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Bubble bursting: universal cavity and jet profiles

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After a bubble bursts at a liquid surface, the collapse of the cavity generates capillary waves, which focus on the axis of symmetry to produce a jet. The cavity and jet dynamics are primarily controlled by a non-dimensional number that compares capillary inertia and viscous forces, i.e. the Laplace number \( \text{La}= \frac{\rho \gamma R_0/\mu^2}{\rho \gamma R_0/\mu^2} \), where \( \rho, \mu, \gamma \) and \( R_0 \) are the liquid density, viscosity, interfacial tension, and the initial bubble radius, respectively. In this paper, we show that the time-dependent profiles of cavity collapse \( (t < t_0) \) and jet formation \( (t > t_0) \) both obey a \( (t - t_0)^{2/3} \) inviscid scaling, which results from a balance between surface tension and inertial forces. Moreover, we present a scaling law, valid above a critical Laplace number, which reconciles the time-dependent scaling with the recent scaling theory that links the Laplace number to the final jet velocity and ejected droplet size. This leads to a self-similar formula which describes the history of the jetting process, from cavity collapse to droplet formation.

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Bubbles bursting is ubiquitous in everyday life and is important for the climate, owing to the exchange of gas, water, heat, and chemical species between ocean and atmosphere [1–7]. After a bubble bursts, a jet forms and liquid drops detached from the jet are emitted to the atmosphere. The aerosol of drops (size ranges from 1 to 100 \( \mu m \)) that stays in the atmosphere is crucial since it regulates atmospheric chemistry [8], threatens human health by sending marine biotoxins and viruses to the atmosphere [9–11], and affects earth’s radiation balance, cloud and ice crystals formation and precipitation [7, 12–15]. Due to the broad impact across a range of research fields, the dynamics of bubble bursting has been an active area of research for the past 60 years.

Much recent progress was made linking jet dynamics with the physical properties of the liquid. Ghabache et al. [16, 17] developed scaling laws for the jet velocity as a function of the size of the jet drop, the liquid properties and the initial size of the mother bubble. A set of scaling laws for the jet velocity, the radial and axial length of the jet as a function of the liquid properties have been developed using a force and energy argument [18] and the effects of gravity were investigated theoretically [19]. The effect of gravity on jet velocity and the critical conditions for ejection of jet drops were examined numerically by Deike et al. [20].

Zeff et al. [21] showed that a liquid-water interface, before a surface wave collapses at time \( t_0 \), is self-similar and obeys a \( (t_0 - t)^{2/3} \) inertial-capillary scaling law, which has been shown to apply to cavity collapse [16, 22] and jet formation in certain regime [23]. However, the existing \( (t_0 - t)^{2/3} \) scaling laws only describe the bursting dynamics for a given set of fluid properties. The connection between the time-dependent self-similar scaling and a global scaling that involves fluid properties (e.g. the Ganan-Calvo scaling [18]) has not been addressed. Here, we present a universal scaling law for the dynamics of both cavity collapse \( (t < t_0) \) and jet formation \( (t > t_0) \), which incorporates both the time-dependent 2/3 scaling law and the Ganan-Calvo scaling [18], to describe the liquid-gas interface as a function of time \( t \), liquid properties (viscosity \( \mu \), interfacial tension \( \gamma \), and density \( \rho \)), and the initial bubble radius \( R_0 \).

We simulate numerically the dynamics of bubbles bursting using the open source solver Gerris with an adaptive mesh [24, 25], which has yielded excellent agree-

FIG. 1: The time evolution of the liquid-gas interface of a bursting bubble of initial radius \( R_0 \) at \( \text{La} = 2000 \) and \( \text{Bo} = 10^{-3} \). The color bar indicates the time corresponding to different profiles. (a) The formation of a jet after the collapse of capillary waves. The time difference between the curves are \( \delta t/\tau = 0.038 \), where \( \tau = \sqrt{\rho R_0/\gamma} \) is the inertia-capillary time scale. The time when a jet ejects a drop is \( t_d \). (b) The interface profiles right before a bubble is entrained. The time difference between the curves is \( \delta t/\tau = 0.0003 \). As the front of the capillary wave approaches \( r = 0 \) the interface steepens, snaps, entrains a bubble, and forms a jet at \( t = t_0 \). The bottom location of the bubble at the entrapment is denoted with \( h_0 \).
ment with experimental results [20, 23]. We assume an axisymmetric system and solve the full two-phase Navier-Stokes equation. For a bubble with initial radius $R_0$ in a liquid with density $\rho$, interfacial tension $\gamma$ and viscosity $\mu$, the relevant dimensionless numbers are the Bond number $Bo \equiv \rho g R_0^2 / \gamma$ (relative importance of gravitational forces compared to surface tension forces) and the Laplace number $La \equiv \rho \gamma R_0 / \mu^2 = 1/Oh^2$ [20] (relative importance of surface tension forces to viscous forces), where $Oh$ is the Ohnesorge number. The initial static bubble shape depends only on $Bo$, and is computed by solving the Young-Laplace equations [26]. The time evolution of the liquid-gas interface $h(r,t)$ for $La = 2000$ and $Bo = 10^{-3}$, solved with a grid size up to 4096$^2$ and 819 grid points across the bubble diameter, is plotted in Fig. 1(a).

After a bubble bursts, capillary waves travel along the interface towards the bottom of the bubble cavity. A jet is formed when the capillary waves collapse and the curvature of the interface reverses at a time $t_0$. For $La \geq 500$ [20, 27], a jet drop detaches from the top of the jet at $t_d$ after the jet grows to a certain length. For $La < 500$ no drops detach from the liquid jet. The cavity profiles near the curvature reversal (dashed window in Fig. 1(a)) right before the jet forms at $t_0$ is plotted in Fig. 1(b). The time difference between the curves is $\delta t / \tau = 0.0003$, where $\tau \equiv \sqrt{\rho R_0^3 / \gamma}$ is the inertial-capillary time scale. As time approaches $t_0$, the interface steepens and snaps, entraps a bubble and forms a jet at $t_0$. The lowest position of the profile at $t_0$ is located at $h = h_0$ and $r = 0$. In this paper we focus on the cases where $Bo = 10^{-3}$ (corresponds to a bubble of radius 85 $\mu$m in water) so that the effects from gravity are negligible. Note that the jet velocity for $Bo = 10^{-3}$ and $10^{-2}$ are the same [20] and converge to the asymptotic limit where $Bo = 0$.

Assuming that during the curvature reversal the inertial forces are of the same orders of magnitude as the surface tension forces and the viscous forces, and that the initial surface energy of the bubbles supplies the viscous dissipation in the capillary waves and the kinetic energy in the jet formation, Ganan-Calvo obtained relationships involving the dimensionless parameter $\varphi \equiv \sqrt{La} (\sqrt{Ln/La^2} - 1)$ [18]. For a jet near curvature reversal with a typical vertical speed $V$, radial speed $V'$, radial length scale $R$, and vertical length scale $L$,

$$V/V_\mu = k_v \varphi^{-3/4},$$

$$V'/V_\mu = k_v \varphi^{-1/2},$$

$$R/\ell_\mu = k_d \varphi^{5/4},$$

$$L/\ell_\mu = k_f \varphi,$$

where $V_\mu \equiv \gamma / \mu$ and $\ell_\mu \equiv \mu^2 / \rho \gamma$. For different $La$ and $Bo = 10^{-3}$, we obtain numerically the drop radius $R_d$, the length of the jet $L_d$ (defined in Fig. 2(a)) and the velocity of the drop $V_d$ when a jet drop detaches at $t_d$.

The numerical and experimental results of drop velocity $V_d$ and radius $R_d$ have been shown to agree well with Eqsns. (1) and (3), respectively, with fitted pre-factors $k_v \approx 16$ and $k_d \approx 0.6$ [18, 20]. For Eqn. (4), we choose the length of the jet $L_d$ at $t_d$ to be the axial length scale $L$ and obtain a good agreement between the numerical result and the scaling law with a numerical pre-factor $k_f \approx 50$, as shown in Fig. 2 (a) by the solid line.

A typical time scale for jet formation can be quantified using the time difference between the jet formation at $t_0$ and drop ejection $t_d$, i.e. $t_d - t_0$. A natural time scale for the flow in the axial direction, using Eqsns. (1) and (4), is $L/V \sim \varphi^{7/4}$. Therefore, we obtain a scaling relation for the axial time scale $(t_d - t_0)$,

$$\frac{(t_d - t_0)}{\ell_\mu/V_\mu} = k_t \varphi^{7/4},$$

which is in excellent agreement with our numerical results. Eqn. (5) fitted to the numerical results ($k_t \approx 2$) is shown by the solid line in Fig. 2(b). This discussion confirms the robustness of Ganan-Calvo’s scaling law [18] to characterize the jet variables at drop detachment.

Now, we include the time-dependent dynamics in the scaling arguments (Eqsns. (1)-(5)). The free surface of the
liquid-air interface \((z = h(r, t))\) prior to cavity collapse \((t = t_0)\), assuming the flows are incompressible and irrotational, have been shown numerically and experimentally to be self-similar,

\[
h(r, t) = (t_0 - t)^{2/3} f(r(t_0 - t)^{-2/3}), \tag{6}
\]

where \(f\) is a function of the shape of the surface profile. Below we show that Eqn. (6) not only applies to cavity collapse but also the formation of the liquid jet for a wide range of \(La\).

First, we combine the time-dependent scaling (Eqn. (6)) with Eqns. (4) and (5), which includes the dependence of jet profiles \(h(r, t)\) on liquid properties \((\mu, \gamma, \rho)\). In Fig. 2 we show that the axial length scale \(L\) of the jet scales like \(\ell \varphi\) and the axial time scale of the jet \((t_d - t_0) \approx \varphi^{7/4} \ell / V\varphi\). Therefore, length and time in

\[
\frac{r}{\ell \varphi} / \left(\frac{t-t_0}{\varphi^{7/4} \ell / V\varphi}\right)^{2/3}.
\]

FIG. 3: \((a)\) The time evolution of the liquid-gas interface during jet formation \((t > t_0)\) for \(La = 1000, 1500, 2000, 5000, 20000, 50000\) at times \(t - t_0 = t_j, 3/2t_j, 2t_j, 5/2t_j\), where \(t_j \equiv \varphi^{7/4} \ell / V\varphi\) is the characteristic time of jet formation. \((b)\) The dimensionless jet profiles rescaled using Eqn. (7). The profiles are shifted in the axial direction with respect to the bottom of the jet \(h_0\). The dimensionless profiles for different times and \(La = 5000 - 50000\) collapse, except for the region of the rounded tip. When \(La \leq 2000\) (i.e. \(\varphi \leq 45\)) the time scale \(t_j\) used here deviates lightly from the jet lifetime \(t_d - t_0\) (see Fig. 2(b)), and therefore affects the collapse of the jet onto the universal profile.

Eqn. (6) can be rescaled using the characteristic length scale \(\ell \varphi\) and time scale \(t_j \equiv \varphi^{7/4} \ell / V\varphi\), respectively. Therefore the dimensionless interface profiles during jet formation \((t > t_0)\) can be written as

\[
\frac{h - h_0}{\ell \varphi} = \left(\frac{t-t_0}{t_j}\right)^{2/3} g_a \left[\frac{r}{\ell \varphi} \left(\frac{t-t_0}{t_j}\right)^{-2/3}\right], \tag{7}
\]

in which \(g_a\) is the dimensionless shape of the profile after the curvature reversal and \(h_0\) is the bottom location of the jet. To test Eqn. (7) we plot the jet profiles for a range of Laplace number \(La = 1000 - 50000\) at times \(t - t_0 = t_j, 3/2t_j, 2t_j, 5/2t_j\) in Fig. 3(a). After non-dimensionalizing Fig. 3(a) using Eqn. (7), the dimensionless jet profiles for different parameters \(La\) and different times collapse except for a region near the rounded tip.
as shown by Fig. 3(b). Very close to the curvature reversal time during \((t - t_0) < t_c\) the profiles fail to collapse, since the jet is comprised mostly by its rounded tip. We note that the size of the rounded tip of the jet does not vary much with time and is roughly the same size as the jet drop \(R_d\), and thus scales as \(R \approx \ell \phi^{3/4}\) (Eqn. (3)). While the size of the rounded tip and the jet drop are set by the radial length scale \(R\) at \(t = t_0\), the radius of the jet body at long times scales the same way as the axial capillary length \(L \approx \ell \phi\), as predicted by Eqn. (6).

On the other hand, before the cavity collapses \((t < t_0)\), capillary waves travel in the radial direction with a characteristic velocity \(V'\) (Eqn. (2)). The time scale for the capillary wave to reach the center, according to Eqns. (2) and (4), can be estimated as \(L/V' \approx \phi^{3/2} \ell \mu / V\). Therefore we define a characteristic time for the traveling capillary wave as \(t_c = \phi^{3/2} \ell \mu / V\). The cavity profiles at \(La = 1000 - 20000\) and \(t_0 - t = 6t_c, 8t_c, 10t_c\) are plotted in Fig. 4(b). For \(La = 1000 - 2000\) only one capillary wave travels on the free surface, while for \(La \geq 5000\) multiple capillary waves are observed. Since the length scale of the capillary wave near the curvature reversal is set by the Ganan-Calvo length scale \(L \approx \ell \phi\) (Eeqn. (4)), we non-dimensionalize the \((t_0 - t)^{2/3}\) self-similar cavity profiles (Eqn. (6)) with the characteristic length scale \(\ell \phi\) and radial time scale \(t_c\) of the capillary waves.

We propose that the dimensionless cavity profiles \((t < t_0)\) right before the capillary waves collapse obey

\[
\frac{h - h_0}{\ell \phi} = \left(\frac{t_0 - t}{t_c}\right)^{2/3} g_b \left[\frac{r}{\ell \phi} \left(\frac{t_0 - t}{t_c}\right)^{-2/3}\right],
\]

where \(g_b\) is the dimensionless shape of the profile before the curvature reversal and \(h_0\) (defined in Fig. 2(a)) is the location of the bottom of the bubble, also the location at which the capillary wave approaches at \(t_0\). The cavity profiles in Fig. 4(a) rescaled using the universal self-similar scaling (Eqn. (8)) collapse for different \(La\) and a period of time \((6t_c \leq t_0 - t \leq 10t_c)\) near \(t_0\), as shown in Fig. 4(b). At \(La \geq 20000\) multiple capillary waves appear and Eqn. (8) fails to collapse the profiles. During \(t_0 - t < 6t_c\) the time is too close to the moment of curvature reversal, and the profiles deviate from the universal shape due to bubble pinch-off.

The differences between the dimensionless profiles for cavity collapse (Eqn. (8), where \(t < t_0\)) and jet formation (Eqn. (7), where \(t > t_0\)) are the choice of the timescale. Therefore, we can rewrite both dimensionless profiles, Eqns. (8) and (7), as

\[
\mathcal{H}(\mathcal{R}, \mathcal{T}) = \mathcal{T}^{2/3} g_{a,b}(\mathcal{R} \mathcal{T}^{-2/3})
\]

where \(\mathcal{R} \equiv r/(\ell \phi)\) is the dimensionless width. For cavity collapse the dimensionless length and time are \(\mathcal{H} \equiv (h(r, t) - h_0)/(\ell \phi)\) and \(\mathcal{T} \equiv (t_0 - t)/t_c\), respectively, where \(t_c \equiv \ell \phi^{3/2}/V\) is the time scale of the traveling capillary wave. For jet formation, \(\mathcal{H} \equiv (h(r, t) - h_b)/(\ell \phi)\) and \(\mathcal{T} \equiv (t_0 - t)/t_j\), where \(t_j \equiv \ell \phi^{7/4}/V\) is the time scale of jet formation. The parameters that were found numerically are the bottom position \(h_0(t)\) of the jet, the bottom position \(h_0\) of the bubble, the time \(t_0\) when the curvature reverses, and the critical \(La\) for ejection of jet drops \(La^*\). Eqn. (9) agrees well with the dynamics of both cavity collapse (Fig. 3) and jet formation (Fig. 4) and connects the dynamics with time \(t\), liquid properties \((\mu, \gamma, \rho)\) and initial bubble radius \(R_0\).

The dynamics of the free surface away from the bubble entrainment time \(t_0\) obeys the \([t - t_0]^{2/3}\) scaling for inviscid, incompressible and irrotational flows. In our numerical simulation, we estimate the forces in the Navier-Stokes equation and find that at times far away from \(t_0\), the viscous forces are small compared with the inertia and surface tension forces. At \(t_0\) the viscous forces reach the same order of magnitude as the inertia and surface tension forces, in regions near high interface curvature. The effects of viscosity come into play during the curvature reversal at \(t_0\), which set the length and velocity scales (Eqns. (1)-(4)) of the flows. Therefore, Eqn. (9) successfully collapse the interface profiles for both \(t < t_0\) and \(t > t_0\) at different times and across a range of \(La\).

In conclusion, we study the self-similar dynamics of a bursting bubble. We report the power-law dependence of the jet length and the time scale of the jet formation on the dimensionless parameter \(\phi\). The jet length \(h_d\) at the moment when a jet drop detaches is proportional to \(\phi\), agreeing well with Ganan-Calvo’s scaling. The time from jet formation to drop detachment obeys \(t_d - t_0 \propto \phi^{7/4}\). Using proper length and time scales, we propose a scaling law to describe the dynamics of both cavity collapse and jet formation as a function of time, liquid properties and the initial bubble size. We show that for a certain range of time and Laplace number, the dimensionless interfacial profiles collapse, exhibiting self-similar dynamics and good agreement with the universal scaling law.

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