Full perturbative calculation of spectral correlation functions for chaotic systems in the unitary symmetry class

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Starting from a semiclassical approach recently developed for spectral correlation functions of quantum systems whose classical dynamics is chaotic, we focus on the case of broken time-reversal symmetry, the so-called unitary class. We obtain to all orders in perturbation theory the non-oscillatory parts of all correlation functions, showing that the off-diagonal contributions to these correlation functions cancel and the conjectured universality holds. The innovation that allows this calculation to be performed is the introduction of an auxiliary matrix model which is governed by the same diagrammatic rules as the semiclassical approach and which can be exactly solved.

I. INTRODUCTION

One of the central problems in the field of quantum chaos has always been to show that, in the semiclassical limit, the energy levels of systems with chaotic classical dynamics have local statistics that agree with the universal predictions made by random matrix theory (RMT). This conjecture was put forward about 30 years ago [1], and is supported by many numerical, experimental and theoretical results.

RMT proceeds by considering an ensemble of Hamiltonians and computing statistical properties of the spectrum [2]. It relies only on the overall symetries of the system, e.g. whether the Hamiltonian is real symmetric (as in systems which are time-reversal invariant) or complex hermitian (otherwise). These ensembles of Hamiltonians are invariant under orthogonal or unitary transformations, respectively, and define the orthogonal/unitary symmetry classes. RMT is therefore a kind of minimal-information approach and its predictions are supposed to describe 'generic' systems with no special features (in particular, having completely chaotic dynamics).

Historically, the most popular quantity to consider has been the nearest-neighbor spacing distribution $P(s)$, but theoretically it is more convenient to work with spectral correlation functions or their Fourier transforms. The n-point correlation function $\mathcal{R}_n(\epsilon_1, \cdots, \epsilon_n)$ measures the likelihood that n energy levels will be located around positions $E+\epsilon_1$, $\cdots$, $E+\epsilon_n$, averaged over E.

The 2-point correlation function has received most of the attention. It depends on a single parameter $(\epsilon_1 - \epsilon_2)$, and a perturbation theory was developed starting from [3]. The leading correction was derived in [4, 5], and all orders of perturbation theory were eventually obtained [6–8]. It was even considered beyond standard perturbation theory [9–11], recovering oscillatory contributions to the correlation functions. Higher correlation functions were addressed to leading order in [12], see also [13] for non-oscillatory contributions.

We have made progress on this problem [14] by obtaining the semiclassical diagrammatic rules that govern the calculation of spectral correlation functions, and showing that the simplest of these functions indeed agree with RMT, at least to the leading orders in perturbation theory.

In the present work we improve on this and show that, for the unitary symmetry class, the agreement between RMT and semiclassics persists to all correlation functions and to all orders in perturbation theory. This is achieved through an auxiliary matrix model which is equivalent to the semiclassical theory and which can be solved exactly.

II. SEMICLASSICAL DIAGRAMMATIC RULES

The Gutzwiller trace formula [15] expresses the density of quantum stationary states as a sum over classical periodic orbits. When it is used to evaluate $\mathcal{R}_n$, it leads to multiple sums over periodic orbits, and the energy average selects correlations: in order to have constructive interference one must find two sets of orbits which have nearly the same total action.

In [14] we focused attention on the case when there are J orbits, denoted $p_1, ..., p_J$, correlated with another K orbits, denoted $q_1, ..., q_K$, so that $J + K = n$. This leads to a kind of partial correlation function $\widetilde{\mathcal{R}}_{J,K}$, and the total correlation function can be easily reconstructed from the partial ones.

The simplest possibility, called the diagonal approximation, is to have identical orbits, i.e. the p and q orbits coincide pairwise, which of course is only possible for even n. For systems with broken time-reversal symmetry, this was considered in [12], where it was shown that the diagonal approximation (in a variant that also captures oscillatory contributions) agrees with the prediction from RMT for all n. That means all corrections to this case, coming from non-identical orbits, must ultimately give a vanishing result. This is a non-trivial fact, however, which is precisely what we want to show.

Correlated sets of periodic orbits can be organized into diagrams. The edges of a diagram represent long periods of time during which a p orbit almost coincides with a q orbit. The vertices are comparatively small regions where the orbits exchange partners (known as 'encounters' in the literature). The diagrams record only the topology of the orbits, and their contribution to the correlation

function requires integrating over all possible action differences between the sets of orbits. This was carried out in [14] and the result is that

$$\widetilde{\mathcal{R}}_{J,K}(\epsilon,\eta) = D_{J,K}\Big[\sum_{\text{struc}}\prod_{jk}\frac{(-1)^V}{(-2\pi i(\epsilon_j-\eta_k))^{M_{jk}}}\Big]. \quad (1)$$

Here the energy increments included in $\epsilon = (\epsilon_1,\ldots,\epsilon_J)$ are associated to the p orbits and the increments included in $\eta = (\eta_1,\ldots,\eta_K)$ are associated to the q orbits. The latter are identified with the parameters $\epsilon_{J+1},\ldots,\epsilon_n$ of $\mathcal{R}_n$ using $\eta_k = \epsilon_{J+k}$. Furthermore, M_{jk} is the number of times orbits p_j and q_k run together but are not the first ones to arrive at an encounter, and the derivative operator is

$$D_{J,K} = \frac{(-1)^K}{(2\pi i)^n}\prod_{j=1}^{J}\frac{\partial}{\partial\epsilon_j}\prod_{k=1}^{K}\frac{\partial}{\partial\eta_k}. \quad (2)$$

The sum in (1) is over all the possible structures that can be associated with a given diagram. A structure corresponds to a particular choice of order in the sequence of traversed edges. They can be put in bijection with some equivalence classes of factorizations of permutations, as discussed in [14]. We will not follow that approach here, but we will come back to structures in the next section.

We showed by examples in [14] that, when we use the diagrammatic rule presented above to compute $\mathcal{R}_n$, there are two general mechanisms leading to a vanishing result at a particular order in perturbation theory: 1) the action of the derivative in $D_{J,K}$ gives zero because it acts on a function with less than n variables; 2) several diagrams cancel each other. Clearly, the second mechanism is the hardest to realize in practice. Fortunately, the model we introduce here for the unitary class implements this mechanism automatically, leaving only the simpler possibility 1) to be considered.

III. DIAGRAMMATICS OF A MATRIX MODEL

A direct exact evaluation of the expression in Eq. (1) is a hard task, because of the complicated combinatorial problem underlying it. However, for the unitary symmetry class we are able to make progress by following an indirect route: we postulate a matrix model which can be treated diagrammatically with precisely the same rule.

Matrix models have a rich history, starting with the work of t' Hooft [16], who noticed that Feynman diagrams in QCD could be recovered from them, and that a perturbation theory in the inverse dimension could be arranged so as to control the topology of these diagrams. Since then, they have been much studied [17–21], especially in the context of 2D quantum gravity [22–27].

Matrix models of a similar type to the one used here have been applied to transport problems [28–31]. Our matrix model also has a close connection to work relating the semiclassics of 2-point functions to the nonlinear sigma model of RMT [8, 11].

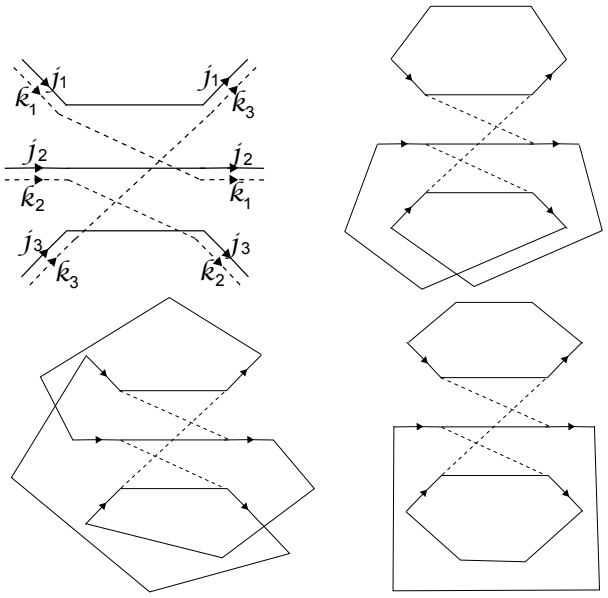

FIG. 1. Diagrammatics of $Z_{j_1k_1}Z^\dagger_{k_1j_2}Z_{j_2k_2}Z^\dagger_{k_2j_3}Z_{j_3k_3}Z^\dagger_{k_3j_1}$ (Left). Solid lines connect incoming j_m to outgoing j_m, while dashed lines connect incoming k_m to outgoing k_m. Notice how k's are cyclically permuted from left to right. Right: Sketch of a diagram contributing to $\mathcal{R}_4$ with $J = K = 2$; notice how there are two periodic orbits correlated with two others (encounter is grossly exaggerated).

We start by considering $N_1 \times N_2$ complex matrices, with no constraints – matrix elements are independent random variables. If we choose a Gaussian distribution

$$d\widetilde{\mu}(Z) = e^{-\alpha\mathrm{Tr}(ZZ^\dagger)}dZ, \quad (3)$$

then

$$\frac{1}{\mathcal{Z}}\int d\widetilde{\mu}(Z)Z_{jk}Z^\dagger_{st} = \frac{\delta_{jt}\delta_{ks}}{\alpha}, \quad (4)$$

where $\mathcal{Z} = \int d\widetilde{\mu}(Z)$ is a normalization constant.

The Gaussian nature of the measure leads to the nice property that the average value of a general product of matrix elements can be recovered from (4). This is known as the Wick rule. It says we must sum, over all possible pairings between Z's and $Z^\dagger$'s, the product of the averages of the pairs. Namely,

$$\left\langle\prod_{m=1}^{q}Z_{j_mk_m}Z^\dagger_{s_mt_m}\right\rangle = \sum_{\pi\in S_q}\prod_{m=1}^{N}\langle Z_{j_mk_m}Z^\dagger_{s_{\pi(m)}t_{\pi(m)}}\rangle. \quad (5)$$

The sum here is over all $q!$ permutations of the numbers from 1 to q.

A diagrammatic formulation can be introduced in order to compute the average value of $\mathrm{Tr}(ZZ^\dagger)^q$. First, we expand this trace as

$$\sum_{j_1,\ldots,j_k}\sum_{k_1,\ldots,k_q}Z_{j_1k_1}Z^\dagger_{k_1j_2}Z_{j_2k_2}Z^\dagger_{k_2j_3}\cdots Z_{j_qk_q}Z^\dagger_{k_qj_1}, \quad (6)$$

where the first sum runs from 1 to N_1 and the second from 1 to N_2. Then, each matrix element Z_{jk} is represented as a pair of arrows, one depicted with solid line and associated with j, the other depicted with dashed line and associated with k. These arrows have a marked end at the tail. The matrix elements of $Z^\dagger$ are represented in the same way, but the marked end is the head. We show an example in Figure 1. We arrange the arrows coming from Z and $Z^\dagger$ as two parallel columns, and draw lines connecting coinciding indices. Finally, the diagrammatic content of Wick's rule is that we must draw all possible connections between marked ends. The sum over pairings becomes a sum over diagrams and the contribution of each diagram to the average $\langle \mathrm{Tr}(ZZ^\dagger)^q \rangle$ will result from the sum over indices of the products of covariances like (4).

In the example of Figure 1 there are 6 possible ways to make the connections allowed by Wick's rule. Three of them are shown in the Figure. The covariances from diagram a) are such that we get the identifications $j_2 = j_3$ and $k_1 = k_3$; the sum over indices gives then $N_1^2 N_2^2$. Diagram b) results in $j_1 = j_2 = j_3$ and $k_1 = k_2 = k_3$, so that its contribution is $N_1 N_2$. In diagram c) the j indices have no identification, while $k_1 = k_2 = k_3$, so that its contribution is $N_1^3 N_2$. Considering all connections one can show that the final result is $\langle \mathrm{Tr}(ZZ^\dagger)^3 \rangle = \alpha^{-3}(N_1^3 N_2 + N_1 N_2^3 + 3N_1^2 N_2^2 + N_1 N_2)$.

When computing the average of a product of traces, each trace like $\mathrm{Tr}(ZZ^\dagger)^q$ is represented by a vertex of valence $2q$ with a specific internal structure, such that incoming j_m is followed by outgoing j_m, and incoming k_m is followed by outgoing k_m. Wick's rule then allows connections between different vertices, producing a diagram which may contain more than one connected component.

IV. OUR MATRIX MODEL

A. Definition

The matrix integral we postulate is

$$F(X,Y) = \frac{1}{\mathcal{Z}} \int d\mu(Z) e^{-\sum_{q\geq 2} \mathrm{Tr}[X(ZZ^\dagger)^q - (Z^\dagger Z)^q Y]}, \quad (7)$$

where $\mathcal{Z} = \int d\mu(Z)$ is again a suitable normalization constant. For the purposes of this section, it is sufficient to consider Z as a square matrix, of dimension N. The measure $d\mu(Z)$ is still Gaussian, but given by

$$d\mu(Z) = e^{-\mathrm{Tr}[XZZ^\dagger - Z^\dagger Z Y]} dZ. \quad (8)$$

The matrices X and Y are constant and diagonal. Let us denote their eigenvalues by x's and y's, respectively. The covariances of this model depend on these eigenvalues and are given by

$$\frac{1}{\mathcal{Z}} \int d\mu(Z) Z_{jk} Z^\dagger_{st} = \frac{\delta_{jt}\delta_{ks}}{x_j - y_k}. \quad (9)$$

B. Semiclassical interpretation

We want to give a semiclassical interpretation of our model. This will also fix a choice for the matrices X and Y and a procedure to extract the correlation functions from $F(X,Y)$. Notice that our matrix model only produces diagrams with encounters, i.e. corrections to the diagonal approximation of [12].

The semiclassical interpretation arises when we Taylor expand the exponential and integrate term by term using Wick's rule to get a sum over diagrams. Then each trace/vertex can be interpreted as an encounter, e.g. it is natural to interpret the full lines in Fig. 1 as topological representations of how connections inside an encounter look for the p-orbits, and the dashed lines as changed connections inside the q-orbits.

The pairwise contractions (9) due to Wick's theorem become links. As discussed previously, for a given diagram with structure, some of the j indices must be identified and likewise for the k indices. In the encounter (see Fig. 1) the j indices coincide for points on the same p orbit and the k indices coincide for points on the same q orbit. For a link/contraction line both indices of the connected points coincide, in line with the fact that both must belong to the same p orbit as well as the same q orbit. So it is natural to interpret the independent indices as periodic orbits, but this identification has a twist to be discussed later.

Considering the contributions, note that the traces involve quantities of the kind $A_q = X(ZZ^\dagger)^q - (Z^\dagger Z)^q Y$. Notice that

$$\mathrm{Tr} A_q = \sum_{j_1,\ldots,j_q} \sum_{k_1,\ldots,k_q} (x_{j_1} - y_{k_1}) \prod_{m=1}^{q} Z_{j_m k_m} Z^\dagger_{k_m j_{m+1}}, \quad (10)$$

so the first indices of X and Y play a distinguished role, analogous to the 'first orbits to arrive at an encounter' entering the definition of M_{jk} on the semiclassical side. The denominator in (9) produces a product of terms of the form $(x_j - y_k)$ for each pair of free indices. When we take into account the similar factor in the numerator, arising from links/contractions according to (9), we see that the contribution of a given diagram with structure ends up being

$$(-1)^V \prod_{jk=1}^{N} \frac{1}{(x_j - y_k)^{M_{jk}}}, \quad (11)$$

where M_{jk} is the number of times indices j and k belong to the same link, but they are not the first ones to arrive at a vertex. The factor $(-1)^V$ comes from the sign in the exponent, which produces a negative sign for each vertex.

The above diagrammatic rule looks similar to the semiclassical one (1). Indeed if we could fully identify j and k with p and q orbits we could also identify x_j, y_k with the energy increments ϵ_j, η_k (up to a constant). This

is because each orbit sum arises from the trace formula for specific energy increments ϵ_j, η_k. However there is a crucial difference, which arises because X and Y are (diagonal) square matrices rather than rectangular. In (1) the product over j runs from 1 to J, and the one over k runs from 1 to K. On the other hand, the products in (11) both run from 1 to N. Therefore, the eigenvalues of X and Y cannot be directly interpreted as energies. We must relate them to ϵ's and η's in a non-trivial way.

We thus extend the range of j, k up to $n-1$ with $n = J + K$. (This is the maximal range of indices we need as j runs up to $J = n-1$ for $K = 1$, and k runs up to $K = n-1$ for $J = 1$.) Moreover we include as diagonal elements of X and Y several 'replicas' of these energy increments. We thus choose $N = (n-1)r$, where n is the index of the correlation function we are interested in and r is some parameter. The eigenvalues of X and Y are then taken to be degenerate according to

$$x_{m+(j-1)r} = -2\pi i\epsilon_j, \quad y_{m+(k-1)r} = -2\pi i\eta_k, \tag{12}$$

where $m \in \{1, ..., r\}$. That is, X has $n-1$ variables ϵ as eigenvalues, all r times degenerate, and analogously for Y and the η variables. When we take into account this degeneracy the diagrammatic factor (11) becomes

$$r^n(-1)^V \prod_{jk=1}^{n} \frac{1}{(-2\pi i(\epsilon_j - \eta_k))^{M_{jk}}}. \tag{13}$$

This is not exactly equal to (1), but it is quite close. The difference is that here we cannot discriminate the different decompositions of n as a sum $n = J + K$. Instead, we have that if the total number of periodic orbits in a diagram is n, i.e. if it must be taken into account in the calculation of $\mathcal{R}_n$, its contribution always gets multiplied by r^n. More concretely, the function $\mathcal{R}_n$ can be obtained from the function $F(X, Y)$ by first computing $[r^n]F$, the coefficient of r^n in F. The function $F(X, Y)$ is, by construction, a symmetric function of the ϵ's and a symmetric function of the η's. If we want, we can apply the appropriate derivative in order to find the partial correlation function we considered before,

$$\widetilde{\mathcal{R}}_{J,K}(\epsilon, \eta) = D_{J,K}\left([r^n]F\right). \tag{14}$$

Note that on the l.h.s. $\widetilde{\mathcal{R}}_{J,K}$ should depend only on ϵ_j with $j = 1, \ldots, J$ and η_k with $k = 1, \ldots, K$. However on the r.h.s. we have not done anything to explicitly exclude contributions depending on the remaining increments, associated to p_j with $j > J$ and q_k with $k > K$. This will be justified at a later stage, when we will see that such contributions vanish and that they do so (in a sense to be clarified then) more immediately than $\widetilde{\mathcal{R}}_{J,K}$ itself.

A similar model was discussed for semiclassical chaotic transport in [28]. In that case, all classical trajectories had the same energy. This peculiarity allowed for a simpler matrix integral, which did not require external matrices with degenerate eigenvalues.

C. Exact solution

Let us now proceed to the exact solution of our matrix integral $F(X, Y)$ in Eq. (7).

We start by computing the normalization constant. Let $Z = UDV$ be the singular value decomposition (SVD) of Z. Here D is a diagonal matrix such that $DD^\dagger$ has the same eigenvalues of $ZZ^\dagger$, let us denote them by λ, while U and V are unitary matrices. The matrix U is uniformly distributed over the unitary group with Haar measure. The matrix V takes values in the coset space $U(N)/[U(1)]^N$, with a measure induced from the Haar measure; this difference is irrelevant, leading only to a constant factor which cancels later on.

The Jacobian of the SVD transformation was obtained in [32] and is given by $|\Delta(\lambda)|^2$, the square of the Vandermonde determinant $\Delta(\lambda) = \prod_{i<j}(\lambda_j - \lambda_i)$. We get

$$\mathcal{Z} = \int dU dV d\lambda |\Delta(\lambda)|^2 e^{-\mathrm{Tr}[XUTU^\dagger] + \mathrm{Tr}[V^\dagger TVY]}, \tag{15}$$

where $T = DD^\dagger$. Using the well-known Harish-Chandra-Itzykson-Zuber integral [33, 34],

$$\int dU e^{-\mathrm{Tr}[XUTU^\dagger]} = c_N \frac{\det(e^{-x_i\lambda_j})}{\Delta(x)\Delta(\lambda)}, \tag{16}$$

where c_N is a constant depending only on the dimension N, we can now perform the integral over U. The integral over V is similar, but with a different constant d_N.

In order to do the integral over the eigenvalues, we resort to the Andréief identity [35]: given two sets of N functions, ϕ_i, ψ_i, the multidimensional integral of a product of determinants is the determinant of a matrix whose elements are one-dimensional integrals,

$$\int \det\left(\phi_j(\lambda_k)\right) \det\left(\psi_j(\lambda_k)\right) d\lambda = N! \det \int \phi_j(\lambda)\psi_k(\lambda) d\lambda.$$

Applying it to

$$\int \det(e^{-x_j\lambda_k}) \det(e^{y_j\lambda_k}) d\lambda \tag{17}$$

leads to the final result for the normalization constant:

$$\mathcal{Z}(X, Y) = \frac{c_N d_N N!}{\Delta(x)\Delta(y)} \det C, \tag{18}$$

where C, sometimes known as the Cauchy matrix, has elements given by $C_{jk} = (x_j - y_k)^{-1}$.

In order to compute $F(X, Y)$ we must first evaluate the sum in the exponent. This gives

$$F(X, Y) = \frac{1}{\mathcal{Z}} \int dU dV d\lambda |\Delta(\lambda)|^2 e^{-\mathrm{Tr}[XU\tilde{T}U^\dagger] + \mathrm{Tr}[V^\dagger \tilde{T}VY]} \tag{19}$$

with $\tilde{T} = \sum_{q\geq 1} T^q$ which has eigenvalues $\sum_{q\geq 1} \lambda_i^q$. As these diverge for $\lambda_i > 1$ the integral for each λ_i is effectively reduced to the interval $[0, 1]$, and the eigenvalues

of the matrix $\tilde{T}$ become $\tilde{T}_i = \lambda_i/(1-\lambda_i)$ which runs from 0 to ∞. The angular integrals are computed exactly as above, leading to

$$F = \underbrace{\frac{1}{\mathcal{Z}} \frac{c_N d_N}{\Delta(x)\Delta(y)}}_{=\frac{1}{N!\det C}} \int \underbrace{d\lambda \left|\frac{\Delta(\lambda)}{\Delta(\tilde{T})}\right|^2}_{=d\tilde{T}\frac{1}{\prod_i (1+\tilde{T}_i)^{2N}}} \det(e^{-x_j \tilde{T}_K}) \det(e^{y_j \tilde{T}_k}). \tag{20}$$

Here we have changed variables to $\tilde{T}_i$, using $d\lambda_i = d\tilde{T}_i/(1+\tilde{T}_i)^2$ and

$$\frac{\Delta(\lambda)}{\Delta(\tilde{T})} = \prod_i \frac{1}{(1+\tilde{T}_i)^{N-1}}. \tag{21}$$

One last use of the Andréief identity (absorbing a factor $\frac{1}{(1+\tilde{T})^N}$ each in ϕ_j and ψ_k) then leads to our final result for $F(X,Y)$ as a ratio of determinants,

$$F(X,Y) = \frac{\det A}{\det C}. \tag{22}$$

Here the matrix A has elements given by

$$\begin{aligned} A_{jk} &= \int d\tilde{T} \frac{1}{(1+\tilde{T})^{2N}} e^{-(x_j - y_k)\tilde{T}} \\ &= e^{x_j - y_k} \mathrm{Ei}(2N, x_j - y_k), \end{aligned} \tag{23}$$

in terms of the incomplete exponential integral,

$$\mathrm{Ei}(a,z) = \int_1^\infty e^{-zt} t^{-a} dt. \tag{24}$$

The matrices A and C both have zero determinant when there are degeneracies among the x's or the y's. The identification between these variables and the ϵ's and η's must therefore be performed only after computing the ratio $\det A/\det C$.

Integration by parts now allows us to write the result in a slightly different form,

$$F(X,Y) = \frac{\det(C-B)}{\det C} \tag{25}$$

where

$$B_{jk} = 2NC_{jk} e^{x_j - y_k} \mathrm{Ei}(2N+1, x_j - y_k). \tag{26}$$

In fact, the method of successive integration by parts can be used in order to produce an asymptotic series for each B_{jk} in terms of $C_{jk} = (x_j - y_k)^{-1}$, in which higher powers of C_{jk} will be accompanied by high powers of N. Importantly for the following considerations, we can thus expand $F(X,Y)$ in terms of higher and higher powers of $N = (n-1)r$. The leading term of order N^0 is $\frac{\det C}{\det C} = 1$ which vanishes after taking derivatives according to (14). All other terms involve higher powers in N.

D. Recovering the RMT result

Our final result (22) can be expanded in inverse powers of $(x-y)$ and, upon use of the degeneracy condition (12), provides all spectral correlation functions. Using Eq. (14) we will show that the off-diagonal contributions to all correlation functions vanish, because the terms that contain all variables ϵ_j and η_k appearing in the derivatives $D_{J,K}$ must necessarily be of a higher order in r than r^n. Hence the corresponding coefficient $[r^n]F$ vanishes.

We start by showing this in an example. The first nontrivial term in the expansion of (25) is $\mathrm{Tr}[C^{-1}B]$, taken to leading order. For $n=3$ we obtain

$$Nr^2 \left(\frac{1}{\epsilon_1 - \eta_1} + \frac{1}{\epsilon_1 - \eta_2} + \frac{1}{\epsilon_2 - \eta_1} + \frac{1}{\epsilon_2 - \eta_2} \right). \tag{27}$$

This corresponds semiclassically to the leading order correction to $\mathcal{R}_3$, as discussed in our previous paper [14]. Notice that it is indeed proportional to r^3 since $N = (n-1)r$. However, the contribution actually vanishes after we take the required derivatives with respect to three different variables, since all terms inside the parenthesis depend only on two variables.

Terms involving all energy increments that appear in the derivatives must be of a higher order in N. However due to $N \propto r$ this leads to an overall order in r of at least 4, and the contribution vanishes after taking the r^3 coefficient.

As seen in [14] the second order correction to $\mathcal{R}_3$ involves cancellation among several diagrams. It is not possible to see that cancellation in action in the present matrix model: it has been performed automatically and imperceptibly. This is precisely the merit of the model, that it has all contributions built in, and gives only the final result.

Fortunately, we do not need to understand the function $F(X,Y)$ in its full complexity, because of the general cancellation mechanism observed above. In general the contributions $\widetilde{\mathcal{R}}_{J,K}$ to the n-point correlation function can be accessed from F using derivatives w.r.t. n variables. Hence if we expand in $(x-y)^{-1}$ the contributing terms must involve at least n different variables x_j, y_k. Because of the degeneracy the contribution is thus proportional to r^n. However, as we have mentioned the series development of the Ei function involves further powers of $N = (n-1)r$. For all terms except the trivial leading term this leads to further factors proportional to N and thus r, and this makes the exponent of r always larger than n. Therefore, when taking the coefficient of order r^n as required in (14), the result automatically vanishes. This is in complete agreement with the prediction from random matrix theory for the unitary symmetry class and shows that the off-diagonal contributions to all correlation functions vanish.

To complete a technical point mentioned earlier, we can now also understand why on the r.h.s. of Eq. (14) we did not have to explicitly exclude contributions from

orbits associated to ϵ_j with $j > J$ and η_k with $k > K$. Any such contributions involve one additional degeneracy factor r for each additional energy increment. Hence these contributions vanish even more immediately than the others and their treatment does not affect our result.

V. CONCLUSIONS

We have introduced a matrix model that mimics the semiclassical diagrammatic formulation of spectral correlation functions of chaotic systems in the unitary symmetry class. This matrix model was then solved, recovering the RMT prediction that the diagonal approximation is exact.

This new approach avoids cumbersome combinatorial analysis that were necessary in previous approaches to show cancellations among diagrams. Instead, these cancellations are built in the model from the start.

This is an important step forward in establishing the conjecture that spectral statistics of quantum chaos are universal. The problem still remains to settle this conjecture for other symmetry classes, and for the oscillatory terms which are not accessible from perturbation theory. We hope the ideas introduced here may pave the way to the full solution.

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[1] O. Bohigas, M.J. Giannoni and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[2] F. Haake, *Quantum Signatures of Chaos* (Springer, 2001).
[3] M.V. Berry, Proc. R. Soc. London A **400**, 229 (1985).
[4] M. Sieber and K. Richter, Phys. Scr., T **90**, 128 (2001).
[5] M. Sieber, J. Phys. A **35**, L613 (2002).
[6] S. Heusler, S. Müller, P. Braun, and F. Haake, J. Phys. A **37**, L31 (2004).
[7] S. Müller, S. Heusler, P. Braun, F. Haake, and A. Altland, Phys. Rev. Lett. **93**, 014103 (2004).
[8] S. Müller, S. Heusler, P. Braun, F. Haake, and A. Altland, Phys. Rev. E **72**, 046207 (2005).
[9] S. Heusler, S. Müller, A. Altland, P. Braun, and F. Haake, Phys. Rev. Lett. **98**, 044103 (2007).
[10] J. P. Keating and S. Müller, Proc. R. Soc. Lond. A **463**, 3241 (2007).
[11] S. Müller, S. Heusler, A. Altland, P. Braun and F. Haake, New J. Phys. **11**, 103025 (2009).
[12] T. Nagao and S. Müller, J. Phys. A **42**, 375102 (2009).
[13] P. Shukla, Phys. Rev. E **55** (1997) 3886.
[14] S. Müller and M. Novaes, submitted.
[15] M. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990).
[16] G. 't Hooft, Nucl. Phys. B **72** 461 (1974).
[17] E. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber, Comm. Math. Phys. **59**, 35 (1978).
[18] D. Bessis, C. Itzykson and J.B. Zuber, Adv. Appl. Math. **1**, 109 (1980).
[19] E. Witten, Nucl. Phys. B **340**, 281 (1990).
[20] E. Witten, Surv. in Diff. Geom. **1**, 243 (1991).
[21] M. Kontsevich, Comm. Math. Phys. **147**, 1 (1992).
[22] E. Brézin and V. Kazakov, Phys. Lett. B **236**, 144 (1990).
[23] M. Douglas and S. Shenker, Nucl. Phys. B **335**, 635 (1990).
[24] D. Gross and A. Migdal, Phys. Rev. Lett. **64**, 127 (1990).
[25] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, Phys. Rep. **254**, 1 (1995).
[26] A. Gerasimov, A. Marshakov, A.D. Mironov, A. Morozov and A. Orlov, Nuclear Physics B **357**, 565 (1991).
[27] A.S. Alexandrov, A.D. Mironov and A. Morozov, Theoretical and Mathematical Physics **150**, 153 (2007).
[28] M. Novaes, J. Phys. A **46**, 502002 (2013).
[29] M. Novaes, Ann. Phys. **361**, 51 (2015).
[30] M. Novaes. J. Math. Phys. **56**, 062109 (2015).
[31] M. Novaes. J. Math. Phys. **57**, 122105 (2016).
[32] T.R. Morris, Nucl. Phys. B **356**, 703 (1991).
[33] Harish-Chandra, Proc. Nat. Acad. Sci. **42**, 252 (1956).
[34] C. Itzykson and J.-B. Zuber, J. Math. Phys. **21**, 411 (1980).
[35] P.J. Forrester, preprint arXiv:1806.10411, 2018.