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CONSTRUCTING A POLYNOMIAL WHOSE NODAL SET IS ANY PRESCRIBED KNOT OR LINK

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ABSTRACT

We describe an algorithm that for every given braid B explicitly constructs a function $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ such that f is a polynomial in u, v and \bar{v} and the zero level set of f on the unit three-sphere is the closure of B . The nature of this construction allows us to prove certain properties of the constructed polynomials. In particular, we provide bounds on the degree of f in terms of braid data.

Keywords: applied topology; real algebraic knot theory; braids; Morse-Novikov number; knotted fields; constructive approach to knot theory

Mathematics Subject Classification 2000: 57M25

1. Introduction

There are various settings in which links can arise in zero level sets of polynomials. Among these the most prominent one is the study of links of isolated singularities [27]. While only a very restricted class of links can be the link of an isolated singularity, for some types of polynomials their existence is known for any link. It is for example known that for any link there is a polynomial $f : \mathbb{R}^3 \rightarrow \mathbb{C}$ in three real variables with complex coefficients such that $f^{-1}(0)$ is the given link. However, explicit constructions are known for only very few links. Dealing with explicit functions can be advantageous for studying some aspects of knot theory more closely. The critical points of the argument $\arg(f) : S^3 \setminus L \rightarrow S^1$ for example relate to circle-valued Morse-Novikov theory, which then connects to Reidemeister torsion and Seiberg-Witten invariants [20]. Explicit polynomials are also very useful for applications in the construction of knotted field configurations in physics, in \mathbb{R}^3 . The goal of this paper is to describe a construction of such polynomials that works for any link.

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Knots as vanishing sets of polynomials have been mostly studied in the context of isolated singularities of a polynomial $\mathbb{C}^2 \rightarrow \mathbb{C}$. For all small $\varepsilon > 0$ the intersection of the zero level set with the three-sphere of radius ε around the singularity is ambient isotopic to the same link, the link of the singularity [27]. For the links that arise from isolated singularities of complex polynomials, that is certain iterated cables of torus links [10,12,13,23,21,36], one can explicitly write down the corresponding polynomial. However, this procedure only covers a very restricted class of links. For a more detailed account of this topic we point the reader to Refs. [27,18].

Similar considerations apply to polynomials $\mathbb{R}^4 \rightarrow \mathbb{C}$ with (weakly) isolated singularities [1,27], but while some knots like the figure-eight knot have been explicitly constructed in this setting [29,31,24], a constructive method that allows the generation of polynomials with a zero level set of any given link explicitly is still missing. Akbulut and King showed [1] that all links arise as links of weakly isolated singularities of real polynomials. However, their proof does not offer an explicit construction of these functions either.

Knots have also been studied as real algebraic curves in $\mathbb{R}\mathbb{P}^3$ in an area that is now known as real algebraic knot theory. The knots that arise in this way for polynomials of degree less than or equal to five or equal to six with genus at most one have been classified [6,26].

It follows from results in real algebraic geometry [7,1] that for any link L there exists a polynomial in three real variables with complex coefficients, whose vanishing set is L , but the proofs do not come with a method of finding these functions without using algebraic approximation, which is highly impractical in general. In particular, the degree of these polynomials could be arbitrarily large. The functions constructed by Brauner, Perron and Rudolph can be used to generate such knotted algebraic sets in \mathbb{R}^3 simply by applying stereographic projection and multiplying by a nowhere-vanishing common denominator for the special cases discussed above. In general, however, we are confronted with a similar situation as in the case of the result by Akbulut and King. While the existence of the polynomials on \mathbb{R}^3 vanishing on L is known for any link L , explicit constructions are known for only very few of them. In this paper we describe a way of constructing such polynomials that works for any link.

In fact, as in the previous constructions, we create a function which is defined on four-dimensional real space and is known to have the desired nodal set on S^3 , then we apply stereographic projection to it. In order to do this, an isolated singularity at the origin is not necessary. All that is required is that the intersection of the unit three-sphere with the zero level set of the function, which is defined on \mathbb{R}^4 (or \mathbb{C}^2 or $\mathbb{C} \times \mathbb{R}^2$), is the desired knot or link.

Rudolph introduced the notion of transverse \mathbb{C} -links [33]. These are the links that arise as transverse intersections of zero level sets of complex polynomials $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ with the unit three-sphere. These functions do not necessarily have an isolated singularity at the origin and the zero level set on three-spheres of small

radii does not have to be that same link. The set of transverse \mathbb{C} -links is the same as the set of quasipositive links, which are closures of braids that are products of conjugates of positive Artin generators σ_i of the braid group [33,9].

We weaken Rudolph's condition on the functions whose nodal sets are transverse \mathbb{C} -links to complex-valued polynomials in complex variables u, v and \bar{v} . We call these polynomials *semiholomorphic*, since they are holomorphic in u , but not in v .

In Refs. [8,15] we described a construction of semiholomorphic polynomials with zero level sets on S^3 in the form of lemniscate knots, a family of knots generalising the family of torus links. The main ingredient of this procedure is that lemniscate knots are closures of braids that have a particularly simple parametrisation in terms of trigonometric functions. In Ref. [14] we showed using the example of the knot 5_2 that by using geometric considerations one can find a similar trigonometric parametrisation for a braid even if its closure is not a lemniscate knot. However, this approach is not algorithmic and can be very time consuming.

In this paper we show that the idea behind the construction of lemniscate knots can be extended to all links. We prove the following theorem.

Theorem 1.1. *For every link L there is a function $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ such that $f^{-1}(0) \cap S^3 = L$ and f is a polynomial in complex variables u, v and \bar{v} , i.e. $f(u, v, \bar{v})$ is a semiholomorphic polynomial.*

The proof provides an algorithm that for any given braid B constructs a semiholomorphic polynomial with the closure of B as its zero level set on the unit-three sphere. Note that stereographic projection from S^3 to \mathbb{R}^3 then allows us to turn the constructed polynomial $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ into a polynomial $\tilde{f} : \mathbb{R}^3 \rightarrow \mathbb{C}$ with the desired shape of its nodal lines.

Having an explicit formalism to generate polynomials comes with the advantage of being able to show certain properties about them. We can for example give upper and lower bounds on the degree of the resulting polynomial f in terms of braid data, the number of strands in each link component and the number of crossings between them. This result carries over to bounds on the degree of \tilde{f} , improving results associated to the Nash-Tognoli theorem [7], which do not contain any information about the degree. We prove the following theorem.

Theorem 1.2. *Let B be a braid on s strands of length ℓ and let L denote its closure. Then there exists a function $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ such that*

- $f^{-1}(0) \cap S^3 = L$,
- f is a polynomial in u, v and \bar{v} , i.e. it is semiholomorphic,
- f is harmonic, i.e. $(\partial_u \partial_{\bar{u}} + \partial_v \partial_{\bar{v}})f = 0$,
- the intersection of $f^{-1}(0)$ and S^3 is transverse and 0 is a regular value of $f|_{S^3}$,
- $\max\{s, c_1\} \leq \deg(f) \leq \max\{s, c_2\}$,

where c_1 and c_2 are expressions involving the number of strands in each link component and the number of crossings between them and are defined later in Eq. (4.16)

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and Eq. (4.15) respectively.

The constructed polynomials should also find applications in the study of a wide range of physical systems, including quantum-mechanical wavefunctions [30], complex scalar optical fields [15], nematic liquid crystals [25] and the Skyrme-Faddeev model [35], where they are used as initial configurations. Our algorithm hence provides a way of creating a knotted initial condition for any link type in any of these systems.

The structure of this paper is as follows. Section 2 introduces the concept of Fourier parametrisations of braids, which are essential for the construction. We then show how we can use such a parametrisation to construct a semiholomorphic polynomial with a zero level set of the desired form. In Section 3 we discuss a method of finding a Fourier braid parametrisation for any braid, making our construction fully algorithmic. In Section 4 we use the nature of our construction to prove bounds for the degree of the resulting polynomials, among other results. We also give an upper bound for the Morse-Novikov number of a link and show that if the desired link is the closure of a strictly homogeneous braid, then there exists a semiholomorphic polynomial f such that the argument $\arg(f)$ of the restriction of f to the unit three-sphere is a fibration over S^1 .

2. Semiholomorphic polynomials with knotted nodal sets

The idea behind the construction of the polynomials is very much in the spirit of Refs. [8,14], but is not restricted to any particular class of knots and is fully algorithmic. We use certain trigonometric parametrisations of a given braid B to define a 1-parameter family of functions from $\mathbb{C} \times [0, 2\pi]$ to \mathbb{C} whose zero level sets are B . Identifying the 2π -periodic variable with the argument of a complex number v allows us to close the braid and define a 1-parameter family of functions whose zero level set is the closure of B for appropriate choices of the parameter. The parametrisation of the braid is chosen in such a way that the resulting function is a polynomial in u , v and \bar{v} .

We will think of a braid B on s strands as the union of s disjoint parametric curves in \mathbb{R}^3 , parametrised by their z -coordinate between 0 and 2π . This means that each strand is given by

$$(X_j(t), Y_j(t), t), \quad j = 0, 1, \dots, s-1, \quad (2.1)$$

where each X_j and each Y_j is a smooth function $[0, 2\pi] \rightarrow \mathbb{R}$.

Additionally, we demand that there is a permutation $\pi_B \in S_s$, such that $X_j(0) = X_{\pi_B(j)}(2\pi)$ and $Y_j(0) = Y_{\pi_B(j)}(2\pi)$ for all $j \in \{1, 2, \dots, s\}$. Given these equalities, we can identify the $z = 0$ plane and the $z = 2\pi$ plane, which closes the braid to a link L . Note that there is a one-to-one correspondence between the cycles of the permutation π_B and the components of the link L . Furthermore, the length

of each cycle is equal to the number of strands making up the corresponding link component.

Projecting the strands $(X_j(t), Y_j(t), t)$ on the $y = 0$ plane will typically result in a braid diagram that allows to read off the braid word of B , that is the element of the braid group on s strands B_s corresponding to B in terms of the Artin generators σ_j .

We want to describe an algorithm that for any given link L constructs a semi-holomorphic polynomial $f : \mathbb{C} \times \mathbb{R}^2 \rightarrow \mathbb{C}$ such that $f^{-1}(0) \cap S^3 = L$. This algorithm is a generalisation of the methods used in Refs. [29,15,8]. Let L be the link we want to construct and B be a braid on s strands that closes to L .

The first step of the algorithm is to find a parametrisation of the braid B as in Eq. (2.1). Then we can define a family of functions $g_{a,b} : \mathbb{C} \times [0, 2\pi] \rightarrow \mathbb{C}$,

$$g_{a,b}(u, t) = \prod_{j=0}^{s-1} (u - aX_j(t) - i bY_j(t)) \quad (2.2)$$

which has the braid B as its zero level set $g_{a,b}^{-1}(0)$ for any $a > 0, b > 0$.

Every $g_{a,b}$ is by definition a polynomial in the complex variable u , but its dependence on t is determined by the parametrisation. In Section 2.1 we show that if the braid parametrisation is of a certain form, $g_{a,b}$ is a polynomial in u, e^{it} and e^{-it} . As such it is the restriction of a semiholomorphic polynomial $f_{a,b} : \mathbb{C}^2 \rightarrow \mathbb{C}$ to $\mathbb{C} \times S^1$. Note that $f_{a,b}$ can be obtained from $g_{a,b}$ simply by replacing every instance of e^{it} by a complex variable v and e^{-it} by \bar{v} , the complex conjugate of v . Section 2.2 shows that for small $\lambda > 0$ the zero level set of $f_{\lambda a, \lambda b}$ on the unit three-sphere is ambient isotopic to L , i.e. $f_{\lambda a, \lambda b}^{-1}(0) \cap S^3 = L$.

2.1. Fourier Braids

In this section we describe a braid parametrisation as in Eq. (2.1) of a certain form that guarantees that $g_{a,b}$ defined by Eq. (2.2) can be written as a polynomial in u, e^{it} and e^{-it} for all a and b .

Recall that we can associate, to every braid B with s strands, an element π_B of the symmetric group S_s on s elements. The cycles of π_B correspond to the link components of the closure of B . We denote the set of cycles of π_B , or equivalently the set of components of L , by \mathcal{C} . For a given cycle C let s_C denote the length of C or equivalently the number of strands that form the link component C .

The condition that $X_j(0) = X_{\pi_B(j)}(2\pi)$ and $Y_j(0) = Y_{\pi_B(j)}(2\pi)$ for all j ensures that the projection of the braid on the xy -plane is a collection of closed curves, one for each link component. Thus for each link component C there exist 2π -periodic, continuous real functions F_C and G_C such that for every strand j we have $X_j(t) = F_C\left(\frac{t+2\pi k}{s_C}\right)$, $Y_j(t) = G_C\left(\frac{t+2\pi k}{s_C}\right)$, $X_{\pi_B(j)}(t) = F_C\left(\frac{t+2\pi(k+1)}{s_C}\right)$ and $Y_{\pi_B(j)}(t) = G_C\left(\frac{t+2\pi(k+1)}{s_C}\right)$ for some $C \in \mathcal{C}$ and $1 \leq k \leq s_C$.

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Let F_C and G_C be trigonometric polynomials, i.e.

$$F_C(t) = \sum_{k=-N_C}^{N_C} a_{C,k} e^{ikt} \quad \text{and} \quad G_C(t) = \sum_{k=-M_C}^{M_C} b_{C,k} e^{ikt} \quad (2.3)$$

with $a_{C,-k} = \bar{a}_{C,k}$ and $b_{C,-k} = \bar{b}_{C,k}$ for all $C \in \mathcal{C}$ and all k . Suppose

$$\bigcup_{C \in \mathcal{C}} \bigcup_{j=0}^{s_C-1} \left(F_C \left(\frac{t+2\pi j}{s_C} \right), G_C \left(\frac{t+2\pi j}{s_C} \right), t \right), \quad t \in [0, 2\pi] \quad (2.4)$$

is a parametrisation of the braid B as in Eq. (2.1). Then as in Eq. (2.2) this parametrisation with $X_j(t) = F_C \left(\frac{t+2\pi k}{s_C} \right)$ and $Y_j(t) = G_C \left(\frac{t+2\pi k}{s_C} \right)$ for some $C \in \mathcal{C}$ and $1 \leq k \leq s_C$ leads to a function $g_{a,b} : \mathbb{C} \times [0, 2\pi] \rightarrow \mathbb{C}$ with B as its zero level set for all $a > 0$, $b > 0$ and we claim it is in fact a polynomial.

Lemma 2.1. *The function $g_{a,b}$ can be written as a polynomial in u , e^{it} and e^{-it} .*

Proof. Obviously all exponents of u are natural numbers, so we only have to show that after expanding the product in Eq. (2.2) all exponents of e^{it} are integers.

We will show this for knots, so there is only one cycle in π_B . Since $g_{a,b}$ is the product over the functions associated with each cycle, this will show that $g_{a,b}$ is a product of polynomials and hence a polynomial itself for any braid. We now write s for s_C , the number of strands of the component C that we focus on.

Since $g_{a,b}$ has exactly s factors, every term of $g_{a,b}$, after expanding the product, consists of exactly s terms which can each be u or $c_m e^{\frac{im(t+2\pi j)}{s}}$ for some non-zero integer m and $c_m \in \mathbb{C}$. Hence every summand has the form

$$u^{s-k} T_{j_1, \dots, j_k}^{m_1, \dots, m_k} = u^{s-k} \prod_{p=1}^k c_{m_p} e^{\frac{im_p(t+2\pi j_p)}{s}} \quad (2.5)$$

with $j_p \neq j_r$ if $p \neq r$.

Moreover, if $T_{j_1, \dots, j_k}^{m_1, \dots, m_k}$ appears with the factor u^{s-k} , then so do all $T_{\tilde{j}_1, \dots, \tilde{j}_k}^{m_1, \dots, m_k}$ with $\tilde{j}_i \in \{0, \dots, s-1\}$, $\tilde{j}_i \neq \tilde{j}_r$ if $i \neq r$. Note that if $m_i = m_w$, then $T_{j_1, \dots, j_i, \dots, j_w, \dots, j_k}^{m_1, \dots, m_i, \dots, m_w, \dots, m_k} = T_{j_1, \dots, j_w, \dots, j_i, \dots, j_k}^{m_1, \dots, m_i, \dots, m_w, \dots, m_k}$. This turns the proof of the lemma into a combinatorial problem. Summing over all summands with the same m_i becomes summing over all possibilities of choosing p distinct values j_p between 1 and s .

We will use induction on $k = s - l$, where l is the exponent of u in the relevant term, to show that all terms $T_{j_1, \dots, j_k}^{m_1, \dots, m_k}$ with $s \nmid \sum_{i=1}^k m_i$ cancel each other, i.e.

$$\sum_{\substack{(j_1, j_2, \dots, j_k) \in (\mathbb{Z}/s\mathbb{Z})^k \\ j_i \text{ disjoint}}} T_{j_1, \dots, j_k}^{m_1, \dots, m_k} = 0. \quad (2.6)$$

Note that this implies that all summands that involve non-integer exponents of e^{it} enter the sum with a factor equal to zero. Hence $g_{a,b}$ can be written as a polynomial in u , e^{it} and e^{-it} .

Consider a term T_j^m which comes with a factor of u^{s-1} , so there is only one term which is not u (i.e. $k = 1$), so $T_j^m = c_m e^{\frac{i m(t+2\pi j)}{s}}$ for some $j = 0, \dots, s-1$. Then T_j^m must appear with the factor u^{s-1} for all values of j . We get

$$\sum_{j=0}^{s-1} T_j^m = c_m e^{i m t/s} \sum_{j=0}^{s-1} e^{2\pi i m j/s}, \quad (2.7)$$

where the last sum is 0 if $s \nmid m$ which shows the statement for $k = 1$.

Assume now there is a $k \in \{1, 2, \dots, s-1\}$ such that all terms $T_{j_1, \dots, j_k}^{m_1, \dots, m_k}$ with $s \nmid \sum_{i=1}^k m_i$ cancel each other, i.e.

$$\sum_{\substack{j_1=1 \\ j_2 \neq j_1}}^s \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^s \dots \sum_{\substack{j_k=1 \\ j_k \neq j_1, j_2, \dots, j_{k-1}}}^s T_{j_1, \dots, j_k}^{m_1, \dots, m_k} = 0. \quad (2.8)$$

Let $T_{j_1, \dots, j_{k+1}}^{m_1, \dots, m_{k+1}}$ be a term with $s \nmid \sum_{i=1}^k m_i$. Summing over all terms with the same (m_1, \dots, m_{k+1}) , but different choices of (j_1, \dots, j_{k+1}) yields

$$\sum_{\substack{(j_1, j_2, \dots, j_{k+1}) \\ \in (\mathbb{Z}/s\mathbb{Z})^{k+1} \\ j_i \text{ disjoint}}} T_{j_1, \dots, j_{k+1}}^{m_1, \dots, m_{k+1}} = e^{i t \sum_{p=1}^{k+1} \frac{m_p}{s}} \left(\prod_{p=1}^{k+1} c_{m_p} \right) \sum_{\substack{(j_1, j_2, \dots, j_{k+1}) \\ \in (\mathbb{Z}/s\mathbb{Z})^{k+1} \\ j_i \text{ disjoint}}} \prod_{p=1}^{k+1} e^{\frac{2\pi i m_p j_p}{s}}. \quad (2.9)$$

There is at least one m_p which is not divisible by s . Otherwise $s \mid \sum_{p=1}^{k+1} m_p$. Without loss of generality, we can assume that $s \nmid m_{k+1}$. Then $\sum_{j=1}^s e^{\frac{2\pi i m_{k+1} j}{s}} = 0$ and hence $e^{\frac{2\pi i m_p j_p}{s}} = - \sum_{p=1}^k e^{\frac{2\pi i m_{k+1} j_p}{s}} - \sum_{\substack{w=1 \\ w \neq j_1, \dots, j_{k+1}}}^s e^{\frac{2\pi i m_{k+1} w}{s}}$. Thus Eq. (2.9) equals

$$\begin{aligned} & e^{i t \sum_{p=1}^{k+1} \frac{m_p}{s}} \left(\prod_{p=1}^{k+1} c_{m_p} \right) \sum_{\substack{(j_1, j_2, \dots, j_{k+1}) \\ \in (\mathbb{Z}/s\mathbb{Z})^{k+1} \\ j_i \text{ disjoint}}} \left[\prod_{p=1}^k e^{\frac{2\pi i m_p j_p}{s}} \right. \\ & \left. \times \left(- \sum_{p=1}^k e^{\frac{2\pi i m_{k+1} j_p}{s}} - \sum_{\substack{w=1 \\ w \neq j_1, \dots, j_{k+1}}}^s e^{\frac{2\pi i m_{k+1} w}{s}} \right) \right] = T_1 + T_2, \quad (2.10) \end{aligned}$$

with

$$T_1 = - e^{i t \sum_{p=1}^{k+1} \frac{m_p}{s}} \left(\prod_{p=1}^{k+1} c_{m_p} \right) \sum_{\substack{(j_1, j_2, \dots, j_{k+1}) \\ \in (\mathbb{Z}/s\mathbb{Z})^{k+1} \\ j_i \text{ disjoint}}} \sum_{r=1}^k e^{2\pi i (m_r + m_{k+1}) j_r} \prod_{\substack{p=1 \\ p \neq r}}^k e^{\frac{2\pi i m_p j_p}{s}} \quad (2.11)$$

and

$$\begin{aligned}
 T_2 &= -e^{it} \sum_{p=1}^{k+1} \frac{m_p}{s} \left(\prod_{p=1}^{k+1} c_{m_p} \right) \sum_{\substack{(j_1, j_2, \dots, j_{k+1}) \\ \in (\mathbb{Z}/s\mathbb{Z})^{k+1} \\ j_i \text{ disjoint}}} \sum_{\substack{w=1 \\ w \neq j_1, \dots, j_{k+1}}}^s e^{\frac{2\pi i m_{k+1} w}{s}} \prod_{p=1}^k e^{\frac{2\pi i m_p j_p}{s}} \\
 &= -e^{it} \sum_{p=1}^{k+1} \frac{m_p}{s} \left(\prod_{p=1}^{k+1} c_{m_p} \right) \sum_{\substack{(j_1, j_2, \dots, j_{k+1}, w) \\ \in (\mathbb{Z}/s\mathbb{Z})^{k+2} \\ w, j_i \text{ all disjoint}}} e^{\frac{2\pi i m_{k+1} w}{s}} \prod_{p=1}^k e^{\frac{2\pi i m_p j_p}{s}} \\
 &= -(s - (k + 1)) e^{it} \sum_{p=1}^{k+1} \frac{m_p}{s} \left(\prod_{p=1}^{k+1} c_{m_p} \right) \sum_{\substack{(j_1, j_2, \dots, j_k, w) \\ \in (\mathbb{Z}/s\mathbb{Z})^{k+1} \\ w, j_i \text{ all disjoint}}} e^{\frac{2\pi i m_{k+1} w}{s}} \prod_{p=1}^k e^{\frac{2\pi i m_p j_p}{s}}, \tag{2.12}
 \end{aligned}$$

where the last equality follows from the fact that the summand

$$e^{\frac{2\pi i m_{k+1} w}{s}} \prod_{p=1}^k e^{\frac{2\pi i m_p j_p}{s}} \tag{2.13}$$

in Equation (2.12) does not depend on j_{k+1} and for each $(j_1, j_2, \dots, j_k, w)$, there are $s - (k + 1)$ choices for j_{k+1} .

For every fixed r the first term T_1 describes $-\sum_{\substack{(j_1, j_2, \dots, j_k) \\ \in (\mathbb{Z}/s\mathbb{Z})^k \\ j_i \text{ disjoint}}} T_{j_1, \dots, j_k}^{m_1, \dots, m_r + m_{k+1}, \dots, m_k}$,

which vanishes by the induction hypothesis (2.8), since $s \nmid \sum_{p=1}^{k+1} m_p = (m_r + m_{k+1}) + \sum_{\substack{p=1 \\ p \neq r}}^k m_p$ for all r .

Hence Eq. (2.10) becomes

$$\sum_{\substack{(j_1, j_2, \dots, j_{k+1}) \\ \in (\mathbb{Z}/s\mathbb{Z})^{k+1} \\ j_i \text{ disjoint}}} T_{j_1, \dots, j_{k+1}}^{m_1, \dots, m_{k+1}} = -(s - (k + 1)) \sum_{\substack{(j_1, j_2, \dots, j_{k+1}) \\ \in (\mathbb{Z}/s\mathbb{Z})^{k+1} \\ j_i \text{ disjoint}}} T_{j_1, \dots, j_{k+1}}^{m_1, \dots, m_{k+1}}. \tag{2.14}$$

It is $k + 1 \leq s$ and hence

$$\sum_{j_1=1}^s \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^s \dots \sum_{\substack{j_{k+1}=1 \\ j_{k+1} \neq j_1, \dots, j_k}}^s T_{j_1, \dots, j_{k+1}}^{m_1, \dots, m_{k+1}} = 0. \tag{2.15}$$

This completes the proof for $k + 1$ and proves the lemma by induction.

Hence all terms of $g_{a,b}$ where e^{it} has a non-integer exponent cancel each other when we expand the product. Therefore $g_{a,b}$ is a polynomial in u , e^{it} and e^{-it} . \square

As a polynomial in u , e^{it} and e^{-it} the function $g_{a,b}$ is the restriction of a polynomial $f_{a,b} : \mathbb{C}^2 \rightarrow \mathbb{C}$ in u, v and \bar{v} to $\mathbb{C} \times S^1$. Thus $f_{a,b}(u, e^{it}) = g_{a,b}(u, t)$ and

by construction the zero level set of $f_{a,b}$ on $\mathbb{C} \times S^1$ is equal to the closure of the braid $g_{a,b}^{-1}(0)$ for every $a > 0$, $b > 0$. The polynomial $f_{a,b}$ can be constructed from $g_{a,b}$ in the following way. We expand the product in Eq. (2.2), so that $g_{a,b}$ has the form of a polynomial in u , e^{it} and e^{-it} . Then we replace every instance of e^{it} by v and every instance of e^{-it} by \bar{v} . Since $f_{a,b}(u, e^{it}) = g_{a,b}(u, t)$ for all $u \in \mathbb{C}$ and $t \in [0, 2\pi]$, the zero level set of $f_{a,b}$ on $\mathbb{C} \times S^1$ is the closure of the braid B parametrised by Eq. (2.4). Performing this substitution in Eq. (2.2) before the product is expanded results in a different function, which is typically not a polynomial in u , v and \bar{v} .

Note that substituting t with nt in F_C and G_C for some $n \in \mathbb{N}$ results in a parametrisation of n repeats of the original braid. Thus if $f_{a,b}$ has a zero level set on $\mathbb{C} \times S^1$ which is of the form of the closure of a braid B , then multiplying each exponent of v and \bar{v} by r gives a semiholomorphic polynomial with zero level set on $\mathbb{C} \times S^1$ which is the closure of B^n .

The classes of knots that were discussed in Ref. [8] can now be seen as particularly simple examples, since lemniscate knots are exactly the closures of repeats of braids given by parametrisations of the form of Eq. (2.4) with $F(t) = \cos(t)$ and $G(t) = \sin(\ell t)$ for some $\ell \in \mathbb{N}$ coprime to the number of strands s , generalising the torus knots with $\ell = 1$. The spiral knots, first introduced in Ref. [11], are exactly the closures of repeats of braids that can be parametrised by Eq. (2.4) with $F(t) = \cos(t)$ and $G(t)$ some trigonometric polynomial.

By Alexander's Theorem [2] every link is the closure of a braid and since trigonometric polynomials are dense in the set of continuous 2π -periodic functions from \mathbb{R} to \mathbb{R} , every braid can be parametrised as in Eq. (2.4). Hence for every link L there exists a family of semiholomorphic polynomials $f_{a,b}$ with $f_{a,b}^{-1}(0) \cap (\mathbb{C} \times S^1) = L$ for all $a > 0$, $b > 0$.

In Section 2.2 we show that for small enough a and b , it is in fact $f_{a,b}^{-1}(0) \cap S^3 = L$ as desired.

2.2. The proof of Theorem 1.1

Let $f_{a,b} : \mathbb{C}^2 \rightarrow \mathbb{C}$ be the polynomial constructed from $g_{a,b}$, so $f_{a,b}^{-1}(0) \cap (\mathbb{C} \times S^1)$ is the closure of the braid that is $g_{a,b}^{-1}(0)$. We will make this statement more precise. Consider the map $\Psi : \mathring{D} \times (\mathbb{C} \setminus \{0\}) \rightarrow S^3$, $\Psi(u, r e^{it}) = (u, \sqrt{1 - |u|^2} e^{it})$, where \mathring{D} is the interior of the unit disk in the complex plane. We think of $f_{a,b}$ as a family of polynomials, parametrised by $v = r e^{it}$, a and b , in one complex variable u . In the following we set $a = \lambda a_1$ and $b = \lambda b_1$, leave a_1 and b_1 as real constants and only vary the real parameter λ . With this notation we write f_λ instead of $f_{a,b}$, slightly abusing notation. For fixed r , t and λ , we denote the s roots of $f_\lambda(\bullet, r e^{it})$ by $u_{\lambda,j}(r, t)$. Note that by definition $u_{\lambda,j}(r, t) = \lambda u_{1,j}(r, t)$, $j = 1, 2, \dots, s$. Thus if λ is small enough, all roots $u_{\lambda,j}(1, e^{it})$ of $f_\lambda(\bullet, e^{it})$ lie in \mathring{D} . It is intuitively clear and can be shown that $\Psi(f_\lambda^{-1}(0) \cap (\mathbb{C} \times S^1)) = \Psi((u_{\lambda,j}(1, e^{it}), e^{it}))$ is the closure of $B = g_{a,b}^{-1}(0)$, which is L . This idea can be found among others in Ref. [29].

In order to show that $f_\lambda^{-1}(0) \cap S^3$ is L as well, we need to show that there is an ambient isotopy between $L_1 = \Psi(f_\lambda^{-1}(0) \cap (\mathbb{C} \times S^1))$ and $L_2 = f_\lambda^{-1}(0) \cap S^3$.

Definition 2.1. Two links L_1 and L_2 are ambient isotopic if there is a continuous function $F : S^3 \times [0, 1] \rightarrow S^3$, such that $F(\cdot, 0)$ is the identity map, $F(L_1, 1) = L_2$ and $F(\cdot, s)$ is a homeomorphism for all $s \in [0, 1]$.

In fact by the Isotopy Extension Theorem, it is enough to construct a smooth isotopy between the two sets of curves.

Theorem 2.2. [16] *Let $I : S^1 \cup \dots \cup S^1 \times [0, 1] \rightarrow S^3$ be a smooth 1-parameter family of embeddings of n circles in S^3 , such that $I(S^1, \dots, S^1, 0) = L_1$ and $I(S^1, \dots, S^1, 1) = L_2$, then there exists an ambient isotopy $\tilde{I} : S^3 \times [0, 1] \rightarrow S^3$ with $\tilde{I}(L_1, s) = I(S^1, \dots, S^1, s)$.*

For fixed λ , j and t we think of the roots $u_{\lambda,j}(r, t)$ as functions of r . Note that the union of the intersection points of the roots $u_{\lambda,j}(r, t)$ with S^3 is equal to the zero level set of f_λ on S^3 . We would like to see that for every fixed λ , j and t there is a unique intersection point of $(u_{\lambda,j}(r, t)r e^{i h})$ with S^3 , so that there is a one-to-one correspondence between the points in $f_\lambda^{-1}(0) \cap S^3$ and $f_\lambda^{-1}(0) \cap (\mathbb{C} \times S^1)$. To do this we first need to restrict the range of r to a domain where all the roots are disjoint. This allows us to treat the roots $u_{\lambda,j}(r, t)$ as smooth functions of r .

Lemma 2.2. *There is a $\delta > 0$ independent of λ such that if $u_{\lambda,j}(r_1, t) = u_{\lambda,k}(r_2, t)$ with $r_1, r_2 \in [1 - \delta, 1]$, then $j = k$.*

Proof. Note that it is enough to show the lemma for $\lambda = 1$, since $u_{\lambda,j}(r, t) = \lambda u_{1,j}(r, t)$, so that $u_{\lambda,j}(r_1, t) = u_{\lambda,k}(r_2, t)$ if and only if $u_{1,j}(r_1, t) = u_{1,k}(r_2, t)$. In particular the value that we find for δ when $\lambda = 1$ will be sufficient for any choice of $\lambda > 0$.

Since the roots of $g_{\lambda a, \lambda b}(\cdot, t)$ are by definition disjoint for every t and every $\lambda > 0$, the roots $u_{\lambda,j}(1, t)$ of $f_\lambda(\cdot, e^{i t})$ are disjoint. By continuity of the roots of a polynomial with respect to its coefficients, they remain disjoint on an open neighbourhood of $r = 1$ for all λ and t .

Let R be the biggest value of r for which different strands intersect for $\lambda = 1$, that is $R = \max_{t \in [0, 2\pi], j \neq k} \{r \mid u_{1,j}(r, t) = u_{1,k}(r, t), r \leq 1\}$. Since $u_{\lambda,j}(r, t) = \lambda u_{1,j}(r, t)$ for all λ , j , r and t , all the roots $u_{\lambda,j}(r, t)$ are simple roots of $f_\lambda(\bullet, r e^{i t})$ as long as $r \in [R, 1]$ and hence $\text{Re}(u_{\lambda,j}(r, t))$ and $\text{Im}(u_{\lambda,j}(r, t))$ are differentiable functions with respect to $r \in [R, 1]$. Note that by the remark above $R < 1$, so $[R, 1]$ is a proper interval.

The Implicit Function Theorem allows us to calculate these derivatives in terms of $\frac{\partial f_a(\cdot, r e^{i t})}{\partial u}$ and $\frac{\partial f_a(\cdot, r e^{i t})}{\partial r}$. Let $D_1(\delta) = \max_{t \in [0, 2\pi], j, r \in [1 - \delta, 1]} \left\{ \left| \frac{\partial \text{Re}(u_{1,j}(r, t))}{\partial r} \right| \right\}$ and $D_2(\delta) = \max_{t \in [0, 2\pi], j, r \in [1 - \delta, 1]} \left\{ \left| \frac{\partial \text{Im}(u_{1,j}(r, t))}{\partial r} \right| \right\}$ for $\delta \in [0, 1 - R]$.

Take $\tilde{\delta}$ to be $\frac{1}{2} \min_{t \in [0, 2\pi]} \{ |u_{1,j}(1, t) - u_{1,k}(1, t)| \}$ and choose $\delta < \frac{\tilde{\delta}}{\sqrt{D_1(\delta)^2 + D_2(\delta)^2}}$.

Note that as δ approaches zero, the right hand side of the inequality converges to a non-zero value. Hence such a δ always exists. The Lemma follows then from

$$\begin{aligned} |u_{1,j}(r_1, t) - u_{1,k}(r_2, t)| &\geq |u_{1,j}(1, t) - u_{1,k}(1, t)| - |u_{1,j}(1, t) - u_{1,j}(r_1, t)| \\ &\quad - |u_{1,k}(1, t) - u_{1,k}(r_2, t)| \\ &\geq |u_{1,j}(1, t) - u_{1,k}(1, t)| - 2\delta |D_1(\delta) + i D_2(\delta)| \\ &> |u_{1,j}(1, t) - u_{1,k}(1, t)| - 2\tilde{\delta} \geq 0 \end{aligned} \quad (2.16)$$

for all $r \in [1 - \delta, 1]$ and $j \neq k$. \square

We use the following lemma to show that for every fixed λ , j and t the intersection point of $(u_{\lambda,j}(r, t), r e^{it})$ with S^3 is unique if λ is small enough.

Lemma 2.3. *There exists $\varepsilon_1 > 0$ such that for every fixed $\lambda < \varepsilon_1$ and all fixed j and t there is at most one intersection of $(u_{\lambda,j}(r, t), r e^{it})$ with S^3 where $1 - \delta \leq r \leq 1$ with δ as in Lemma 2.2.*

Proof. Lemma 2.2 implies that the roots $u_{\lambda,j}(r, t)$, $j = 1, 2, \dots, s$ are disjoint for $r \in [1 - \delta, 1]$. As simple roots of the polynomial $f_\lambda(\bullet, r e^{it})$, they depend smoothly on its coefficients, in particular $u_{\lambda,j}(r, t)$ is a smooth function of r for all $r \in [1 - \delta, 1]$.

Consider the function $|u_{\lambda,j}(r, t)|^2 + r^2 - 1 = \lambda^2 |u_{1,j}(r, t)|^2 + r^2 - 1$. This is a smooth function of r on $[1 - \delta, 1]$ and its zeros correspond to intersection points of $u_{\lambda,j}(r, t)$ with S^3 .

Now suppose a curve $(u_{\lambda,j}(r, t), r e^{it})$ with fixed λ , j and t has multiple intersection points with S^3 while $r \in [1 - \delta, 1]$. Then between these the function $|u_{\lambda,j}(r, t)|^2 + r^2 - 1$ must have an extremum.

We can choose λ small enough such that the derivative $\frac{d}{dr}(\lambda^2 |u_{1,j}(r, t)|^2 + r^2 - 1)$ is strictly positive for all $r \in [1 - \delta, 1]$. Hence for small λ , there is a unique intersection point with r in that interval.

In fact, we can find a sufficient value for ε_1 as follows.

We need to make sure that $\frac{d}{dr}(\lambda^2 |u_{1,j}(r, t)|^2 + r^2 - 1) > 0$ for all $r \in [1 - \delta, 1]$ and all $\lambda < \varepsilon_2$.

Since $f_\lambda(\bullet, r e^{it})$ is a polynomial in the first variable u , we may write $f_\lambda(u, r e^{it}) = \sum_{j=0}^s c_j \lambda^{s-j} u^j$, where $c_j \in \mathbb{C}$ depends on r and t . Note that it follows from Rouché's Theorem that

$$U = \max_{\substack{r \in [1 - \delta, 1] \\ j \in \{0, 1, \dots, s-1\} \\ t \in [0, 2\pi]}} \left\{ 1, \sum_{j=0}^{s-1} |c_j| \right\} \quad (2.17)$$

is a bound on the modulus of all roots of all polynomials $f_1(\bullet, r e^{it})$ with $r \in [1 - \delta, 1]$. Hence λU is an upper bound on the modulus of all roots

of all polynomials $f_\lambda(\bullet, r e^{it})$, which are by definition $u_{\lambda,j}(r, t)$. Define $\tilde{D}_1 = \max_{\substack{t \in [0, 2\pi] \\ j=0,1,\dots,s-1 \\ r \in [1-\delta, 1]}} \left\{ \left| \frac{\partial |\operatorname{Re}(u_{1,j}(r,t))|}{\partial r} \right| \right\}$ and $\tilde{D}_2 = \max_{\substack{t \in [0, 2\pi] \\ j=0,1,\dots,s-1 \\ r \in [1-\delta, 1]}} \left\{ \left| \frac{\partial |\operatorname{Im}(u_{1,j}(r,t))|}{\partial r} \right| \right\}$.

Note that in general $D_i \neq \tilde{D}_i$. With $\varepsilon_1 = \sqrt{\frac{(1-\delta)}{U(\tilde{D}_1 + \tilde{D}_2)}}$ we find

$$\frac{d}{dr}(\lambda^2 |u_{1,j}(r, t)|^2 + r^2 - 1) > 2r - \lambda^2 2U(\tilde{D}_1 + \tilde{D}_2) > 0, \quad (2.18)$$

as long as $\lambda < \varepsilon_1$ and $r \in [1 - \delta, 1]$. \square

Recall that δ from Lemma 2.2 does not depend on λ . This allows us to prove the following lemma.

Lemma 2.4. *There exists $\varepsilon_2 > 0$ such that for all $\lambda < \varepsilon_2$ all intersections of the curves $(u_{\lambda,j}(r, t), r e^{it})$ with S^3 occur with $r \in [1 - \delta, 1]$.*

Proof. With the definition of U above we can make sure that all intersection points of the roots $u_{\lambda,j}(r, t)$ with S^3 happen in the region where $r \in [1 - \delta, 1]$ if λ is small enough. We set $\varepsilon_2 = \frac{\sqrt{\delta(2-\delta)}}{U}$. Then for all $\lambda < \varepsilon_2$, we have

$$|u_{\lambda,j}(r, t)|^2 + r^2 < \lambda^2 U^2 + (1 - \delta)^2 < 1 \quad (2.19)$$

for all $r < 1 - \delta$ and hence all intersections with S^3 happen for $r \in [1 - \delta, 1]$. \square

Note that Eq. (2.19) also shows that for every $\lambda < \varepsilon_2$ and every fixed j and t , the curve $(u_{\lambda,j}(r, t), r e^{it})$ intersects S^3 , since $|u_{\lambda,j}(r, t)|^2 + r^2 \geq 1$ at $r = 1$. Combining this with Lemma 2.3 means that this intersection point is unique for every curve $(u_{\lambda,j}(r, t), r e^{it})$. Denote the value of r at which the intersection occurs by $r_{\lambda,j}(t)$.

Recall the definition of the map $\Psi : \mathring{D} \times (\mathbb{C} \setminus \{0\}) \rightarrow S^3$, $\Psi(u, r e^{it}) = (u, \sqrt{1 - |u|^2} e^{it})$. We now define a smooth isotopy between $\Psi(f_\lambda^{-1}(0) \cap (\mathbb{C} \times S^1))$, which we know to be the desired link, and $f_\lambda^{-1}(0) \cap S^3$.

Let $I : \underbrace{(S^1 \cup \dots \cup S^1)}_{|C| \text{ copies}} \times [0, 1]$ be defined by

$$I(\Psi((u_{\lambda,j}(1, t), e^{it})), s) = \Psi(u_{\lambda,j}(1 - s + sr_{\lambda,j}(t), t), (1 - s + sr_{\lambda,j}(t)) e^{it}). \quad (2.20)$$

Lemma 2.5. *There is a $\varepsilon > 0$ such that for all $\lambda < \varepsilon$ the function I is a smooth isotopy from $\Psi(f_\lambda^{-1}(0) \cap (\mathbb{C} \times S^1))$ to $f_\lambda^{-1}(0) \cap S^3$.*

Proof. We have to show that for every s the function $I(\bullet, s)$ is an embedding and that it is smooth with respect to s . Suppose there is an s such that $I(\bullet, s)$ is not an embedding, then there exist complex numbers $u_{\lambda,j}(1 - s + sr_{\lambda,j}(t), t) = u_{\lambda,k}(1 - s + sr_{\lambda,k}(t), t)$ with $j \neq k$, but Lemma 2.2 and Lemma 2.4 tell us that this does not happen if $\lambda < \min\{\varepsilon_1, \varepsilon_2\}$. Furthermore, since simple roots of a polynomial depend smoothly on its coefficients and the map Ψ is smooth, t is a smooth isotopy.

Note that

$$I(\Psi((u_{\lambda,j}(1,t), e^{it})), 0) = \Psi(u_{\lambda,j}(1,t), e^{it}) = \Psi(f_{\lambda}^{-1}(0) \cap (\mathbb{C} \times S^1)) \quad (2.21)$$

and

$$I(\Psi((u_{\lambda,j}(1,t), e^{it})), 1) = \Psi(u_{\lambda,j}(r_{\lambda,j}(t), t), r_{\lambda,j}(t) e^{it}) = f_{\lambda}^{-1}(0) \cap S^3 \quad (2.22)$$

which finishes the proof. Note that I is well-defined if $\lambda < \min\{\varepsilon_1, \varepsilon_2\}$, so we can set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. \square

The Isotopy Extension Theorem says that I extends to an ambient isotopy. Thus $f_{\lambda}^{-1}(0) \cap S^3$ is ambient isotopic to the desired link if λ is small enough.

Since every link is the closure of a braid by Alexander's Theorem [2] and for every braid B there is a family of functions f_{λ} , this concludes the proof of Theorem 1.1. We have shown that for every link L there is a function $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ such that $f^{-1}(0) \cap S^3 = L$ and f is a polynomial in complex variables u, v and \bar{v} , i.e. f is a semiholomorphic polynomial.

In Ref. [14] we describe the constructed semiholomorphic polynomials for many knots and links of low minimal crossing number. It turns out that in practice λ can be chosen to be a lot larger than the bound $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, which is given in the proof.

3. Finding Fourier parametrisations

In this section we present an algorithm that generates a parametrisation as in Eq. (2.4) for any given braid.

For every link component $C \in \mathcal{C}$ we obtain F_C and G_C by trigonometric interpolating data points that can be obtained from a presentation of the braid diagram. For background on trigonometric interpolation we point the reader to Ref. [3].

Step 1: Finding the data points for the trigonometric interpolation for F_C

We need to perform a trigonometric interpolation for F_C for each link component $C \in \mathcal{C}$. The data points for this interpolation are chosen such that they contain the information about the position of every strand of C between two crossings in the braid diagram.

Let s denote the number of strands and ℓ the length of the braid word. The given braid word allows us to draw a braid diagram of B . For convenience we draw the strands as piecewise linear curves with all crossing points evenly distributed along the t -axis as in Figure 1 (a). If we neglect the signs of the crossings, the diagram consists of s curves $(D_{C,j}(t), 0, t)$, $t \in [0, 2\pi]$, $C \in \mathcal{C}$, $j = 0, 1, \dots, s_C$ in the tx -plane, each of which can be interpreted as the graph of a function $D_{C,j}(t)$ (Figure 1 (b)). Since the crossing points are evenly distributed, they occur at $t_k = 2\pi(2k-1)/(2\ell)$ with $k = 1, 2, \dots, \ell$. The braid diagram is drawn in such a way that the value of

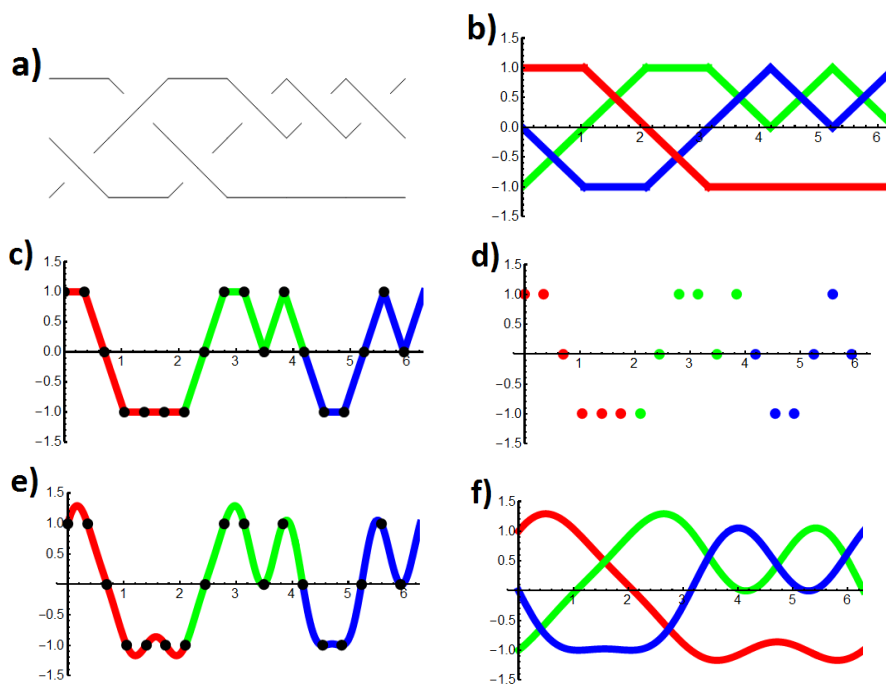


Fig. 1. The interpolation for F . a) The braid diagram of a minimal braid of 52. b) The projection of strands of the braid as piecewise linear curves in a coordinate system. c) The graph of the function D_C . d) The data points for the trigonometric interpolation. These can be read off from the graph of D_C . e) The graph of F , the trigonometric polynomial interpolating the data points. f) The graphs of the different X_j , when F is used to parametrise the x -coordinate of the braid. The crossing points in this plot give rise to the data points for the interpolation for G .

$D_{C,j}(2\pi)$ is equal to $D_{C,k}(0)$ for some k , say $k = j + 1$. This way each strand obtains a label (C, j) . This label is unique once for each component C an arbitrary strand of that component is chosen as the strand $(C, 1)$. We can draw the graph of the piecewise-linear function $D_C : [0, 2\pi] \rightarrow \mathbb{R}$, $D_C((t + 2\pi j)/s_C) = D_{C,j}(t)$ as in Figure 1 (c).

We can now perform a trigonometric interpolation through the points $(t_k/s_C - 2\pi/(2s_C\ell), D_C(t_k - 2\pi/(2\ell)))$, $k = 1, 2, \dots, s_C\ell$ (shown in Figure 1 (d)) for every component C to obtain $|C|$ trigonometric polynomials F_C that satisfy $F_C(t_k/s_C - 2\pi/(2s_C\ell)) = D_C(t_k - 2\pi/(2\ell))$ for every $k = 1, 2, \dots, s_C\ell$ and every link component $C \in \mathcal{C}$ (cf. Figure 1 (e)).

Step 2: Trigonometric interpolation for F_C

The trigonometric polynomials F_C can be explicitly constructed by using the

discrete Fourier transform

$$\tilde{D}_{C,k} = \frac{1}{s_C \ell} \sum_{n=0}^{s_C \ell - 1} D_C \left(t_n - \frac{2\pi}{2s_C \ell} \right) e^{-i \left(\frac{t_n}{s_C} - \frac{2\pi}{2s_C \ell} \right) k}. \quad (3.1)$$

We get

$$F_C(t) = \begin{cases} \tilde{D}_{C,0} + \sum_{k=1}^{s_C \ell / 2 - 1} 2 \operatorname{Re}(\tilde{D}_{C,k}) \cos(kt) - \sum_{k=1}^{s_C \ell / 2 - 1} 2 \operatorname{Im}(\tilde{D}_{C,k}) \sin(kt) \\ \quad + \tilde{D}_{C, s_C \ell / 2} \cos\left(\frac{s_C \ell}{2} t\right), & \text{if } s_C \ell \text{ is even,} \\ \tilde{D}_{C,0} + \sum_{k=1}^{(s_C \ell - 1) / 2} 2 \operatorname{Re}(\tilde{D}_{C,k}) \cos(kt) - \sum_{k=1}^{(s_C \ell - 1) / 2} 2 \operatorname{Im}(\tilde{D}_{C,k}) \sin(kt), \\ \quad \text{if } s_C \ell \text{ is odd.} \end{cases} \quad (3.2)$$

Since the example in Figure 1 is the knot 5_2 , there is only one link component. Figure 1 (e) shows the graph of the trigonometric polynomial F_C . The graphs of $F_C((t + 2\pi j)/s_C)$ in Figure 1 (f) form a braid diagram B' with unspecified signs of crossing. Note that in contrast to usual braid diagrams there might in general be more than two strands involved in a crossing and crossing strands might not be transverse.

Since the trigonometric polynomials F_C are interpolating the data points, there is a bijection between the strands of B' and the strands of the original braid B and for every k there is a crossing of B' in the interval $[t_k - 2\pi/(2\ell), t_{k+1} - 2\pi/(2\ell)]$ that involves the same strands as the crossing of B at t_k . This crossing might not be unique. However, for every pair of strands of B that is not crossing at t_k , there is an even number of crossings in the diagram B' between them in the interval $[t_k - 2\pi/(2\ell), t_{k+1} - 2\pi/(2\ell)]$ (counting multiplicities if a crossing is not transverse). For the pair of strands of B that is crossing at t_k , the corresponding strands of B' will cross an odd number of times in the interval $[t_k - 2\pi/(2\ell), t_{k+1} - 2\pi/(2\ell)]$ (counting multiplicities). This is due to the choice of data points for the trigonometric interpolation that store the information about the position of every strand of B between two crossings.

Step 3: Finding the data points for the trigonometric interpolation for G_C

Again we need to perform a trigonometric interpolation for each link component $C \in \mathcal{C}$. Recall that Step 2 resulted in B' , a braid diagram whose signs of crossings are not specified. By choosing data points whose t -coordinates are the positions of crossings in the braid diagram B' and choosing the y -coordinate appropriately, we attach signs to the crossings of B' in such a way that it becomes the diagram of a braid isotopic to B .

This can be achieved as follows. For every $k = 1, 2, \dots, \ell$ we choose a bijection w_k between the strands of B' and a set of s distinct real numbers such that the strand

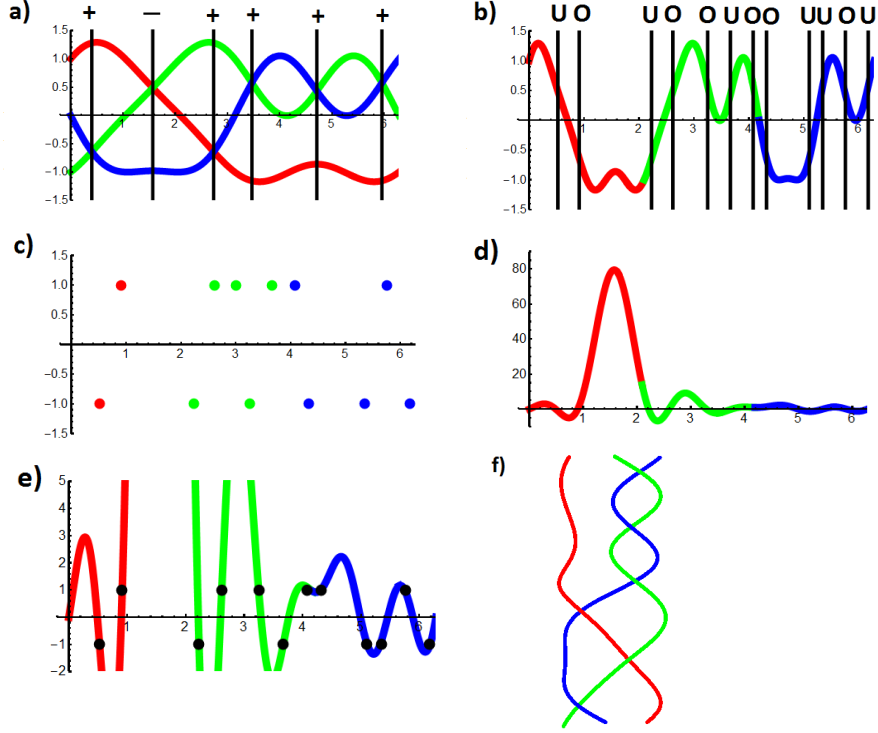


Fig. 2. The trigonometric interpolation to get G_C . a) The x -coordinate of s strands parametrised by F_C with marked positions of crossings and signs of crossings. b) The graph of F_C with marked positions of crossings and a label indicating whether the corresponding strand is the overpassing (O) or the underpassing (U) strand. c) The set of data points to be interpolated by G_C . Overpassing strands should have value +1 at the position of the crossings and underpassing strands the value -1. d) The graph of the interpolating trigonometric polynomial G_C . e) The graph of G_C passes through all data points. f) The braid parametrised by F_C and G_C (plotted in a left-handed coordinate system).

corresponding to the strand of B which is overcrossing at t_k gets assigned a larger number than the strand corresponding to the strand of B which is undercrossing at t_k .

This time the t -coordinate of the data points used for the trigonometric interpolation for G_C are the positions of the crossings that the strands $F_C(t/s_C + 2\pi j/s_C)$ are involved in. Let the strand (C, j) be involved in a crossing of B' at $t = t_0$, i.e. $F_C(t_0/s_C + 2\pi j/s_C) = F_{C'}(t_0/s_{C'} + 2\pi m/s_{C'})$ for some $C' \in \mathcal{C}$ and $m \in \{0, 1, \dots, s_{C'} - 1\}$. Then $((t_0 + 2\pi j)/s_C, w_k((C, j)))$ is a data point. Note that if a strand is not involved in a crossing in the k th interval, then there is no data point for this strand in this interval and hence its image under w_k does not affect the result at all. We denote the data points by (t'_k, y_k) . The number of data points for G_C depends on the number of crossings of B' that involve strands of the link

component C .

Each crossing creates a data point for each strand involved in the crossing. This choice of data points implies that any family of trigonometric polynomials G_C that interpolates these together with F_C from Step 1 gives a parametrisation of a braid as in Eq. (2.4) and this braid is braid isotopic to B . This can be seen as follows. Since the y -coordinate of the data points only depends on the strand and on k , all crossings between one pair of strands in the interval $[t_k - 2\pi/(2\ell), t_{k+1} - 2\pi/(2\ell)]$ in B' have the same strand as an overcrossing strand (and the other as the undercrossing strand). This and the previous observation about the parity of numbers of crossings between strands means that in every interval $[t_k - 2\pi/(2\ell), t_{k+1} - 2\pi/(2\ell)]$ all crossings but one can be canceled. The two strands that are involved in the remaining crossing correspond to the two strands of B crossing at t_k and by definition of w_k , they cross with the required sign.

Step 4: Trigonometric interpolation for G_C

Since the positions of the crossings of B' are in general not equidistributed, the trigonometric interpolation does not directly translate to a discrete Fourier Transform.

Finding a trigonometric polynomial that interpolates the data points (t'_k, y_k) is equivalent to finding a function $q(z) = \sum_{k=-M_C}^{M_C} c_k z^k$ with $q(e^{it'_k}) = y_k$ for all k , which is equivalent to finding a polynomial $\tilde{q} \sum_{k=0}^{2M_C} c_{k-K} z^k$ such that $q(e^{it'_k}) = y_k e^{iKt'_k}$ for all k and $c_{-k} = \bar{c}_k$ for all k .

We can find such a function by using the Lagrange formula for polynomial interpolation. This allows us to explicitly calculate G_C . If the number N of data points (t'_k, y_k) is odd, say $N = 2K + 1$, we get

$$q(e^{it}) = G_C(t) = \sum_{k=1}^N y_k e^{-iKt+iKt'_k} \prod_{\substack{m=1 \\ m \neq k}}^N \frac{e^{it} - e^{it'_m}}{e^{it'_k} - e^{it'_m}}. \quad (3.3)$$

If N is even, say $N = 2K$, the result is

$$q(e^{it}) = G_C(t) = \sum_{k=1}^N y_k e^{-iKt+iKt'_k} \frac{e^{it} - e^{i\alpha_k}}{e^{it'_k} - e^{i\alpha_k}} \prod_{\substack{m=1 \\ m \neq k}}^N \frac{e^{it} - e^{it'_m}}{e^{it'_k} - e^{it'_m}}, \quad (3.4)$$

where α_k is a free parameter that can be chosen to be 0 for all k .

Writing $q(e^{it}) = \sum_{k=-K}^K c_k e^{ikt}$ as a power series in e^{it} allows the coefficients of G_C to be computed for every link component $C \in \mathcal{C}$.

Then Eq. (2.4) with the trigonometric polynomials F_C and G_C obtained by trigonometric interpolation is a finite Fourier parametrisation of the desired braid.

In this construction several choices must be made, each of which leads to different trigonometric polynomials. There is first of all the choice of braid. We will see in Section 4 how the choice of braid affects the possible degrees of the resulting

trigonometric polynomials. It is not true in general that the simplest braid (in terms of number of crossings ℓ or number of strands s) leads to the simplest trigonometric polynomials (in terms of degree). However, we compute bounds for the degree of the trigonometric polynomials in terms of ℓ and s and these are strictly increasing with ℓ and s .

Secondly, there is the choice of embedding the linear braid diagram in \mathbb{R}^2 , which corresponds to the functions $D_{C,j}$. Demanding that between crossings the strands are equidistant and are arranged symmetrically around $x = 0$ seem like reasonable conditions, since a rather symmetric arrangement of data points suggests an interpolating trigonometric polynomial with few terms. The choice of function D_C also means that we choose a first strand for each link component. The algorithm works for any such choice.

Furthermore, the choice of functions w_k that are used to determine the data points for the interpolation for G_C do not affect the topology of the braid parametrised by the obtained trigonometric polynomials. Like in the choice of the embedding it appears reasonable to place the strands that are involved in crossings in the relevant interval symmetrically and with equal distances around $y = 0$.

In Ref. [8] we described two families of knots, lemniscate knots and spiral knots, whose particular symmetric braid words allowed comparatively simple trigonometric braid parametrisations.

Figure 1 illustrates our method for the example of the knot 5_2 , which is the simplest knot (in terms of minimal crossing number) that is not a lemniscate knot or a spiral knot. For this example we choose the braid $\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^3$ (Figure 1 (a)) which closes to the knot 5_2 . In fact it is a minimum braid word of 5_2 [19]. We read off braid words from braid diagram from left to right and strand positions are numerated from top to bottom. A positive crossing σ_i corresponds to the strand in position i passing over the strand in position $i + 1$ from the top.

Since 5_2 is a knot, there is only one component. To find F_C , we interpolate the data points (\tilde{t}_k, x_k) for $k = 0, 1, \dots, 17$, where $\tilde{t}_k = 2\pi k/18$ and x_k is the k th entry of $k = (1, 1, 0, -1, -1, -1, -1, 0, 1, 1, 0, 1, 0, -1, -1, 0, 1, 0)$. These data points can be read off from Figure 1 (c) and are displayed in Figure 1 (d).

The resulting interpolating trigonometric polynomial is

$$\begin{aligned} F(t) &= 0.108784 e^{-8it} + 0.0846189 i e^{-7it} + 0.24365 i e^{-5it} - 0.0886815 e^{-4it} \\ &\quad + 0.479898 e^{-2it} - 0.129644 i e^{-it} + 0.129644 i e^{it} + 0.479898 e^{2it} \\ &\quad - 0.0886815 e^{4it} - 0.24365 i e^{5it} - 0.0846189 i e^{7it} + 0.108784 e^{8it} \\ &= -0.259288 \sin(t) + 0.959796 \cos(2t) - 0.177363 \cos(4t) + 0.4873 \sin(5t) \\ &\quad + 0.169238 \sin(7t) + 0.217568 \cos(8t). \end{aligned} \tag{3.5}$$

The coefficients were computed numerically, so they may not definitely go through the interpolation points, but they give the braid correctly. The graph of F is plotted in Figure 1 (e). In order to find the data points for the trigonometric interpo-

lation of G , we need to compute the values of t for which $F(t+2\pi j/3) = F(t+2\pi k/3)$ for some $j, k \in \{0, 1, 2\}$.

There is only one crossing between the red strand ($j = 0$) and the green strand ($j = 1$), namely at $t = 0.523599$. Since this crossing corresponds to the crossing σ_1^{-1} in the original braid word, the green strand is supposed to pass over the red strand. Hence the points $(0.523599, -1)$ and $(0.523599 + 2\pi/3, 1)$ will be included in the list of data points used for the interpolation of G . Note that this corresponds to a bijection w_1 that sends the red strand to -1 and the green strand to 1 .

Similarly, we compute the intersection points of the other pairs of strands and obtain the list of data points $(0.523599, -1)$, $(0.912415, 1)$, $(0.134782 + 2\pi/3, -1)$, $(0.523599 + 2\pi/3, 1)$, $(1.15567 + 2\pi/3, 1)$, $(1.5708 + 2\pi/3, -1)$, $(1.98592 + 2\pi/3, 1)$, $(0.134782 + 4\pi/3, 1)$, $(0.912415 + 4\pi/3, -1)$, $(1.15567 + 4\pi/3, -1)$, $(1.5708 + 4\pi/3, 1)$, $(1.98592 + 4\pi/3, -1)$.

Performing a trigonometric interpolation through these points yields

$$\begin{aligned} G(t) = & 19.0248 - 0.823358 \cos(t) + 17.1048 \sin(t) - 15.2722 \cos(2t) - 0.13139 \sin(2t) \\ & - 0.454434 \cos(3t) - 12.8637 \sin(3t) + 11.9691 \cos(4t) - 1.02332 \sin(4t) \\ & - 0.823379 \cos(5t) + 8.6227 \sin(5t) - 4.10823 \cos(6t) - 0.818417 \sin(6t). \end{aligned} \quad (3.6)$$

Using F and G in Eq. (2.4) results in a Fourier parametrisation of the desired braid, closing to 5_2 . This allows us to construct $g_{a,b}$ and then $f_{a,b}$. In Ref. [14] we describe the construction of a semiholomorphic polynomial with zero level set of the form of the knot 5_2 . This involves finding a Fourier parametrisation for its braid through geometric considerations alone. The parametrisation found by our algorithm is significantly more complicated than the one described in Ref. [14]. However, the procedure in Ref. [14] is not algorithmic and the braid was sufficiently simple to find by inspection.

As discussed in Ref. [14] braids whose words are identical up to signs, i.e. $B_1 = \sigma_{i_1}^{\varepsilon_{i_1,1}} \sigma_{i_2}^{\varepsilon_{i_2,1}} \dots \sigma_{i_\ell}^{\varepsilon_{i_\ell,1}}$ and $B_2 = \sigma_{i_1}^{\varepsilon_{i_1,2}} \sigma_{i_2}^{\varepsilon_{i_2,2}} \dots \sigma_{i_\ell}^{\varepsilon_{i_\ell,2}}$ with all $\varepsilon_i \in \{\pm 1\}$, can be parametrised using the same F_C . For example the function F computed above can also be used to parametrise the x -coordinate of the braid $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-3}$, which closes to the knot 6_2 .

In Ref. [14] we use geometric inspection to find a finite Fourier parametrisation for the braid word $\sigma_1^{-1} \sigma_2 \sigma_1^3 \sigma_2$, which is obviously also a minimal braid word for the knot 5_2 , since it differs from the one given above only by conjugation by σ_2 . There are some fundamental differences between the approach in Ref. [14] and the algorithm described here. The method from Ref. [14] requires a lot of testing. Choosing a braid representative which is as symmetric as possible typically allows us to parametrise the braid using only few Fourier terms and low spatial frequencies. Geometric considerations give a rough idea of a possible range of coefficients and spatial frequencies, so that it becomes feasible to find a finite Fourier parametrisation for the given braid. The desire for symmetry makes $\sigma_1^{-1} \sigma_2 \sigma_1^3 \sigma_2$ a better choice

of braid word, in particular if we place the crossings in equal distances along the t -axis and have the first crossing at $t = 0$. Note that this is different from the positions of the crossings in the description of our algorithm. In order to calculate the trigonometric interpolation efficiently by using the discrete Fourier transform, we want the data points to be equally distributed along the t -axis starting at $t = 0$. Since the data points should lie between crossings, we explicitly do not want a crossing at $t = 0$.

For short and symmetric braids with few strands the method from Ref. [14] might lead to simpler polynomials in the sense of fewer terms and lower orders, for example the parametrisation for the knot 5_2 in Ref. [14] only has two terms for F and G each and the highest order is five. However, since the method is not algorithmic, it is in general not a good way of finding parametrisations for more complicated braids.

With the described algorithm we can find a finite Fourier parametrisation for any braid B . Then $g_{a,b}$ defined as in Eq. (2.2) can be written as a polynomial in u , e^{it} and e^{-it} by Lemma 2.1. Constructing $f_{a,b}$ as in Section 2.1 gives a semiholomorphic polynomial which by the results from Section 2.2 has the closure of B as its zero level set on the three-sphere of radius one if a and b are small enough. This makes our proof of the existence of semiholomorphic polynomials with knotted zero level sets fully algorithmic.

4. Properties of the constructed polynomials

In this section we prove some properties of the constructed polynomials. We use the notation from Section 2, in particular the symbol f_λ will always denote a polynomial that was constructed using the method described in Section 2.

Corollary 4.1. *If λ is small enough, 0 is a regular value of $f_\lambda|_{S^3}$.*

Proof. Let $(u_*, v_*) \in S^3$ be a point on the link, so $f_\lambda(u_*, v_*) = 0$. Then for fixed $v = v_*$, the function f_λ is a polynomial in u with u_* being a simple root. Hence $\frac{\partial f_\lambda}{\partial u}(u_*, v_*) \neq 0$, so in particular $\nabla f_\lambda(u_*, v_*)$ has full rank.

Furthermore, for small fixed λ a straightforward calculation shows that the intersections of the curves $(u_{\lambda,j}(r, t), r e^{it})$, $j = 1, 2, \dots, s$, $t \in [0, 2\pi]$ fixed, with S^3 are transverse. Hence (u_*, v_*) is a regular point of $f|_{S^3}$. \square

This corollary is important for applications which focus on functions $\mathbb{R}^3 \rightarrow \mathbb{C}$. In order to make the functions satisfy extra physical conditions, which depend on the individual setting, coefficients might have to be slightly changed. The fact that 0 is a regular value offers a certain stability under small perturbations of the originally constructed polynomial. This means that if we do not change the coefficients too much, the zero set of interest will remain the desired link.

In Ref. [33] Rudolph discusses transverse \mathbb{C} -links. As described above, these are links which can be realised as the transverse intersection of the zero level set

of a complex polynomial $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ with the unit three-sphere, and a link is a transverse \mathbb{C} -link if and only if it is quasi-positive. Corollary 4.1 shows that if we relax the condition from f being a complex polynomial to being semiholomorphic, then every link can be realised as the transverse intersection of the zero level set of such a polynomial with S^3 .

Corollary 4.2. *The polynomial f_λ is harmonic, that is $(\partial_u \partial_{\bar{u}} + \partial_v \partial_{\bar{v}})f_\lambda = 0$.*

Proof. This is immediate from the construction. Since f_λ is a polynomial in u , there is no dependence on \bar{u} . Recall that f_λ was obtained from $g_{a,b}$ by replacing e^{it} by v and e^{-it} by \bar{v} and $g_{a,b}$ is a polynomial in u , e^{it} and e^{-it} . Now suppose there was a monomial containing both v and \bar{v} , say $v^n \bar{v}^m$, as a factor. Then in the polynomial expression of $g_{a,b}$ these would have simplified to $e^{it(n-m)}$, which would give v^{n-m} or \bar{v}^{m-n} depending on whether n or m is larger. In either case, there is no monomial with v and \bar{v} as a factor. Thus $\partial_{\bar{v}} f_\lambda$ does not depend on v which proves the corollary. \square

Not every polynomial of the form constructed in Section 2 will be of practical use. In particular with regard to the engineering of physical knotted fields, a polynomial with too many terms or of too high degree might be too hard to control to be applicable to some of the systems described in Ref. [8]. This is the reason why a naive algebraic approximation of a link given as a parametric curve is not particularly useful.

The advantage of our construction is that it allows for an upper bound of the degree of the constructed polynomial f_λ in terms of the braid data.

We use the notation from Section 2. The algorithm in Section 3 finds a finite Fourier parametrisation (2.4) of a braid B that closes to the desired link L . Using trigonometric interpolation to find such a parametrisation allows us to give a bound on the highest order $D_C = \max\{N_C, M_C\}$ in terms of the length ℓ of the braid and the number of strands s_C in each link component C .

Lemma 4.1. *Let \mathcal{C} be again the set of all link components of L , s_C be the number of strands that form $C \in \mathcal{C}$, ℓ be the length of the braid word and $s = \sum_{C \in \mathcal{C}} s_C$ the number of strands of the braid B . Then*

$$D_C \leq \left\lfloor \frac{(s_C + 1)(s_C \ell - 1) + \ell s_C (s - s_C) - 1}{2} \right\rfloor. \quad (4.1)$$

Proof. The degrees of the trigonometric polynomials F_C and G_C can be directly calculated from the number of data points used for the interpolation. Note that the number of data points needed for the trigonometric interpolation of each of the polynomials F_C is equal to $s_C \ell$. Thus N_C is equal to $\lfloor \frac{s_C \ell - 1}{2} \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x .

The union of the graphs of $F_C \left(\frac{t+2\pi j}{s_C} \right)$ form a braid diagram B' with unspecified signs of crossings. We can choose the signs in such a way that the resulting braid

is identical to the input braid B , although the braid word might be different. The number of crossings of B' is equal to the length ℓ' of its braid word and hence by the discussion in Section 3 at least ℓ .

For the trigonometric interpolation of one trigonometric polynomial G_C we need a data point for every crossing that involves a strand of C , counting with multiplicities. The crossings of B' correspond to intersections of certain trigonometric polynomials related to the different F_C . Since the trigonometric polynomials can be associated with complex polynomials on the unit circles, by the fundamental theorem of algebra their number of intersections can be bounded in terms of their degree.

Let $C \in \mathcal{C}$ be a component consisting of s_C strands. We first consider the number of intersections between different strands of the same component C . Recall that for a trigonometric polynomial $F_C(h) = \sum_{k=-N_C}^{N_C} c_{k,C} e^{ikt}$ we can construct a complex polynomial $p_C(z)$ of degree $2N_C$ with $p_C(e^{it}) = e^{itN_C} F_C(t)$ by defining $p_C(z) = z^{N_C} \sum_{k=-N_C}^{N_C} c_{k,C} z^k$. Note that an intersection point of a strand with index j and a strand with index $j+1$ corresponds to a number $t \in [0, 2\pi]$ such that $F_C(t) = F_C(t + 2\pi/s_C)$, which is equivalent to $p_C(e^{it}) = p_C(e^{it} e^{i2\pi/s_C}) e^{-i2\pi N_C/s_C}$.

Therefore the sum of the number of intersections between strands with index j and $j+1 \pmod{s_C}$ is the exactly the number of points on the unit circle where the complex polynomials $p_C(z)$ and $p_C(z e^{i2\pi/s_C}) e^{-i2\pi N_C/s_C}$ are equal. Since both have degree $2N_C$, this number is at most $2N_C \leq s_C \ell - 1$. In general, the sum of intersections between strands with index j and $j+k$ correspond to points on the unit circle where $p_C(z)$ and $p_C(z e^{i2\pi k/s_C}) e^{-i2\pi k N_C/s_C}$ coincide. Thus there are again at most $2N_C \leq s_C \ell - 1$ many of these. In order to capture all pairs of strands in the component C , we have to sum over k from 1 to $\lfloor \frac{s_C+1}{2} \rfloor$. Thus there are at most $\lfloor \frac{s_C+1}{2} \rfloor (s_C \ell - 1)$ many intersection points between two different strands of the same component C . Hence the number of data points needed to achieve the correct signs for these crossings are at most $2 \lfloor \frac{s_C+1}{2} \rfloor (s_C \ell - 1) \leq (s_C + 1)(s_C \ell - 1)$.

Intersections involving two different link components, especially if the components consist of different numbers of strands, are more complicated to count. We denote by $\text{lcm}(s_C, s_{C'})$ the least common multiple of s_C and $s_{C'}$ and by $\text{gcd}(s_C, s_{C'})$ their greatest common divisor. A crossing between a strand of the component C with a strand of component C' is equivalent to the existence of $t_1, t_2 \in [0, 2\pi]$ such that $F_C(t_1) = F_{C'}(t_2)$ and $t_1 s_C \equiv t_2 s_{C'} \pmod{2\pi}$. This is equivalent to

$$\begin{aligned} & p_C \left(\left(e^{\frac{it_1 s_C}{\text{lcm}(s_C, s_{C'})}} \right)^{\text{lcm}(s_C, s_{C'})/s_C} \right) e^{-iN_C t_1} \\ &= p_{C'} \left(\left(e^{\frac{it_2 s_{C'}}{\text{lcm}(s_C, s_{C'})}} \right)^{\text{lcm}(s_C, s_{C'})/s_{C'}} \right) e^{-iN_{C'} t_2} \end{aligned} \quad (4.2)$$

and also

$$\begin{aligned} p_C & \left(\left(e^{\frac{i(t_1+2\pi m)s_C}{\text{lcm}(s_C, s_{C'})}} \right)^{\text{lcm}(s_C, s_{C'})/s_C} \right) \left(e^{\frac{-i N_C(t_1+2\pi m)s_C}{\text{lcm}(s_C, s_{C'})}} \right)^{\text{lcm}(s_C, s_{C'})/s_C} \\ & = p_{C'} \left(\left(e^{\frac{i(t_2+2\pi n)s_{C'}}{\text{lcm}(s_C, s_{C'})}} \right)^{\text{lcm}(s_C, s_{C'})/s_{C'}} \right) \left(e^{\frac{-i N_{C'}(t_2+2\pi n)s_{C'}}{\text{lcm}(s_C, s_{C'})}} \right)^{\text{lcm}(s_C, s_{C'})/s_{C'}} \end{aligned} \quad (4.3)$$

for all $m = 0, 1, \dots, \frac{\text{lcm}(s_C, s_{C'})}{s_C} - 1$ and $n = 0, 1, \dots, \frac{\text{lcm}(s_C, s_{C'})}{s_{C'}} - 1$.

We want again a one-to-one correspondence between crossings and the zeros of a polynomial on the unit circle. We can turn Eq. (4.3) in an equation of two polynomials by multiplying both sides by $z^{\text{lcm}(s_C, s_{C'}) \max\{N_C/s_C, N_{C'}/s_{C'}\}}$. Then Eq. (4.3) says that there are points $z_1 = e^{i(t_1+2\pi m)s_C/\text{lcm}(s_C, s_{C'})}$ and $z_2 = e^{i(t_2+2\pi n)s_{C'}/\text{lcm}(s_C, s_{C'})}$ where the two polynomials

$$p_C(z^{\text{lcm}(s_C, s_{C'})/s_C}) z^{\text{lcm}(s_C, s_{C'}) (\max\{N_C/s_C, N_{C'}/s_{C'}\} - N_C/s_C)} \quad (4.4)$$

and

$$p_{C'}(z^{\text{lcm}(s_C, s_{C'})/s_{C'}}) z^{\text{lcm}(s_C, s_{C'}) (\max\{N_C/s_C, N_{C'}/s_{C'}\} - N_{C'}/s_{C'})} \quad (4.5)$$

agree.

Therefore the common roots of the two polynomials above are in one-to-one correspondence with crossings between strands (C, j) and (C', k) such that there are m, n with

$$\begin{aligned} (t_1 + 2\pi m)s_C/\text{lcm}(s_C, s_{C'}) & = \left(\frac{t_1+2\pi j}{s_C} + 2\pi m\right)s_C/\text{lcm}(s_C, s_{C'}) \\ & = \left(\frac{t_1+2\pi k}{s_{C'}} + 2\pi n\right)s_{C'}/\text{lcm}(s_C, s_{C'}) = (t_2 + 2\pi n)s_{C'}/\text{lcm}(s_C, s_{C'}) \pmod{2\pi}, \end{aligned} \quad (4.6)$$

which is equivalent to the existence of m, n with

$$j - k = ns_{C'} - ms_C \pmod{\text{lcm}(s_C, s_{C'})}. \quad (4.7)$$

Such m and n exists if and only if $j - k = 0 \pmod{\text{gcd}(s_C, s_{C'})}$.

In general, this means that the crossings of (C, j) with (C', k) with $j - k = M \pmod{\text{gcd}(s_C, s_{C'})}$ are in one-to-one correspondence with the points on the unit circle where

$$\begin{aligned} & p_C \left(z^{\text{lcm}(s_C, s_{C'})/s_C} \right) z^{\text{lcm}(s_C, s_{C'}) (\max\{N_C/s_C, N_{C'}/s_{C'}\} - N_C/s_C)} \\ & = p_{C'} \left(z^{\text{lcm}(s_C, s_{C'})/s_{C'}} e^{-i 2\pi \frac{M}{\text{gcd}(s_C, s_{C'})}} \right) z^{\text{lcm}(s_C, s_{C'}) (\max\{N_C/s_C, N_{C'}/s_{C'}\} - N_{C'}/s_{C'})}. \end{aligned} \quad (4.8)$$

The degrees of the polynomials are

$$\text{lcm}(s_C, s_{C'}) (N_C/s_C + \max\{N_C/s_C, N_{C'}/s_{C'}\}) \quad (4.9)$$

and

$$\text{lcm}(s_C, s_{C'}) (N_{C'}/s_{C'} + \max\{N_C/s_C, N_{C'}/s_{C'}\}) \quad (4.10)$$

Summing over M from 1 to $\text{gcd}(s_C, s_{C'})$ we get that there are at most

$$\text{gcd}(s_C, s_{C'}) \ell(\text{lcm}(s_C, s_{C'})) = \ell s_C s_{C'} \quad (4.11)$$

crossings that involve exactly one strand of the component C and one strand of the component C' . Therefore the total number of crossings that involve exactly one strand from the link component C is at most $\ell s_C \sum_{\substack{C' \in C \\ C' \neq C}} = \ell s_C (s - s_C)$

Thus the total number of data points needed for G_C is at most $(s_C + 1)(s_C \ell - 1) + \ell s_C (s - s_C)$ and hence the degree of G_C is at most

$$M_C \leq \left\lfloor \frac{(s_C + 1)(s_C \ell - 1) + \ell s_C (s - s_C) - 1}{2} \right\rfloor. \quad (4.12)$$

Thus

$$\begin{aligned} D_C &= \max\{N_C, M_C\} \\ &\leq \max \left\{ \left\lfloor \frac{s_C \ell - 1}{2} \right\rfloor, \left\lfloor \frac{(s_C + 1)(s_C \ell - 1) + \ell s_C (s - s_C) - 1}{2} \right\rfloor \right\} \\ &= \left\lfloor \frac{(s_C + 1)(s_C \ell - 1) + \ell s_C (s - s_C) - 1}{2} \right\rfloor, \end{aligned} \quad (4.13)$$

where the last equality holds because s, ℓ and s_C are all positive integers. \square

Lemma 4.2. *The degree of f_λ is equal to $\sum_{C \in C} \max\{D_C, s_C\}$.*

Proof. The polynomial degree of f_λ is equal to the degree of $g_{\lambda a, \lambda b}$, when considered as a polynomial in u, e^{it} and e^{-it} . Since $g_{\lambda a, \lambda b}$ is defined as the product of polynomials that one obtains for each component, its degree is the sum of the degrees of its factors

$$g_C(u, t) := \prod_{j=0}^{s_C} \left(u - \lambda a F_C \left(\frac{t + 2\pi j}{s_C} \right) - i \lambda b G_C \left(\frac{t + 2\pi j}{s_C} \right) \right). \quad (4.14)$$

The degree with respect to u of the factor corresponding to the component C is s_C . Note that the total degree of a monomial of $g_C(u, t)$ for which the degree with respect to u is k is at most $\frac{D_C}{s_C}(s_C - k) + k$. If $D_C \geq s_C$, then $\frac{D_C}{s_C}(s_C - k) + k \leq D_C + k(1 - \frac{D_C}{s_C}) \leq D_C$. If $D_C < s_C$, then $\frac{D_C}{s_C}(s_C - k) + k \leq (s_C - k) + k = s_C$.

Note that there are monomials of degree s_C and D_C respectively, so the degree of g_C is $\max\{s_C, D_C\}$. \square

Using the bound we have for D_C , we get:

Corollary 4.3. *The degree of f_λ is bounded above by*

$$\deg(f_\lambda) \leq c_2 := \sum_{C \in C} \max \left\{ \left\lfloor \frac{(s_C + 1)(s_C \ell - 1) + \ell s_C (s - s_C) - 1}{2} \right\rfloor, s_C \right\}. \quad (4.15)$$

Note that for knots $|\mathcal{C}| = 1$, $s = s_C$ and hence for non-trivial knots $\deg(f_\lambda) \leq \lfloor \frac{(s+1)(s\ell-1)-1}{2} \rfloor$, since in this case $s \geq 2$ and $\ell \geq 3$.

Also note that the bound given in Corollary 4.3 holds for all semiholomorphic polynomials constructed using the algorithm described in Section 2 and 3, in particular using trigonometric interpolation to find the trigonometric braid parametrisation as described in Section 3. Corollary 4.3 is not a statement about the non-existence of polynomials of a certain degree.

We can also give a lower bound for the polynomial degree, which holds for all semiholomorphic polynomials constructed as in Section 2, whether trigonometric interpolation as in Section 3 is used or not. For any braid parametrisation of the form (2.4) the degree of the trigonometric polynomials is bounded below in terms of the number of crossings between pairs of strands. Again we can write the trigonometric polynomials as complex polynomials restricted to the unit circle such that crossings of strands correspond to points on the unit circle where the two corresponding polynomials share the same value.

As before we have to sum over all possible pairs of strands and obtain the following bounds.

Corollary 4.4. *Let C' be the component of the braid B such that the degree $N_{C'}$ of the trigonometric polynomial $F_{C'}$ used to parametrise B as in Equation (2.4) is $\max\{N_C : C \in \mathcal{C}\}$. Then*

$$N_{C'} \geq \frac{\ell - |\mathcal{C}|}{|\mathcal{C}|^2 s_{\tilde{C}}}, \quad (4.16)$$

where ℓ is the length of the braid word (i.e. the number of crossings) and \tilde{C} is the component such that $s_{\tilde{C}} = \max\{s_C : C \in \mathcal{C}\}$.

Proof. We only give a sketch of the proof here, since it is the same principle as the proof of Corollary 4.3. In order for the F_C to provide a parametrisation of the x -coordinate of the braid B as in Eq. (2.4), each pair of strands has to cross at least a prescribed number of times. The values of $t \in [0, 2\pi]$ where these crossings occur, correspond to points on the unit circle where two complex polynomials agree. This yields a lower bound for the degrees of these polynomials which are easily related to the different N_C . \square

If the braid closes a knot, there is only one component, so $N_{C'} \geq \frac{\ell-1}{s}$, where ℓ is the length of the braid word. Corollary 4.4 implies that

$$\begin{aligned} \deg(f_\lambda) &\geq \max\{s, N_{C'} + s - s_{C'}\} \\ &\geq \max\left\{s, \frac{\ell - |\mathcal{C}|}{|\mathcal{C}|^2 s_{\tilde{C}}} + s - s_{\tilde{C}}\right\}. \end{aligned} \quad (4.17)$$

For knots this inequality is $\deg(f_\lambda) \geq \max\{s, (\ell - 1)/s\}$. From the polynomials we have constructed, we find that the bounds given by Corollary 4.4 and Corollary

4.3 are not very tight bounds. The proofs can explain this, since the degree was determined by the number of data points which in turn was determined by the number of points on the unit circle where two complex polynomials agree. This number was bounded by the degree of these complex polynomials, but of course in general a complex polynomial does not have all of its roots on the unit circle. The results of Corollaries (4.1)-(4.4) are summarised in Theorem 1.2 in Section 1.

We have proven the existence of a semiholomorphic polynomial f of bounded degree, whose zero level set on the unit three-sphere is a given link. Applying the standard stereographic projection

$$u = \frac{x^2 + y^2 + z^2 - 1 + 2iz}{x^2 + y^2 + z^2 + 1}, \quad v = \frac{2(x + iy)}{x^2 + y^2 + z^2 + 1} \quad (4.18)$$

to f results in rational function, whose denominator is a constant times some power of $(x^2 + y^2 + z^2 + 1)$. Hence multiplying by the common denominator yields a polynomial $\mathbb{R}^3 \rightarrow \mathbb{C}$ in x, y and z , whose zero level set is L . It follows from Lemma 4.1 that the coefficients of this polynomial can be taken to be Gaussian integers. This shows:

Corollary 4.5. *Let B be a braid on s strands of length ℓ and let L denote its closure. Then there exist $F_1, F_2 \in \mathbb{Z}[x, y, z]$ such that the vanishing set of (F_1, F_2) over the reals $\{(x, y, z) \in \mathbb{R}^3 : F_1(x, y, z) = F_2(x, y, z) = 0\}$ is ambient isotopic to L . Furthermore*

$$\max\{\deg(F_1), \deg(F_2)\} \leq 2 \sum_{C \in \mathcal{C}} \max \left\{ \left\lfloor \frac{(s_C + 1)(s_C \ell - 1) + \ell s_C (s - s_C) - 1}{2} \right\rfloor, s_C \right\}, \quad (4.19)$$

where $s_{\tilde{C}}$ is the number of strands of the link component $\tilde{C} \in \mathcal{C}$ that consists of the most strands.

In Ref. [8] we described the construction of semiholomorphic polynomials for the family of lemniscate knots, a generalisation of torus knots, and discussed applications of these functions to physical systems. We will not go through the details here, but we would like to point out that all results about applications still hold. In particular, for any link L which is the closure of a braid on s strands and any $N \in \mathbb{Z}$ we can construct a rational map $W : S^3 \rightarrow S^2$ of topological degree Ns such that $W^{-1}(0, 0, -1) = L$. Similarly for any link L which is the closure of a braid on s strands and any $N \in \mathbb{Z}$ we can construct a vector field $V : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that some flow lines of V form the link L and the helicity of V is equal to Ns . This allows us to implement any link type as an initial condition in the wide range of physical systems mentioned in Section 1.

Another result that was shown for lemniscate knots in Ref. [8] remains true in this most general setting without changing the proof given in Ref. [8].

Lemma 4.3. [8] *If $g_{a,b}$ has exactly n non-degenerate phase-critical points, i.e. $\nabla_{\mathbb{C} \times S^1} \arg(g_{a,b}(x))$ has full rank for all $x \in \mathbb{C} \times S^1$ except on n points and a and*

b are small, then $f_{a,b}|_{S^3}$ also has exactly n phase-critical points. It follows that if $n = 0$, then $\arg(f_{a,b}) : S^3 \setminus L \rightarrow S^1$ is a fibration of $S^3 \setminus L$ over S^1 .

We call a braid on s strands strictly homogeneous if it is homogeneous and every generator $\sigma_1, \dots, \sigma_{s-1}$ appears at least once in its braid word, either with a positive or a negative sign. Lemma 4.3 can be used to show the following result.

Theorem 4.6. *Let B be a strictly homogeneous braid on s strands and L be its closure. Then L is fibred and there exists a semiholomorphic polynomial $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ such that $\deg_u f = s$, $f^{-1}(0) \cap S^3 = L$ and $\arg(f)$ is a fibration of $S^3 \setminus L$ over S^1 .*

Proof. By Lemma 4.3 it is enough to show that B has a Fourier parametrisation such that $g_{a,b}$ does not have any phase-critical points. Then for any $\lambda > 0$ the function $g_{\lambda a, \lambda b}$ will not have any phase-critical points either. It follows from Theorem 1.1 that if we choose λ small enough, the resulting $f_{a,b}$ will have a zero level set on S^3 that is ambient isotopic to L .

First note that the phase-critical points of $g_{a,b}$ are exactly those points in $\mathbb{C} \times S^1$, where $\partial_u g_{a,b} = 0$ and $\partial_t \arg(g_{a,b}) = 0$. Now if $\bigcup_{j=0}^{s-1} (X_j(t), Y_j(t), t)$ is any parametrisation of B (not necessarily of the form of Eq. (2.4)), we can define $g_{a,b}$ as in Eq. (2.2). We assume that that $g_{a,b}$ has $s - 1$ disjoint non-zero critical values for every t . Then the critical values together with a strand $(0, 0, t)$, $t \in [0, 2\pi]$ form a braid in $\mathbb{C} \times S^1$ on s strands. We call this the critical braid.

Note that the set W of monic polynomials of degree s with disjoint roots and disjoint critical values fibers over $V = \{(z_1, z_2, \dots, z_{s-1}) \in \mathbb{C}^{s-1} | z_i \neq 0 \text{ and } z_i \neq z_j \text{ if } i \neq j\} / S_{s-1}$ via the map θ that sends each such polynomial to its set of critical values [4]. Let H be a homotopy of loops in $S^1 \times V$ such that H starts at a loop $x \in S^1 \times V$ which is in the image of $(\text{id}, \theta) : S^1 \times W \rightarrow S^1 \times V$, where id is the identity map. Say $x = (\text{id}, \theta)(y)$ for some $y \in S^1 \times W$. Then H lifts to a homotopy of loops in $S^1 \times W$ starting at y . Thus if two critical braids B_1 and B_2 are braid isotopic and $p_{t,1}$ is a 1-parameter ($t \in S^1$) family of monic polynomials with disjoint roots and critical values such that $(\text{id}, \theta)(p_{t,1}) = B_1$, then the zeros of $p_{t,1}$ form a closed braid in $S^1 \times \mathbb{C}$ which is isotopic to the braid formed by the roots of some other family of polynomials $p_{t,2}$ with $(\text{id}, \theta)(p_{t,2}) = B_2$.

Let $B = \sigma_{i_1}^{\varepsilon_{i_1}} \sigma_{i_2}^{\varepsilon_{i_2}} \dots \sigma_{i_\ell}^{\varepsilon_{i_\ell}}$, $\varepsilon_i \in \{\pm 1\}$. Then there exists a parametrisation P of B such that if $g_{a,b}$ is defined using P , then the critical braid of $g_{a,b}$ can be isotoped to the braid $A_{i_1, \varepsilon_{i_1}} A_{i_2, \varepsilon_{i_2}} \dots A_{i_\ell, \varepsilon_{i_\ell}}$, where $A_{i, \varepsilon} = (\sigma_1 \sigma_2 \dots \sigma_{i-1}) \sigma_i^{2\varepsilon} (\sigma_1 \sigma_2 \dots \sigma_{i-1})^{-1}$ [32].

For a strictly homogeneous braid B , we can write down a parametrisation $\bigcup_{j=0}^{s-1} (X'_j(t), Y'_j(t), t)$ of the braid of critical values such that $\partial_t \arg(X'_j + iY'_j) > 0$ for all t if ε_j is positive, $\partial_t \arg(X'_j + iY'_j) < 0$ for all t if ε_j is negative and such that this critical braid is braid isotopic to the critical braid derived from the parametrisation P . Hence the parametrisation $\bigcup_{j=0}^{s-1} (X'_j(t), Y'_j(t), t)$ of the critical braid lifts to a parametrisation $(X_j(t), Y_j(t), t)$ of a braid which is isotopic to B and the function

$g_{a,b}$ corresponding to this parametrisation does not have any phase-critical points. Since the roots are disjoint and the critical points are disjoint, the dependence of the critical values on the coefficients is in fact differentiable, so a small perturbation of the parametrisation will not introduce any more phase-critical points. Since trigonometric polynomials are dense in the set of continuous, 2π -periodic real functions, we can approximate the parametrisation $(X_j(t), Y_j(t), t)$ by a Fourier parametrisation of the form of Eq. (2.4) such that the corresponding $g_{a,b}$ does not have any phase-critical points either. \square

Stallings [34] already showed that closures of homogeneous braids are fibred. Our theorem specifies this by providing a certain form of the fibration map. This proof shows in fact more. Let $B = \sigma_{i_1}^{\varepsilon_{i_1}} \sigma_{i_2}^{\varepsilon_{i_2}} \dots \sigma_{i_\ell}^{\varepsilon_{i_\ell}}$ and let $\beta(B)$ denote

$$\begin{aligned} \beta(B) = \sum_{i=1}^{s-1} & |\{j \in \{1, 2, \dots, \ell - 1\} : \exists k \in \{1, 2, \dots, \ell - 1\} \text{ s.t. } i_j = i_{j+k \bmod \ell} = i, \\ & i_{j+m \bmod \ell} \neq i \text{ for all } m < k \text{ and } \varepsilon_{i_j} \varepsilon_{i_{j+k \bmod \ell}} = -1\}| \\ & + 2 |\{j \in \{1, 2, \dots, s - 1 : \text{There is no } k \text{ s.t. } i_k = j\}\}|. \end{aligned} \quad (4.20)$$

Note that for strictly homogeneous braids, it is $\beta(B) = 0$. The β -value measures how far a braid is from being strictly homogeneous. Together with Lemma 4.3 the proof of Theorem 4.6 shows that for any braid B on s strands, there exists a semiholomorphic polynomial $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ such that $f^{-1}(0) \cap S^3$ is the closure of B , $\deg_u f = s$ and $\arg(f)$ has exactly $\beta(B)$ critical points.

It can be arranged that $\arg(f)$ is a smooth circle-valued Morse function on $S^3 \setminus L$. Such a function is called *regular* if there is a diffeomorphism φ of the union of $|\mathcal{C}|$ solid tori, such that the composition of φ and $\arg(f)$ applied to a tubular neighbourhood of the link is just the projection map $L \times (D \setminus \{0\}) \rightarrow S^1 : (x, y) \mapsto y/|y|$. Since $\arg(f)$ is the argument of a semiholomorphic polynomial, it is regular.

The Morse-Novikov number $\mathcal{MN}(L)$ of a link L is the minimal number of critical points of all smooth, regular circle-valued Morse functions [28], so that $\mathcal{MN}(L) = 0$ if and only if L is fibred.

The discussion above then implies

$$\mathcal{MN}(L) \leq \beta(B). \quad (4.21)$$

We should point out that this bound is a strict inequality for some knots. For example the knot 8_{20} is known to be fibred and has thus $\mathcal{MN}(L)$ equal to zero. However, it is not the closure of a homogeneous braid and hence $\beta(B) > 0$ for all braids B that close to 8_{20} [5].

The proof of Theorem 4.6 follows ideas from Ref. [32] very closely, so we believe that we might not be the first to derive this bound. What we have shown here is that the bound can be realised by the argument of a semiholomorphic polynomial.

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