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Transmission and absorption in a waveguide with a metamaterial cavity

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1 The reflection and transmission of acoustic waves along a waveguide of uniform width
2 by a metamaterial cavity is considered. The metamaterial is comprised of a closely-
3 spaced array of micro-channels separated by thin plates between which the field may
4 be damped. Exact equations governing the field in the microstructured metamaterial
5 cavity are replaced by an effective field using homogenisation approach. This allows a
6 solution to be formulated in terms of an integral equation across the interface between
7 the metamaterial cavity and the waveguide. Attention focusses on the resonant and
8 damping effects of a metamaterial cavity of tapered height where rainbow trapping
9 phenomena are encountered. It is shown that near-perfect broadband absorption
10 of the incoming wave energy can be achieved.

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11 I. INTRODUCTION

12 The Helmholtz resonator is a classical device used for suppressing transmission of waves
13 along waveguides by enhancing reflection and/or absorption of wave energy¹. The resonator
14 is usually comprised of a chamber with a narrow neck which connects to the waveguide.
15 The geometry of the Helmholtz resonator determines its resonant frequencies and its inter-
16 action with propagating waveguide modes becomes significant close to these frequencies².
17 For example, when damping is absent total reflection can occur and, with visco-thermal
18 losses accounted for, it is possible to absorb up to half of the incident wave energy close to
19 resonance³. Perfect absorption can be achieved by two resonators^{4,5} and multiple resonators,
20 tuned to different frequencies, extend these effects over multiple frequencies⁶, having a close
21 connection to a phenomenon labelled “rainbow trapping” in Physics⁷.

22 In undamped periodic arrays of scatterers, stop bands are defined as the ranges of fre-
23 quencies over which unattenuated wave propagation is prohibited within the array; these
24 generally depend on scattering geometry and spacing. Rainbow trapping occurs when arrays
25 are designed with a slow modulation of geometry and/or spacing along their length⁸⁻¹⁰ and
26 waves of different frequencies encounter stop bands at different positions along the array.
27 At the edges of stop bands the group velocity is zero and a field of high intensity is locally
28 trapped. Thus, a modulated array acts to block wave transmission over a broad range of
29 wave frequencies. When damping is added, broadbanded absorption of wave energy can be
30 induced^{6,11}.

31 Rainbow trapping can be achieved by passive structures or micro-resonators in 2D or
32 3D^{12,13}. One such device is to use a comb-like grating consisting of an array of grooves of
33 tapered length or width which act as micro-resonators^{8-10,14}.

34 In this paper we consider a two-dimensional waveguide with a cavity attached to one wall.
35 The cavity possesses a microstructure consisting of multiple equally-spaced narrow channels
36 separated by thin parallel plates extending perpendicular to the waveguide. Each micro-
37 channel acts as a Helmholtz resonator whose fundamental resonant frequency depends on
38 its length. By arranging the micro-channels to extend over a range of lengths in a linearly-
39 tapered array we construct a broadbanded resonant cavity. The assumption of narrowness
40 of the micro-channels implies that in the physical setting of acoustics viscous losses will
41 be important and should be included in the governing equations. Within this paper we
42 model these losses by adding a linear damping^{3, 15} which manifests itself as a complex-valued
43 wavenumber within the cavity.

44 The solution to the problem of discrete micro-channels is hard to solve by exact analytical
45 methods and it is typical to use Finite Element Method simulations^{8-10,14}, or asymptotic
46 approximations¹⁶. Instead, here we take advantage of the contrast in lengthscales between
47 the microstructure and the other lengthscales in the problem and use a homogenisation
48 approach to replace the microstructured cavity by an effective medium/continuum. This
49 particular approximation has been shown to work well when compared to exact mathematical
50 description of the array in a related problem¹⁷.

51 Within the framework of linearised acoustics the mathematical solution to the boundary-
52 value problem is treated semi-analytically, by employing Fourier transforms within the

53 waveguide and matching to an exact description of the effective wave field within the cavity.
 54 The matching gives rise to an integral equation for an unknown function across the join
 55 between waveguide and cavity. Application of a standard Galerkin approximation results in
 56 a linear system of equations which is straightforward to compute – details are contained in
 57 Section 2 of the paper. Section 3 considers expressions for the damping coefficient, a measure
 58 of the proportion of wave power absorbed by the cavity. Section 4 contains a range of results
 59 and extended discussion of various features of the solution which arise and conclusions follow
 60 in Section 5.

61 II. DESCRIPTION OF THE PROBLEM

62 In terms of two-dimensional Cartesian coordinates (x, y) a compressible fluid fills a long
 63 uniform waveguide with sound-hard walls along $y = 0$, $-\infty < x < \infty$ and $y = a$, $|x| > c$. A
 64 cavity attaches to the waveguide along a finite length of one wall $\{|x| < c, y = a\}$. Inside
 65 this cavity the compressible fluid fills narrow channels between a closely-spaced cascade of
 66 thin parallel plates aligned with the y -axis. The length of each of the channels can vary as
 67 a function of x as illustrated in Fig. 1.

68 Within the waveguide, $\Re\{\psi(x, y)e^{-i\omega t}\}$ represents time-harmonic variations of the pres-
 69 sure field where the complex-valued function $\psi(x, y)$ satisfies

$$(\nabla^2 + k^2)\psi = 0, \quad -\infty < x < \infty, \quad 0 < y < a. \quad (1)$$

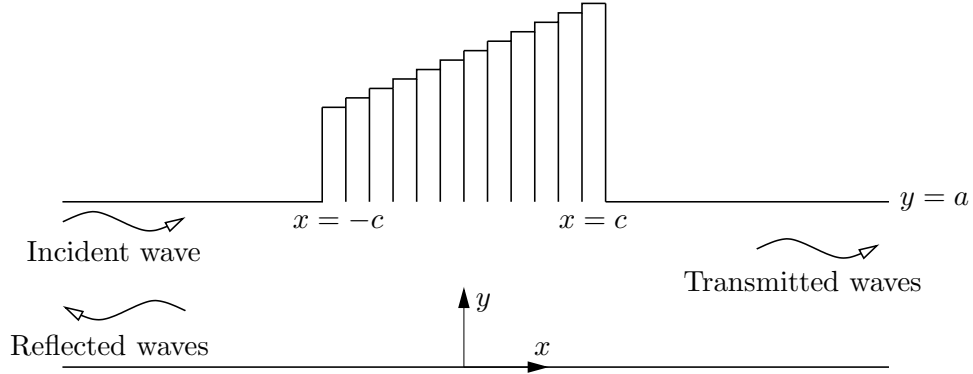


FIG. 1. Definition sketch of the waveguide and plate-array metamaterial cavity.

70 where $k = \omega/c_s$ where c_s is the wave speed in the waveguide. The walls of the waveguide
 71 are sound-hard so

$$\psi_y(x, 0) = 0, \quad \text{and} \quad \psi_y(x, a) = 0, \quad \text{for } |x| > c. \quad (2)$$

72 A wave of unit amplitude is incident from $x = -\infty$ and is partially reflected and partially
 73 transmitted due to the effect of the cavity. Separation of variables applied to (1) with (2)
 74 in $|x| > c$ determines that

$$\psi(x, y) \sim e^{ikx} + \sum_{n=0}^N R_n e^{-i\alpha_n x} \cos(n\pi y/a), \quad x \rightarrow -\infty \quad (3)$$

75 and

$$\psi(x, y) \sim \sum_{n=0}^N T_n e^{i\alpha_n x} \cos(n\pi y/a), \quad x \rightarrow \infty \quad (4)$$

76 where R_n, T_n are reflection and transmission coefficients, to be found, and the higher-order
 77 wavenumbers are defined by the real quantities

$$\alpha_n = \sqrt{k^2 - (n\pi/a)^2}, \quad n = 0, 1, \dots, N \quad (5)$$

78 and $N = \lfloor ka/\pi \rfloor$ is the integer part of ka/π .

79 Within the cavity, the closely-spaced array of plates has the effect of restricting the propa-
 80 gation of waves to the y -direction and the equation governing the fluid/plate microstructure
 81 is represented by

$$(\partial_{yy} + \mu^2)\psi = 0 \quad (6)$$

82 in $y > a$ for $|x| < c$. A formal derivation of (6) can be made by rescaling the x -coordinate
 83 within micro-channels width d where $\epsilon = kd \ll 1$. Equating orders of magnitude in ϵ uses
 84 the local lateral boundary conditions on the micro-channel walls en route to the derivation
 85 of (6); see¹⁶. In (6), $\mu \in \mathbb{C}$ replaces k to allow viscous damping effects within the cavity due
 86 to the narrowness of the micro-channels, and is defined (see¹⁵ §2.7, for example) by

$$\mu = k + i\sqrt{k}\sigma, \quad \sigma = (\nu/2c_s)^{1/2}/(2d) \quad (7)$$

87 where ν is the kinematic viscosity of the fluid and a small adjustment to the real component
 88 of the wavenumber has been neglected.

89 We remark that the current problem has an analogue in electromagnetic setting for TM-
 90 polarised waves in two-dimensional waveguide with perfectly-electric conducting surfaces in
 91 which μ represents the effect of a dielectric^{18,19}. In accordance with the use of a continuum
 92 model (6) to describe the microstructure of the array, the terraced upper boundary of the
 93 metamaterial cavity illustrated in Fig. 1 is represented by the continuous line $y = b + mx$
 94 (such that $b \pm mc > a$); on this boundary we impose

$$\psi_y = 0. \quad (8)$$

95 Solutions of (6) with (8) are given by

$$\psi(x, y) = u(x) \frac{\cos \mu(b + mx - y)}{\mu \sin \mu(b + mx - a)} \quad (9)$$

96 in terms of the unknown function $u(x) = \psi_y(x, a)$ for $|x| < c$.

97 Within the waveguide, solutions are sought using Fourier transforms. Thus we define

$$\Psi(l, y) = \int_{-\infty}^{\infty} (\psi(x, y) - e^{ikx}) e^{-ilx} dx \quad (10)$$

98 to be the Fourier transform of the scattered part of the field and l is the Fourier transform

99 variable. The inverse is

$$\psi(x, y) = e^{ikx} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(l, y) e^{ilx} dl \quad (11)$$

100 in which the contour of integration will be defined to satisfy the radiation condition. That

101 is the contribution to $\psi(x, y)$ as $x \rightarrow \pm\infty$ from the integral must be defined outgoing waves.

102 Taking the Fourier transform of (1) gives

$$\Psi''(l, y) - \gamma^2 \Psi(l, y) = 0, \quad 0 < y < a \quad (12)$$

103 where $\gamma^2 = l^2 - k^2$, whilst the Fourier transform of (2) gives $\Psi'(l, 0) = 0$ and

$$\Psi'(l, a) = \int_{-\infty}^{\infty} \frac{\partial}{\partial y} (\psi(x, y) - e^{ikx})_{y=a} e^{-ilx} dx = U(l) \equiv \int_{-c}^c u(x) e^{-ilx} dx \quad (13)$$

104 using (2). Thus, the transform function can be written

$$\Psi(l, y) = \frac{U(l) \cosh \gamma y}{\gamma \sinh \gamma a} \quad (14)$$

105 and using (11) we have

$$\psi(x, y) = e^{ikx} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{U(l) \cosh \gamma y}{\gamma \sinh \gamma a} e^{ilx} dl. \quad (15)$$

106 It is evident from (15) that there are poles on the axis of integration at $l = \pm\alpha_n$ for

107 $n = 0, \dots, N$ and these relate to propagating modes at $x \rightarrow \infty$. In order that energy is

108 outgoing, the contour is chosen to pass below the poles $l = \alpha_n$ on the positive real l -axis and
 109 above the poles $l = -\alpha_n$ on the negative real l -axis. This definition means that as $x \rightarrow \infty$,

$$\psi(x, y) \sim e^{ikx} + \sum_{n=0}^N \frac{i\epsilon_n (-1)^n U(\alpha_n)}{2\alpha_n a} e^{i\alpha_n x} \cos(n\pi y/a) \quad (16)$$

110 where $\epsilon_0 = 1$, $\epsilon_n = 2$ for $n \geq 1$, an expression found by deforming the contour of integration
 111 into the upper-half plane and evaluating contributions from the poles along $l = \alpha_n$. Similarly,
 112 we find that as $x \rightarrow -\infty$

$$\psi(x, y) \sim e^{ikx} + \sum_{n=0}^N \frac{i\epsilon_n (-1)^n U(-\alpha_n)}{2\alpha_n a} e^{-i\alpha_n x} \cos(n\pi y/a) \quad (17)$$

113 found by deforming the contour of integration into the lower-half plane and evaluating
 114 contributions from poles at $l = -\alpha_n$.

115 Comparing with (3), (4) we find that

$$R_n = \frac{i\epsilon_n (-1)^n U(-\alpha_n)}{2\alpha_n a}, \quad n = 0, 1, \dots, N \quad (18)$$

116 and

$$T_n = \delta_{n0} + \frac{i\epsilon_n (-1)^n U(\alpha_n)}{2\alpha_n a}, \quad n = 0, 1, \dots, N. \quad (19)$$

117 The formulation is completed by matching the two representations of $\psi(x, y)$, (9) and
 118 (15) across the common boundary $y = a$, $|x| < c$. Thus

$$\frac{\cot \mu(b - a + mx)}{\mu} u(x) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\coth \gamma a}{\gamma} e^{ilx} \int_{-c}^c u(x') e^{-ilx'} dx' dl = e^{ikx} \quad (20)$$

119 for $|x| < c$ represents an integral equation for $u(x)$.

120 A numerical solution of this equation will be sought by expanding $u(x)$ using a finite
 121 complex Fourier series over $-c < x < c$. I.e. we write

$$u(x) \approx \sum_{p=-P}^P c_p u_p(x/c), \quad \text{where } u_p(t) = (-1)^p e^{ip\pi t/c} \quad (21)$$

122 in which c_p are coefficients to be determined and the value of P will be chosen to ensure the
 123 numerical solution is sufficiently converged – this is discussed further in the results section.
 124 Substituting (21) into (20) and then multiplying by the conjugate $u_q^*(x/c)$, $q = -P, \dots, P$
 125 and integrating over $-c < x < c$ gives the algebraic system of equations

$$\sum_{p=-P}^P (L_{pq} - M_{pq}) c_p = F_q(kc), \quad q = -P, \dots, P \quad (22)$$

126 for the unknown coefficients c_p where

$$L_{pq} = \frac{(-1)^{p+q}}{2\mu c^2} \int_{-c}^c e^{i\pi(p-q)x/c} \cot \mu(b-a+mx) dx \quad (23)$$

127 and

$$M_{pq} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\coth \gamma a}{\gamma} F_p(lc) F_q(lc) dl \quad (24)$$

128 with

$$F_p(lc) = \frac{1}{2} \int_{-c}^c u_p(x/c) e^{-ilx} dx = \frac{\sin(lc)}{lc - p\pi}. \quad (25)$$

129 Note that if $m = 0$, $L_{pq} = \delta_{pq} \cot[\mu(b-a)]/(\mu c)$. Work is also required to arrange M_{pq} into
 130 a computable form and these details are contained in the Appendix.

131 Using (21) in (17), (18) with (13) gives

$$R_n = \frac{i\epsilon_n(-1)^n}{\alpha_n a} \sum_{p=-P}^P c_p F_p(-\alpha_n c), \quad n = 0, \dots, N \quad (26)$$

132 and

$$T_n = \delta_{n0} + \frac{i\epsilon_n(-1)^n}{\alpha_n a} \sum_{p=-P}^P c_p F_p(\alpha_n c), \quad n = 0, \dots, N. \quad (27)$$

133 **III. DAMPING**

134 The time-averaged flux of energy crossing a boundary S with unit normal $\hat{\mathbf{n}}$ is calculated,
 135 for any pressure field $p(x, y)$ satisfying (1), from

$$\frac{\omega\rho}{2}\Im\left\{-\int_S p(\hat{\mathbf{n}}\cdot\nabla p^*)ds\right\}. \quad (28)$$

136 where ρ is the fluid density, ds is the arclength along S and the asterisk denotes complex
 137 conjugate. When $p(x, y) = e^{ikx}$ and the boundary, S , is the interval $0 < y < a$ for a
 138 constant x , the quantity above equates to $\frac{1}{2}\omega\rho a$; this is the power in the incident wave of
 139 unit amplitude travelling along the waveguide defined in §2.

140 In the scattering problem considered in the previous section, we can evaluate the outgoing
 141 energy flux by application of (28) to the function $p(x, y) \equiv \psi(x, y) - e^{ikx}$ for $x \rightarrow -\infty$ as
 142 given by (3) and by application of $p(x, y) = \psi(x, y)$ as $x \rightarrow \infty$ as in (4).

143 The mean energy absorption ratio – or *damping coefficient* – is defined by the mean
 144 incoming power minus the total mean outgoing power normalised by the mean incoming
 145 power. For the problem considered in §2 this is calculated to be

$$\eta = 1 - \sum_{n=0}^N \frac{\alpha_n}{\epsilon_n} (|R_n|^2 + |T_n|^2). \quad (29)$$

146 I.e. $\eta = 0$ is non-absorbing and $\eta = 1$ represents total absorption of incident wave energy.

147 An independent calculation of η can be obtained by measuring the mean rate of energy
 148 loss across the boundary $y = a$, $-c < x < c$ between the cavity and the waveguide using
 149 (28). Once normalised with respect to the power of the incoming wave of unit amplitude,
 150 this gives

$$\eta = \frac{1}{a}\Im\left\{-\int_{-c}^c \psi(x, a)\psi_y^*(x, a)dx\right\}. \quad (30)$$

151 When used with the definition (9) this gives

$$\eta = \frac{1}{a} \Im \left\{ - \int_{-c}^c |u(x)|^2 \frac{\cot \mu(b + mx - a)}{\mu} dx \right\}. \quad (31)$$

152 In terms of the results of the numerical scheme the above is expressed as

$$\eta \approx -\frac{2c}{a} \Im \left\{ \sum_{p=-P}^P \sum_{q=-P}^P c_p c_q^* L_{pq} \right\}. \quad (32)$$

153 When $m = 0$ the simplification to L_{pq} reduces this to expression to

$$\eta \approx -\frac{2c}{a} \Im \left\{ \cot[\mu(b - a)] / (\mu c) \right\} \sum_{p=-P}^P |c_p|^2. \quad (33)$$

154 Either (29) or (32)/(33) for $m = 0$ can be used to calculate the damping coefficient. Nu-
 155 merically, we find agreement between the two expressions to machine precision in computed
 156 results (indeed, it can be proved with some effort that one does imply the other) and thus
 157 serves only as a check on the implementation of the method, not an indicator of the accuracy
 158 of the numerical results.

159 IV. RESULTS

160 The focus of our results are $|R_n|$, $|T_n|$, the amplitude of the scattering coefficients and
 161 on the damping coefficient η . Numerically these are computed by (26), (27) and (29) or
 162 (32) which depend on the solution to the system of equations (22). Approximations result
 163 from the truncation to $2P + 1$ terms of the system of equations and from the truncation of
 164 the infinite integrals. We have conducted exhaustive tests on convergence of the results and
 165 conclude that truncating integrals to $l = 400$ and using $P = 5$ gives accuracy to more than
 166 four decimal places in all results presented, apart from where special comments apply. For

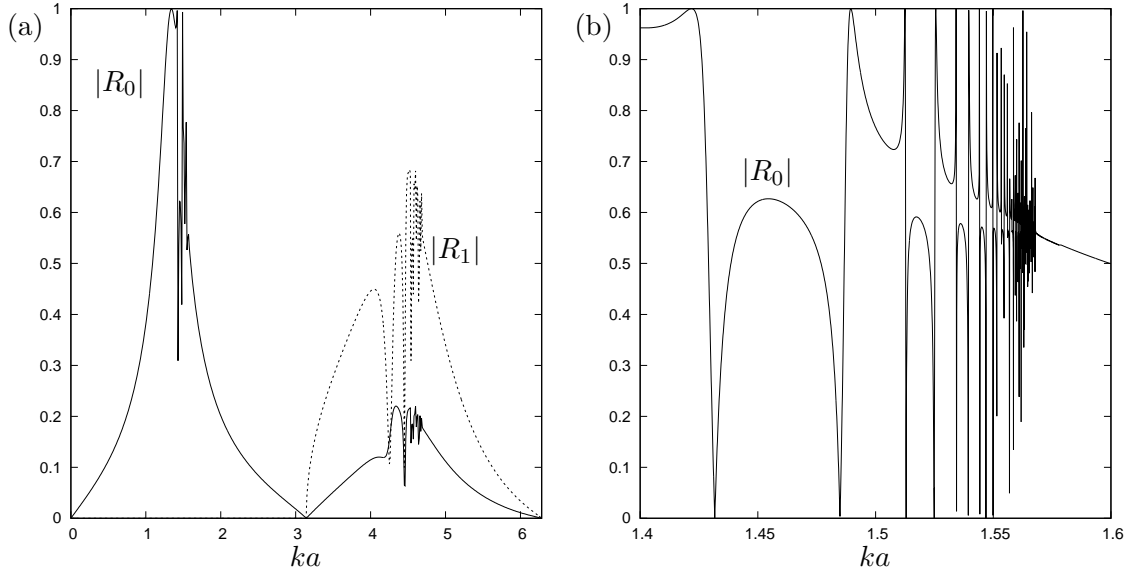


FIG. 2. Variation of reflection coefficients with ka (magnified scale in (b)) for a lossless rectangular metamaterial cavity $b/a = 2$, $c/a = 0.2$, $m = 0$ and $\mu = k$.

167 many, but not all, cases truncation to $l = 10$ and using $P = 1$ are sufficient; the numerical
 168 scheme is generally very quick and efficient to run.

169 We start by considering $m = 0$ so that the metamaterial cavity is rectangular and $\mu = k$
 170 so that there is no damping. We pick an example which illustrates the effect of this basic
 171 cavity by selecting $b/a = 2$, $c/a = 0.2$. Results showing the amplitudes of the reflected and
 172 transmitted wave coefficients $|R_n|$, $|T_n|$ are shown, as ka varies, in Fig. 2(a) with Fig. 2(b)
 173 focussing on results close to $ka = \frac{1}{2}\pi$. The higher order modes are cut-on at $ka = n\pi$,
 174 $n = 1, 2, \dots$ and thus there are two modes shown in $\pi < ka < 2\pi$. We have displayed
 175 only reflected wave amplitudes in order to make the graphs presentable. The behaviour of
 176 scattering coefficients is complicated as ka approaches $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$.

177 These two values have a particular physical significance as they are related to the eigen-
 178 solutions to the 1D wave equation in the channels formed by the metamaterial. That is,

179 at these frequencies the channels within the metamaterial cavity support a resonant wave
 180 with a node at the opening, $y = a$, and an antinode at the end of the channel, $y = b$.
 181 For the rectangular cavity considered in Fig. 2, this resonance condition is the same for all
 182 micro-channels: $ka = (q + \frac{1}{2})\pi/(b/a - 1)$, $q = 0, 1, \dots$. Thus, in the case shown in Fig. 2,
 183 where $b/a = 2$, resonance is predicted at $ka = \frac{1}{2}\pi$ and $ka = \frac{3}{2}\pi$.

184 A higher numerical truncation parameter, P , is required as $ka \rightarrow \frac{1}{2}\pi$ to resolve the
 185 increasingly oscillatory behaviour of the scattering coefficients, suggesting an increasing
 186 frequency in oscillations in the field between the two sidewalls of the cavity. This has been
 187 confirmed by numerical results not shown here.

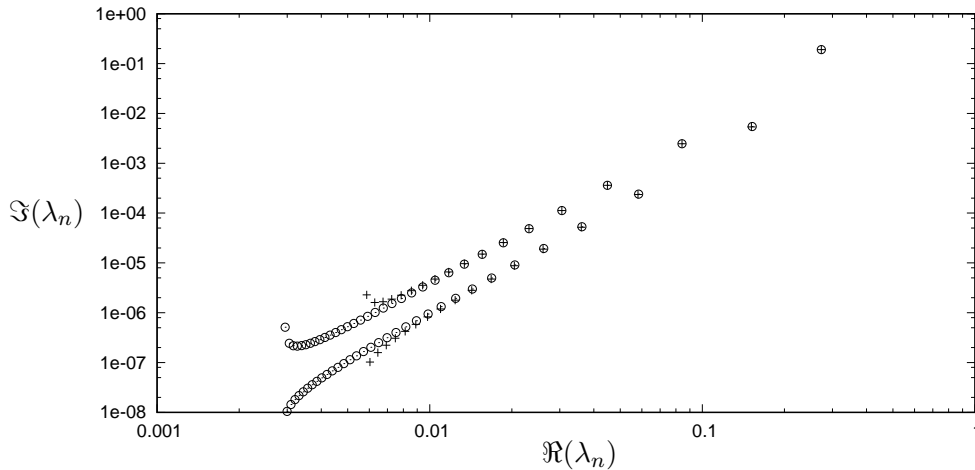


FIG. 3. The eigenvalues of the matrix M with elements M_{pq} for $P = 32$ (circles) and $P = 16$ (crosses) and with $ka = 1.5$, $b/a = 2$, $c/a = 0.2$.

188 In order to understand the complex behaviour seen in the results we need to understand
 189 the integral operator in (20). The operator is non-self-adjoint principally on account of
 190 the particular sense in which deformations have been made to the contour of integration
 191 to avoid poles in $|l| \leq 1$ located on the real integration axis. A self-adjoint version of the

192 integral operator, in which integration is confined to the real l -axis and with integration
 193 across poles are interpreted in the Cauchy principal-value sense, has the property that its
 194 eigenvalues, λ_n , are positive and have zero as a limit point (i.e. with $0 < \lambda_{n+1} < \lambda_n$, $\lambda_n \rightarrow 0$
 195 as $n \rightarrow \infty$)²⁰. In this regard an alternative formulation of the problem is possible in which
 196 this self-adjoint operator takes the part of the existing non-self-adjoint operator in (20) but
 197 happens at the expense of increased algebraic complication elsewhere; the rearrangement of
 198 terms give rise to a scattering matrix formulation reliant on the solution of $2N+2$ uncoupled
 199 integral equations. However, it is not clear that pursuing such an approach brings any clear
 200 advantage or clarity to the problem.

201 In the numerical method the eigenvalues of the non-self-adjoint integral operator are
 202 manifested as eigenvalues of the matrix M (with elements M_{pq} defined by (24)). There are
 203 now a finite number of these eigenvalues which are complex but with imaginary parts smaller
 204 than their real parts – see Fig. 3. The sequence of eigenvalues formed by taking an increased
 205 truncation parameter P tends to zero with positive real and imaginary parts and matches
 206 the behaviour anticipated above. When $m = 0$ and $\mu = k$, $b/a = 2$ the matrix elements
 207 $L_{pq} = \delta_{pq} \cot(ka)$ from (23) and it is clear from (22) that near resonance arises when the
 208 real-valued $\cot(ka)$ passes close to the complex eigenvalues of the matrix M . With reference
 209 to Fig. 5 as $ka \rightarrow \frac{1}{2}\pi$ from below this happens with increasing frequency and the strength
 210 of the near resonance increases; this explains the plot in Fig. 2. Note that the same effect
 211 is replicated at higher frequencies – as $ka \rightarrow (q + \frac{1}{2})\pi/(b/a - 1)$ for any integer $q = 0, 1, \dots$
 212 and any value of b/a .

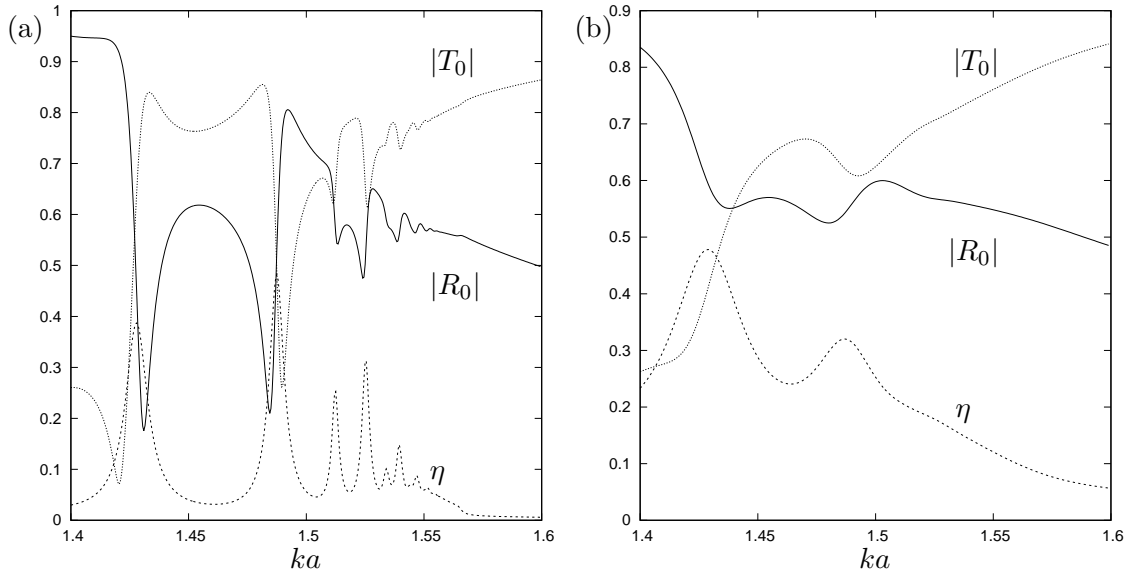


FIG. 4. Variation of scattering and damping coefficients with ka for a rectangular metamaterial cavity $b/a = 2$, $c/a = 0.2$, $m = 0$ and $\mu = k + i\sigma\sqrt{k}$ with $\sigma = 0.001$ in (a) and $\sigma = 0.01$ in (b).

213 It is tempting to conclude that there is no solution in Fig. 2 at $ka = \frac{1}{2}\pi$. However for
 214 $b/a = 2$, $m = 0$, $ka = \frac{1}{2}\pi$ the solution in the metamaterial cavity satisfies $\psi(x, a) = 0$
 215 for $|x| < c$. Thus, the solution in the waveguide must satisfy $\psi(x, a) = 0$ for $|x| < c$ in
 216 addition to (1), (2) and radiation conditions and is therefore decoupled from the solution
 217 in the cavity. This waveguide boundary-value problem is well-posed and the solution can
 218 be expressed using Fourier transforms by (20) but with the first term absent. The solution,
 219 $u(x)$, representing $\psi_y(x, a)$, $|x| < c$, sets the value of $\psi(x, y)$ within metamaterial cavity. On
 220 account of the boundary condition across $|x| < c$ being homogenous Dirichlet and (2) for
 221 $|x| > c$ being homogeneous Neumann, the solution, $u(x)$, is known to possess inverse square
 222 root singularities as $|x| \rightarrow c^-$, (e.g.²). We have used a modified set of functions

$$u_p(t) = \frac{2e^{-ip\pi/2}T_p(t)}{\pi\sqrt{1-t^2}}, \quad (34)$$

223 where $T_n(t)$ are Chebychev functions, in place of those defined in (22)² which results in
 224 $F_p(lc) = J_p(lc)$ replacing (25). The revised numerical scheme has been used to compute
 225 accurate and rapidly-convergent solutions for the specific case relating to $ka = \frac{1}{2}\pi$ in Fig. 2.

226 Computation of results for the problem with parameters used in Fig. 2 evaluated at
 227 exactly $ka = \frac{1}{2}\pi$ returns values for $|R_0|$ of 0.551615 ($P = 8$), 0.551301 ($P = 16$), 0.55115
 228 ($P = 32$) and 0.55113 ($P = 64$). With (34) we find $|R_0| = 0.55105$ to five significant figures
 229 with a truncation parameter of $P = 1$.

230 In Figs. 4(a,b) we consider the effect on the results shown in Fig. 2 of adding small (but
 231 increasing) amounts of damping. Thus we retain the geometrical parameters $m = 0$ and
 232 $b/a = 2$, $c/a = 0.2$, but take $\mu = k + 0.001i\sqrt{k}$ and $\mu = k + 0.01i\sqrt{k}$ in the two plots.
 233 In Figs. 4(a,b) we add the transmission coefficient, $|T_0|$, and the damping coefficient, η . A
 234 small amount of damping smooths out the rapid fluctuations in scattering coefficients.

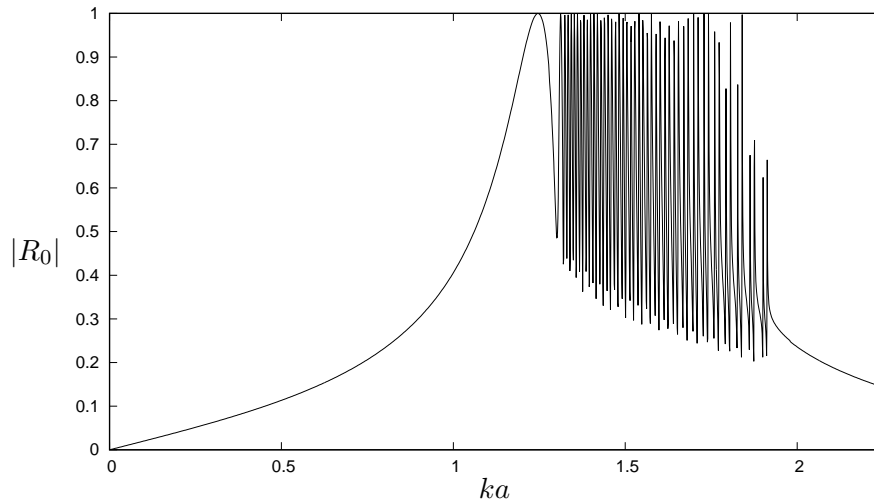


FIG. 5. Variation of $|R_0|$ with ka for a lossless ($\mu = k$) tapered metamaterial cavity $b/a = 2$, $c/a = 0.2$, $m = 1$.

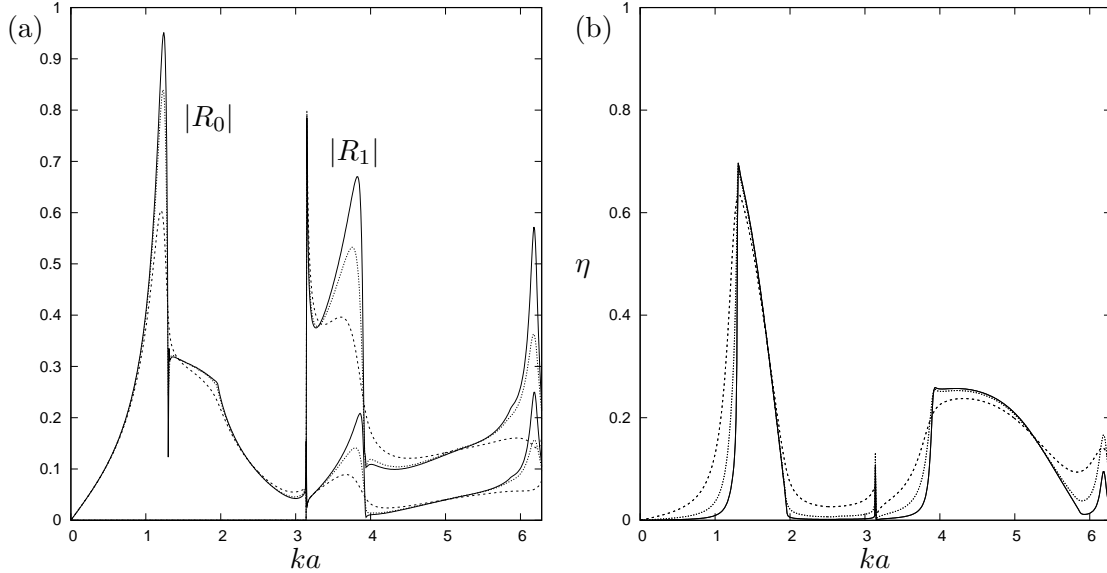


FIG. 6. Variation of (a) scattering and (b) damping coefficients with ka for a lossy tapered metamaterial cavity $b/a = 2$, $c/a = 0.2$, $m = 1$: $\mu = k + 0.0025i\sqrt{k}$ (solid), $\mu = k + 0.01i\sqrt{k}$ (dashed), $\mu = k + 0.04i\sqrt{k}$ (dotted).

235 We stick with $b/a = 2$ and $c/a = 0.2$ in Figs. 5, 6 where the effect of changing cavity
 236 taper, m , is considered. We have shown results for $m = 1$, so that the cavity taper is angled
 237 at 45° . In Fig. 5 results are given for a lossless cavity ($\mu = k$). For these parameters there
 238 is a continuous range ($1.309 < ka < 1.963$) of resonant frequencies embedded within the
 239 metamaterial cavity over which $|R_0|$ oscillates rapidly. The number of oscillations is set by
 240 the truncation parameter – $P = 24$ in Fig. 5. When P is halved or doubled the number of
 241 oscillations in this range is halved or doubled although the vertical extent of the oscillations
 242 forms a robust and well-defined envelope (the resolution of the plot accounts for random
 243 variations in the vertical). Thus, it appears that the numerical solution does not converge
 244 as $P \rightarrow \infty$ and this single issue has been at the centre of most of the work performed on
 245 this paper.

246 Various alternative approaches have been explored to shed light on this. One approach has
 247 been to change the numerical approximation scheme. This has included using collocation
 248 methods and different basis functions. We have also reformulated the integral equation
 249 (20) replacing $u(x) \equiv \phi_y(x, a)$ with $\phi(x, a)$ as the unknown and have used the fact that
 250 $\phi(x_*, a) = 0$ to construct a basis where x_* is a solution of $\tan(\mu(b - a + mx_*)) = 0$ and
 251 the location of the resonant channel in the metamaterial cavity. By taking this approach
 252 we have attempted to remove potential issues with singularities or discontinuities associated
 253 with derivatives. Every attempt had resulted in the same outcome, namely non-convergent
 254 oscillations whose frequency are tied to the numerical scheme. We note that similar results
 255 have been observed in related studies¹¹. Finally, an approximation has been made to the
 256 current problem which involves replacing the continuum model for the cavity by a finite
 257 number of discrete narrow channels, using matched asymptotic expansions to determine
 258 overall scattering. The formulation and results are described in a supplementary report,¹⁶.
 259 Not only is this approach able to accurately reproduce the qualitative behaviour of the
 260 reflection and transmission coefficients seen in Fig. 5, but it indicates that there are as
 261 many zeros of transmission as there are micro-channels in the cavity. Even the envelope of
 262 oscillations suggested by Fig. 5 is captured accurately. Thus the oscillations increase as the
 263 number of finite channels increase (so that their width decreases in proportion) and we are
 264 led to the conclusion that a converged solution to the undamped continuum metamaterial
 265 cavity does not exist.

266 The addition of damping regularises the convergence. In Figs. 6(a,b) we show the reflected
 267 wave coefficients and the damping coefficient, η , for the same parameters as in Fig. 5 but

268 with $\mu = k + 0.04i\sqrt{k}$, $\mu = k + 0.01i\sqrt{k}$ and $\mu = k + 0.0025i\sqrt{k}$. The curves are produced
 269 with truncation parameters $P = 16$, $P = 32$ and $P = 128$, respectively. It can be seen
 270 that as the imaginary part of μ tends to zero, the results converge (although the numerical
 271 scheme has to work harder to achieve this) but *not* to a solution for zero damping. In fact,
 272 $\eta \rightarrow 0$, as the damping parameter tends to zero in all non-resonant intervals of ka . Over
 273 intervals of ka where there is resonance (e.g. $1.309 < ka < 1.963$, $3.927 < ka < 5.890$ in
 274 Fig. 6) in the metamaterial cavity the damping coefficient η converges to non-zero values
 275 and forms a well-defined curve.

276 We now turn our attention to the potential practical application of this device which
 277 is to act as an acoustic damper. In Fig. 7 we have plotted the damping coefficient, η ,
 278 and the scattering coefficients for a tapered array with $b/a = 2$, $m = 0.25$, $c/a = 4$ and
 279 $\mu = k + 0.05i\sqrt{k}$. Thus, the horizontal extent of the cavity is 8 times the waveguide width, the
 280 longest micro-channel is twice the waveguide width and the cavity tapers to micro-channels
 281 of zero length. This configuration means there is resonance in the cavity for all $ka > \frac{1}{4}\pi$ and
 282 we see that damping is close to 100% for a broad range of values of ka extending from $\frac{1}{4}\pi$
 283 dropping slowly as ka increases beyond π . As already noted in relation to Fig. 6, the shape
 284 of the damping coefficient curve is quite robust to changes in the damping parameter.

285 To some extent, the shape and size of the metamaterial cavity does not affect the high
 286 absorption demonstrated in Fig. 7. By way of example, in Fig. 8 we have extended the
 287 depth of the cavity by setting $b/a = 3$, retaining $c/a = 4$ using $m = 0.5$ to taper the length
 288 of the micro-channels from four times the waveguide width down to zero. Cavity resonances
 289 now extend beyond $ka = \frac{1}{8}\pi \approx 0.39$ and in Fig. 8 a damping parameter of $\mu = 1 + 0.1i$ has

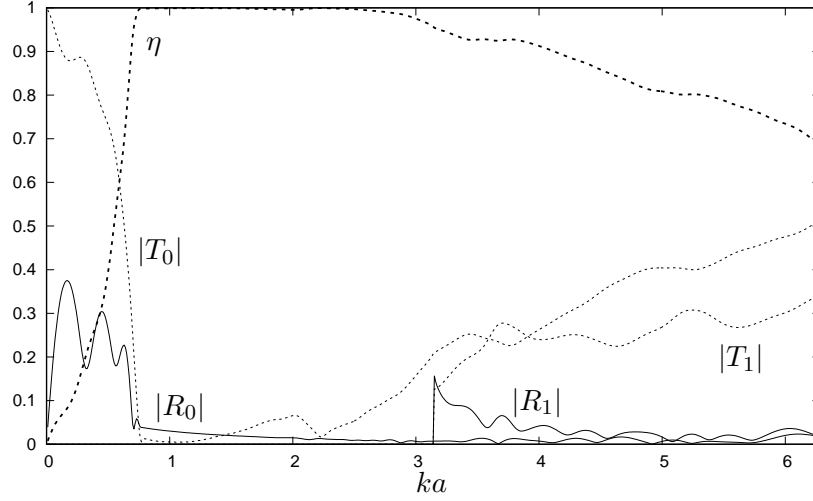


FIG. 7. Variation of η (thick dotted curve) and scattering coefficients with ka for a tapered metamaterial cavity $b/a = 2$, $c/a = 4$, $m = 0.25$ with damping $\mu = k + 0.05i\sqrt{k}$.

290 been used to demonstrate once again that high absorption can be achieved (over 98% of the
 291 acoustic energy is damped over $0.4 < ka < 2.83$).

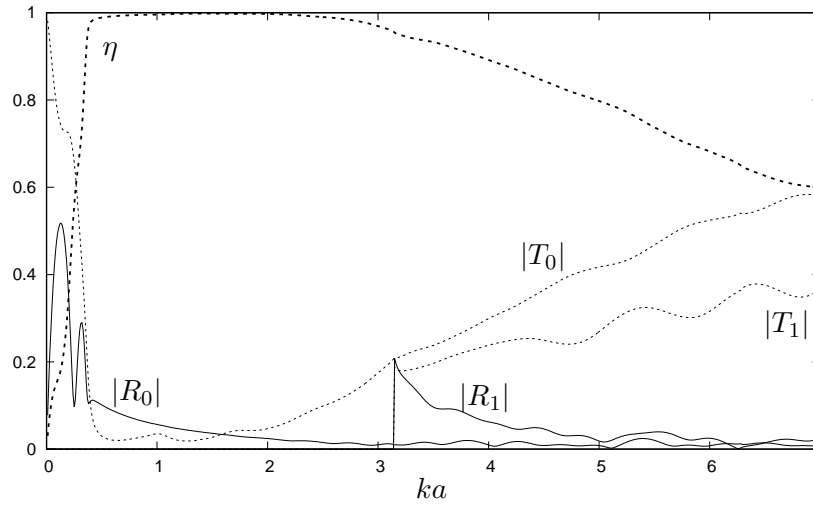


FIG. 8. Variation of η (thick dotted curve) and scattering coefficients with ka for a tapered metamaterial cavity $b/a = 3$, $c/a = 4$, $m = 0.5$ with damping $\mu = k + 0.1i\sqrt{k}$.

292 **V. CONCLUSIONS**

293 We have presented a simplified mathematical model of a microstructured plate-array
294 metamaterial cavity of a type commonly used in applications of rainbow trapping. The
295 cavity has been attached to the sidewall of a waveguide and its effect on acoustic wave
296 propagation has been considered. The simplified model for the cavity has allowed us to
297 express important features of the problem such as the scattering coefficients and acoustic
298 absorption in terms of the solution of a simple integral equation.

299 The main purpose of the problem was to consider the efficacy of a tapered metamaterial
300 cavity as a model of a rainbow trapping absorbing device to provide a broadbanded damping
301 of acoustic energy. However, many interesting features of the solution have emerged in the
302 process, relating to resonance in the case where the damping is set to zero. In particular,
303 we have shown that the effective medium/continuum model produces anomalous results
304 when resonance is encountered; in a rectangular metamaterial cavity oscillations in the
305 scattering coefficients increase in frequency without bound as isolated resonant parameters
306 are approached but the limiting case at resonance parameters is well-defined. On the other
307 hand, for a lossless tapered metamaterial cavity possessing a continuous range of resonant
308 parameters the effective medium model appears to be at fault. Numerical results fail to
309 converge, consistent with discrete models of micro-channelled cavities,¹⁶. A continuum model
310 which includes damping *does* converge numerically for a fixed damping parameter and as
311 this tends to zero results converge, though not to the solution of a zero-damping problem.

312 Other results have demonstrated that close to 100% of ducted acoustic wave energy
 313 can be damped by a tapered array over a broad range of frequencies suggesting that the
 314 metamaterial cavity is an extremely effective broadbanded absorber. Work is ongoing on
 315 using the continuum model to construct absorbing surfaces using tapered metamaterial
 316 cavities.

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321 APPENDIX: COMPUTATION OF INTEGRALS

322 From (23) we substitute $l \rightarrow -l$ for $l < 0$ to write

$$M_{pq} = \frac{2}{\pi} \int_0^\infty \frac{\coth \gamma a}{\gamma} S_{pq}(l) dl \quad (\text{A.1})$$

323 where

$$S_{pq}(l) = \frac{[(lc)^2 + pq\pi^2] \sin^2(lc)}{[(lc)^2 - (p\pi)^2][(lc)^2 - (q\pi)^2]}. \quad (\text{A.2})$$

324 and the contour of integration has been defined to pass below the poles at $l = \alpha_n$, $n =$
 325 $0, 1, \dots, N$. The value of N and hence the number of poles is dependent on ka but there
 326 always exists a pole at $l = k$ corresponding to $n = 0$.

327 Integrals with contours passing below the poles are evaluated as principal-value integrals
 328 plus half-residues from the vanishingly-small semi-circular indentations of the contour around

329 the poles. The principal-value integral at $l = k$ is dealt with by organising the integral in a
 330 form suitable for numerical quadrature with

$$\oint_0^{2k} f(l) dl = \int_0^k (f(l) + f(2k - l)) dl. \quad (\text{A.3})$$

331 To treat any remaining principal-value evaluations at $l = \alpha_n$ for $n \geq 1$ we write

$$\oint_0^k \frac{f(l)}{g(l)} dl = \int_0^k \left(\frac{f(l)}{g(l)} - \frac{f(\alpha_n)}{(k - \alpha_n)g'(\alpha_n)} \right) dl + \log \left(\frac{k - \alpha_n}{\alpha_n} \right) \frac{f(\alpha_n)}{g'(\alpha_n)} \quad (\text{A.4})$$

332 where it is assumed that $g(\alpha_n) = 0$ so that the integrand on the right-hand side is now
 333 bounded as $l \rightarrow \alpha$. With these tricks in place we may write (A.1) as

$$\begin{aligned} M_{pq} = & \frac{2}{\pi} \int_{2k}^{\infty} \frac{\coth(\sqrt{l^2 - k^2}a)}{\sqrt{l^2 - k^2}} S_{pq}(l) dl \\ & + \frac{2}{\pi} \int_0^k \left[\frac{\coth(\sqrt{(2k - l)^2 - k^2}a)}{\sqrt{(2k - l)^2 - k^2}} S_{pq}(2k - l) - \frac{\cot(\sqrt{k^2 - l^2}a)}{\sqrt{k^2 - l^2}} S_{pq}(l) \right. \\ & \left. - \sum_{n=1}^N \frac{S_{pq}(\alpha_n)}{\alpha_n a (l - \alpha_n)} \right] dl + \frac{i}{ka} S_{pq}(k) + \frac{2}{\pi} \sum_{n=1}^N \frac{(\pi i + \log(k/\alpha_n - 1))}{\alpha_n a} S_{pq}(\alpha_n) \end{aligned} \quad (\text{A.5})$$

334 which includes the evaluations from semi-circular indentations below the poles.

335 The integrand in the real integral over $0 < l < 1$ in (A.5) is smooth and bounded
 336 everywhere and can be computed using a standard numerical quadrature. The integrand in
 337 the real semi-infinite integral in (A.5) decays like $O(1/l^3)$ and is approximated by truncating
 338 the upper limit to $l = 400$.

339 In the case of $m = 0$, L_{pq} is explicit. For $m \neq 0$ and μ complex the complex-valued
 340 integral defined by (23) can be performed by numerical quadrature.

341 However, when $m \neq 0$ and μ is real special care may be required owing to the fact
 342 that the integrand may contain singularities. In such an instance (23) will be defined as

343 principal-value type and we use the same procedure outlined above of subtracting and adding
 344 singularities to get

$$\begin{aligned}
 L_{pq} = & \frac{(-1)^{p+q}}{2\mu c^2} \int_{-c}^c \left(e^{i\pi(p-q)x/c} \cot \mu(b-a+mx) - \sum_{r=1}^R \frac{e^{i\pi(p-q)x_r/c}}{\mu m(x-x_r)} \right) dx \\
 & + \frac{(-1)^{p+q}}{2\mu^2 m c^2} \sum_{r=1}^R e^{i\pi(p-q)x_r/c} \log \left(\frac{c-x_r}{c+x_r} \right)
 \end{aligned} \tag{A.6}$$

345 where $x_r \in (-c, c)$ satisfy $\sin \mu(b-a+mx_r) = 0$, $r = 1, \dots, R$. If no such x_r exists the sums
 346 in (A.6) are removed and the original integral in (23) is done directly.

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