\alpha : x \in \underline{\mathcal{N}}^{\alpha}_{\neg\theta}\}) = \delta(\{\alpha : x \in (\overline{\mathcal{N}}^{\alpha}_{\theta})^c\}) = \delta(\{\alpha : x \in \overline{\mathcal{N}}^{\alpha}_{\theta}\}^c)$$
$$= 1 - \delta(\{\alpha : x \in \overline{\mathcal{N}}^{\alpha}_{\theta}\}) = 1 - \overline{\mu}_{\theta}(x)$$

Similarly $\overline{\mu}_{\neg\theta}(x) = 1 - \underline{\mu}_{\theta}(x)$

Also notice from theorem 2 that $\forall \alpha \leq \alpha'$ and $\forall \theta \in LE$, $\overline{\mathcal{N}}^{\alpha'}_{\theta} - \underline{\mathcal{N}}^{\alpha'}_{\theta} \subseteq \overline{\mathcal{N}}^{\alpha}_{\theta} - \underline{\mathcal{N}}^{\alpha}_{\theta}$, and hence, in accordance with the original intuition, the parameter α is a direct indicator of the imprecision associated with the definition of θ.

Theorem 3. *$\forall\theta,\varphi \in LE$, $\forall\alpha\in[0,1]$ the following hold:*

- $\underline{\mathcal{N}}^{\alpha}_{\neg(\neg\theta)} = \underline{\mathcal{N}}^{\alpha}_{\theta}$, *and* $\overline{\mathcal{N}}^{\alpha}_{\neg(\neg\theta)} = \overline{\mathcal{N}}^{\alpha}_{\theta}$
- $\underline{\mathcal{N}}^{\alpha}_{\neg(\theta\wedge\varphi)} = \underline{\mathcal{N}}^{\alpha}_{\neg\theta\vee\neg\varphi}$ *and* $\overline{\mathcal{N}}^{\alpha}_{\neg(\theta\wedge\varphi)} = \overline{\mathcal{N}}^{\alpha}_{\neg\theta\vee\neg\varphi}$
- $\underline{\mathcal{N}}^{\alpha}_{\neg(\theta\vee\varphi)} = \underline{\mathcal{N}}^{\alpha}_{\neg\theta\wedge\neg\varphi}$ *and* $\overline{\mathcal{N}}^{\alpha}_{\neg(\theta\vee\varphi)} = \overline{\mathcal{N}}^{\alpha}_{\neg\theta\wedge\neg\varphi}$
- $\underline{\mathcal{N}}^{\alpha}_{\theta\wedge\neg\theta} = \emptyset$ *and* $\overline{\mathcal{N}}^{\alpha}_{\theta\vee\neg\theta} = \Omega$

Proof.
- $\underline{\mathcal{N}}^{\alpha}_{\neg(\neg\theta)} = (\overline{\mathcal{N}}^{\alpha}_{\neg\theta})^c = ((\underline{\mathcal{N}}^{\alpha}_{\theta})^c)^c = \underline{\mathcal{N}}^{\alpha}_{\theta}$ and similarly $\overline{\mathcal{N}}^{\alpha}_{\neg(\neg\theta)} = (\underline{\mathcal{N}}^{\alpha}_{\neg\theta})^c = ((\overline{\mathcal{N}}^{\alpha}_{\theta})^c)^c = \overline{\mathcal{N}}^{\alpha}_{\theta}$
- $\underline{\mathcal{N}}^{\alpha}_{\neg(\theta\wedge\varphi)} = (\overline{\mathcal{N}}^{\alpha}_{\theta\wedge\varphi})^c = (\overline{\mathcal{N}}^{\alpha}_{\theta} \cap \overline{\mathcal{N}}^{\alpha}_{\varphi})^c = (\overline{\mathcal{N}}^{\alpha}_{\theta})^c \cup (\overline{\mathcal{N}}^{\alpha}_{\varphi})^c = \underline{\mathcal{N}}^{\alpha}_{\neg\theta} \cup \underline{\mathcal{N}}^{\alpha}_{\neg\varphi} = \underline{\mathcal{N}}^{\alpha}_{\neg\theta\vee\neg\varphi}$ and similarly $\overline{\mathcal{N}}^{\alpha}_{\neg(\theta\wedge\varphi)} = (\underline{\mathcal{N}}^{\alpha}_{\theta\wedge\varphi})^c = (\underline{\mathcal{N}}^{\alpha}_{\theta} \cap \underline{\mathcal{N}}^{\alpha}_{\varphi})^c = (\underline{\mathcal{N}}^{\alpha}_{\theta})^c \cup (\underline{\mathcal{N}}^{\alpha}_{\varphi})^c = \overline{\mathcal{N}}^{\alpha}_{\neg\theta} \cup \overline{\mathcal{N}}^{\alpha}_{\neg\varphi} = \overline{\mathcal{N}}^{\alpha}_{\neg\theta\vee\neg\varphi}$
- $\underline{\mathcal{N}}^{\alpha}_{\neg(\theta\vee\varphi)} = (\overline{\mathcal{N}}^{\alpha}_{\theta\vee\varphi})^c = (\overline{\mathcal{N}}^{\alpha}_{\theta} \cup \overline{\mathcal{N}}^{\alpha}_{\varphi})^c = (\overline{\mathcal{N}}^{\alpha}_{\theta})^c \cap (\overline{\mathcal{N}}^{\alpha}_{\varphi})^c = \underline{\mathcal{N}}^{\alpha}_{\neg\theta} \cap \underline{\mathcal{N}}^{\alpha}_{\neg\varphi} = \underline{\mathcal{N}}^{\alpha}_{\neg\theta\wedge\neg\varphi}$ and similarly $\overline{\mathcal{N}}^{\alpha}_{\neg(\theta\vee\varphi)} = (\underline{\mathcal{N}}^{\alpha}_{\theta\vee\varphi})^c = (\underline{\mathcal{N}}^{\alpha}_{\theta} \cup \underline{\mathcal{N}}^{\alpha}_{\varphi})^c = (\underline{\mathcal{N}}^{\alpha}_{\theta})^c \cap (\underline{\mathcal{N}}^{\alpha}_{\varphi})^c = \overline{\mathcal{N}}^{\alpha}_{\neg\theta} \cap \overline{\mathcal{N}}^{\alpha}_{\neg\varphi} = \overline{\mathcal{N}}^{\alpha}_{\neg\theta\wedge\neg\varphi}$
- $\underline{\mathcal{N}}^{\alpha}_{\theta\wedge\neg\theta} = \underline{\mathcal{N}}^{\alpha}_{\theta} \cap \underline{\mathcal{N}}^{\alpha}_{\neg\theta} = \underline{\mathcal{N}}^{\alpha}_{\theta} \cap (\overline{\mathcal{N}}^{\alpha}_{\theta})^c \subseteq \underline{\mathcal{N}}^{\alpha}_{\theta} \cap (\underline{\mathcal{N}}^{\alpha}_{\theta})^c = \emptyset$ by theorem 1.
- $\overline{\mathcal{N}}^{\alpha}_{\theta\vee\neg\theta} = \overline{\mathcal{N}}^{\alpha}_{\theta} \cup \overline{\mathcal{N}}^{\alpha}_{\neg\theta} \supseteq \underline{\mathcal{N}}^{\alpha}_{\theta} \cup \overline{\mathcal{N}}^{\alpha}_{\neg\theta} = \Omega$ by theorem 1

Example 1. Let $\Omega = \mathbb{R}$ and L_i be a label with prototype $P_i = \{10\}$ (i.e. L_i denotes *about 10*). Let $\underline{f}(\alpha) = 2\alpha$ and $\overline{f}(\alpha) = 4 - 2\alpha$ (see figure 1) and also let δ be a gaussian distribution with mean 0.5 and standard deviation 0.15 normalised so as to have integral 1 on $[0, 1]$ (see figure 2). From this we have the following lower and upper neighbourhoods:

$\underline{\mathcal{N}}^{\alpha}_{L_i} = [10 - 2\alpha, 10 + 2\alpha]$ and $\overline{\mathcal{N}}^{\alpha}_{L_i} = [6 + 2\alpha, 14 - 2\alpha]$

Hence, the lower and upper membership functions are given by (see figure 3):

$$\underline{\mu}_{L_i}(x) = \begin{cases} \int_{\frac{10-x}{2}}^{1} \delta(\epsilon)d\epsilon : 8 \le x \le 10 \\ \int_{\frac{x-10}{2}}^{1} \delta(\epsilon)d\epsilon : 10 < x \le 12 \\ 0 : \text{otherwise} \end{cases} \qquad \overline{\mu}_{L_i}(x) = \begin{cases} \int_{0}^{\frac{x-6}{2}} \delta(\epsilon)d\epsilon : 6 \le x \le 10 \\ \int_{0}^{\frac{14-x}{2}} \delta(\epsilon)d\epsilon : 10 < x \le 14 \\ 0 : \text{otherwise} \end{cases}$$

3 Discussion and Conclusions

We have introduced a random set and prototype theory interpretation of lower and upper fuzzy membership functions. In particular, we have proposed lower and upper random set extensions of fuzzy descriptions generated as recursive combinations of random set neighbourhoods of prototypes defined for a set of basic fuzzy labels. Each such extension then identifies those elements of Ω which can be *appropriately described* by the associated fuzzy description. Random sets are defined based on lower and upper threshold distances from prototypes which are taken to be functions of a single parameter α indicating the overall level of imprecision associated with concept definition. Uncertainty associated with

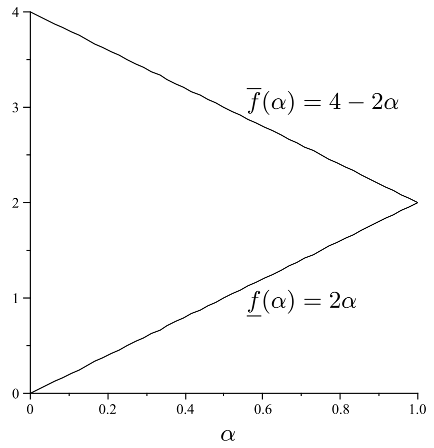

Fig. 1. Example of lower and upper threshold functions

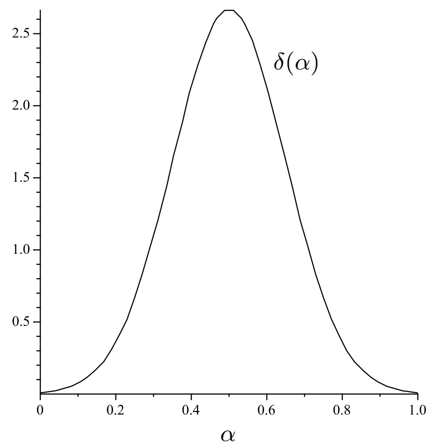

Fig. 2. Normalised gaussian density function δ with mean 0.5 and standard deviation $0,15$

the correct level of α is modelled by a probability density function δ, according to which we can calculate the lower and upper membership functions of x in θ, as the probabilities of those α values for which x is in the lower and upper extensions of θ respectively. In effect these two measures are the single point coverage functions of the lower and upper random sets, and hence according to the random set interpretation of membership functions, can be viewed as lower and upper memberships functions of the extension of the fuzzy concept.

Based on this definition we have then shown that the lower and upper membership functions are fully truth-functional satisfying the min and max rules for conjunction and disjunction as proposed for interval fuzzy sets. However, the interpretation also imposes other additional constraint on the calculus for lower

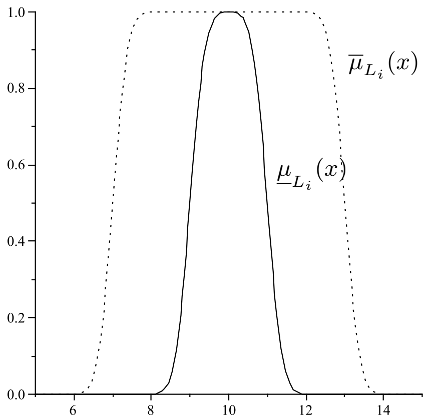

Fig. 3. Example of lower and upper membership functions for label L_i with prototype $P_i = \{10\}$

and upper membership functions. For instance, from theorem 3 it follows immediately that $\forall \theta \in LE$ and $\forall x \in \Omega$, $\underline{\mu}_{\theta \wedge \neg\theta}(x) = 0$ and $\overline{\mu}_{\theta \vee \neg\theta}(x) = 1$. Hence, by corollary 2, it holds that $\min(\underline{\mu}_{\theta}(x), 1 - \overline{\mu}_{\theta}(x)) = 0$. In other words, for any element x and expression θ, either $\underline{\mu}_{\theta}(x) = 0$ or $\overline{\mu}_{\theta}(x) = 1$. Clearly then by applying the mapping $\tau_\theta(x) = \underline{\mu}_{\theta}(x)$ and $\nu_\theta(x) = 1 - \overline{\mu}_{\theta}(x) = \underline{\mu}_{\neg\theta}(x)$ we obtain a calculus for membership and non-membership degree identical to that proposed for IFS by Atanassov [1], but with the additional constraint that either $\tau_\theta(x) = 0$ or $\nu_\theta(x) = 0$ for any x and θ.

Dubois etal. [5] question the interpretation of IFS as an intuitionistic theory. For example, unlike intuitionistic logic, IFS satisfies double negation while it does not satisfy the law of non-contradiction [2]. This criticism would seem to be borne out under the current interpretation, since the fundamental notion underlying the measures $\underline{\mu}$ and $\overline{\mu}$ is that of random set neighbourhoods, which do not seem to be at all intuitionistic in nature. For instance, from theorem 3 we see that the double negation law is strongly validated since it holds for both lower and upper neighbourhoods. Also theorem 3 shows that the behaviour of lower and upper neighbourhoods with regard to the laws of excluded middle and non-contradiction differs significantly from intuitionistic logic. Indeed, while the lower neighbourhood does not satisfy excluded middle, the upper neighbourhood does. Similarly, while the lower neighbourhood satisfies the law of non-contradiction, the upper neighbourhood does not. Indeed the behaviour of lower and upper random set neighbourhoods with regard to these laws is exactly what would be expected from two criteria, one weaker and one stronger, related in a bipolar manner as outlined in [4] , rather than being based on the notion of justifiability as is the case in intuitionistic logic. In particular, lower and upper membership functions would seem to be a special case of what Dubois and Prade [4] refer

[2] We refer here to the standard min, max calculus for IFS where $\tau_{\neg\theta} = \nu_\theta$ and $\nu_{\neg\theta} = \tau_\theta$. Atanassov [3] shows that for other choices of negation operator, the law of non-contradiction may be satisfied.

to as *symmetric bivariate unipolarity*, whereby judgments are made according to two distinct evaluations on unipolar scales. In the current context, we have a strong and a weak evaluation criterion where the former corresponds to definite appropriateness and the latter to possible appropriateness. As with many examples of this type of bipolarity there is a natural duality between the two evaluation criterion in that a description θ is definitely appropriate to describe element x if and only if $\neg\theta$ is not possibly appropriate to describe x.

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