AN INSTANCE WHERE THE MAJOR AND MINOR ARC INTEGRALS MEET

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Dedicated to the memory of Christopher Hooley

Abstract. We apply the circle method to obtain an asymptotic formula for the number of integral points on a certain sliced cubic hypersurface related to the Segre cubic. Unusually, the major and minor arc integrals in this application are both positive and of the same order of magnitude.

1. Introduction

In this communication we concern ourselves with the simultaneous Diophantine equations

$$\sum_{i=1}^{10} x_i^3 = \sum_{i=1}^{6} x_i = \sum_{i=7}^{10} x_i = 0, \tag{1.1}$$

and specifically with the number $N(B)$ of integral solutions $\mathbf{x}$ of this system satisfying $|x_i| \leqslant B$. Associated with (1.1) are the p-adic densities

$$\chi_p = \lim_{h\to\infty} (p^h)^{-7} M_p(h), \tag{1.2}$$

in which, for each prime number p, the expression $M_p(h)$ denotes the number of solutions of (1.1) with $\mathbf{x} \in (\mathbb{Z}/p^h\mathbb{Z})^s$. Also, one has the real density

$$\chi_\infty = \lim_{\eta\to 0+} (2\eta)^{-3} M_\infty(\eta), \tag{1.3}$$

where $M_\infty(\eta)$ denotes the volume of the subset of $\left[-\frac{1}{2}, \frac{1}{2}\right]^{10}$ defined by the simultaneous inequalities

$$\left|\sum_{i=1}^{10} x_i^3\right| < \eta, \quad \left|\sum_{i=1}^{6} x_i\right| < \eta, \quad \left|\sum_{i=7}^{10} x_i\right| < \eta. \tag{1.4}$$

We note that our definition here of the real density employs a unit hypercube, in contrast to the commonly favoured box $[-1, 1]^{10}$.

Our principal goal is an asymptotic formula for $N(B)$.

Theorem 1.1. *One has*

$$N(B) = (45 + \mathcal{C})(2B)^5 + O(B^{5-1/200}), \tag{1.5}$$

where $\mathcal{C} > 0$ is defined by the absolutely convergent product $\mathcal{C} = \chi_\infty \prod_p \chi_p$.

2010 *Mathematics Subject Classification.* 11D72, 11L15, 11L07, 11P55, 11E76.

Key words and phrases. Cubic Diophantine equations, Hardy-Littlewood method.

The term $\mathcal{C}(2B)^5$ in (1.5) arises from the product of local densities in the counting problem under consideration, and is asymptotically equal to the major arc integral in a Hardy-Littlewood (circle) method treatment of $N(B)$. The additional term $45(2B)^5$ stems from 45 linear spaces of affine dimension 5 contained in the variety defined by (1.1), as exemplified by that defined via the equations $x_{2i-1}+x_{2i}=0$ $(1\leqslant i\leqslant 5)$. A notable feature of our treatment of $N(B)$ is that this second term may be seen as arising directly from the minor arc integral in an application of the circle method.

Manin and his collaborators [2, 7] and later Peyre [10] made far-reaching predictions concerning the number of rational points of bounded height on a large class of projective varieties. When interpreted in terms of the circle method, one is led to the general philosophy, made explicit in the appendix to [15], that the major arc integral accounts for the contribution of the generic points in the solution set, this being approximated by the product of local densities. Meanwhile, the minor arc integral should approximate the number of points lying on certain special subvarieties. Examples involving problems of degree exceeding two in which this heuristic has been substantiated are rare. Moreover, hitherto, the derivation of the associated asymptotic formulae did not proceed via the circle method, and the alignment with the conjectured behaviour was identified only *a posteriori* by a formal computation of the major arc integral and comparison with the product of local densities. What we have in mind here are such approaches as that involving the application of torsors (see for example [3, 5, 11]). In contrast, our theorem is obtained by a direct application of the circle method. The special subvarieties in question are the aforementioned 45 linear spaces, each containing approximately $(2B)^5$ integral points in $[-B,B]^{10}$. As our analysis shows, the minor arc integral is also approximately $45(2B)^5$. The contributions arising from the major and minor arc integrals are consequently of the same order of magnitude. Hooley [8, Chapter II.1] has offered evidence in support of the conjecture that a similar phenomenon holds also in a circle method approach to counting integral solutions of the equation

$$x_1^3+x_2^3+x_3^3=x_4^3+x_5^3+x_6^3.$$

A consequence of Theorem 1.1 is an essentially optimal mean value estimate.

Corollary 1.2. *One has*

$$\int_{[0,1)^3}\Bigl|\sum_{1\leqslant x,y\leqslant X} e(\alpha_1x+\alpha_2y+\alpha_3(x^3+y^3))\Bigr|^5\,\mathrm{d}\boldsymbol{\alpha}\asymp X^5.$$

This conclusion is related to recent work of Bourgain and Demeter [4]. Given complex numbers $\mathfrak{a}_{x,y}$, their work establishes the upper bound

$$\int_{[0,1)^3}\Bigl|\sum_{1\leqslant x,y\leqslant X}\mathfrak{a}_{x,y}e(\alpha_1x+\alpha_2y+\alpha_3(x^2+y^2))\Bigr|^4\,\mathrm{d}\boldsymbol{\alpha}\ll X^\varepsilon\Bigl(\sum_{1\leqslant x,y\leqslant X}|\mathfrak{a}_{x,y}|^2\Bigr)^2.$$

With care, one would extract the same conclusion when the binary form x^2+y^2 is replaced by x^3+y^3, and this would deliver the upper bound

$$\int_{[0,1)^3} \Bigg| \sum_{1\leqslant x,y\leqslant X} e(\alpha_1 x+\alpha_2 y+\alpha_3(x^3+y^3)) \Bigg|^4 \mathrm{d}\boldsymbol{\alpha} \ll X^{4+\varepsilon}.$$

The conclusion of Corollary 1.2 implies the more general upper bound

$$\int_{[0,1)^3} \Bigg| \sum_{1\leqslant x,y\leqslant X} e(\alpha_1 x+\alpha_2 y+\alpha_3(x^3+y^3)) \Bigg|^s \mathrm{d}\boldsymbol{\alpha} \asymp X^s + X^{2s-5} \tag{1.6}$$

for all non-negative exponents s. Indeed, when $s>5$, little additional effort is required to obtain an asymptotic formula for this mean value. It would be desirable to obtain an analogue of the upper bound implicit in (1.6) in which general complex weights are present, thereby providing a strengthening of the conclusions of Bourgain and Demeter [4] in this context.

After a brief detour in §2 in which we establish the proof of Corollary 1.2 assuming the conclusion of Theorem 1.1, we attend to the main focus of this paper, namely the confirmation of the asymptotic formula for $N(B)$ presented in the latter theorem. This is achieved by means of the Hardy-Littlewood method in §§3-9. We begin in §3 by describing the infrastructure for our slightly non-standard application of the circle method. The minor arc integral is handled in §4 by utilising what, elsewhere, we have referred to as a complification process. Preliminary work on the major arc integral in §5 is followed in §§6 and 7 by the completion of the truncated singular integral and singular series, respectively. We evaluate these completions in §§8 and 9, thereby completing the proof of Theorem 1.1. The evaluation of the completed singular integral in §9 is pursued in some detail in order to make transparent the connection between the formal singular integral and the quantity χ_∞ defined via the Siegel volume in (1.3). Our treatment should be of sufficient independent interest that scholars may find it to be of utility in quite general analyses employing the circle method.

Our basic parameter is B, a sufficiently large positive number. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon>0$. In this paper, implicit constants in Vinogradov's notation $\ll$ and $\gg$ may depend on ε. We write $X \asymp Y$ when $X \ll Y \ll X$. We make frequent use of vector notation in the form $\mathbf{x}=(x_1,\ldots,x_r)$. Here, the dimension r depends on the course of the argument. As is conventional in analytic number theory, we write $e(z)$ for $e^{2\pi i z}$, and, when $q\in\mathbb{N}$, we put $e_q(z)=e^{2\pi i z/q}$.

Acknowledgements: The authors acknowledge support by Akademie der Wissenschaften zu Göttingen and Deutsche Forschungsgemeinschaft. The second author's work was supported by a European Research Council Advanced Grant under the European Union's Horizon 2020 research and innovation programme via grant agreement No. 695223.

2. Deduction of the corollary

We first establish the lower bound implicit in the conclusion of Corollary 1.2. Let

$$g(\alpha,\beta)=\sum_{1\leqslant x\leqslant X} e(\alpha x^3+\beta x)$$

and write $G(\boldsymbol{\alpha})=g(\alpha_1,\alpha_2)g(\alpha_1,\alpha_3)$, so that

$$G(\boldsymbol{\alpha})=\sum_{1\leqslant x,y\leqslant X} e(\alpha_1(x^3+y^3)+\alpha_2 x+\alpha_3 y).$$

Also, put $\tau=1/24$, and define the box

$$\mathfrak{B}=[0,\tau X^{-3}]\times[0,\tau X^{-1}]\times[0,\tau X^{-1}].$$

Then whenever $1\leqslant x,y\leqslant X$ and $\boldsymbol{\alpha}\in\mathfrak{B}$, one has

$$0\leqslant \alpha_1(x^3+y^3)+\alpha_2 x+\alpha_3 y\leqslant 4\tau,$$

whence

$$|G(\boldsymbol{\alpha})|\geqslant \lfloor X\rfloor^2\cos(8\pi\tau)=\tfrac{1}{2}\lfloor X\rfloor^2.$$

Thus,

$$\int_{\mathfrak{B}}|G(\boldsymbol{\alpha})|^5\,\mathrm{d}\boldsymbol{\alpha}\gg X^{10}\mathrm{mes}(\mathfrak{B})\asymp X^5,$$

and the desired lower bound follows.

In order to derive the corresponding upper bound from Theorem 1.1, we begin by noting that

$$|G(\boldsymbol{\alpha})|^5\leqslant |g(\alpha_1,\alpha_2)^4g(\alpha_1,\alpha_3)^6|+|g(\alpha_1,\alpha_2)^6g(\alpha_1,\alpha_3)^4|.$$

By integrating over the unit cube and applying orthogonality, we thus obtain

$$\int_{[0,1)^3}|G(\boldsymbol{\alpha})|^5\,\mathrm{d}\boldsymbol{\alpha}\leqslant 2N(X),$$

and the desired upper bound follows from Theorem 1.1. Having established Corollary 1.2, the inquisitive reader will find the proof of (1.6) to be routine via suitable applications of Hölder's inequality in combination with the trivial estimate $|G(\boldsymbol{\alpha})|\leqslant X^2$.

3. Preliminary manoeuvres

We now embark on our main mission of establishing Theorem 1.1. We begin by introducing the exponential sum

$$f(\alpha_1,\alpha_2)=\sum_{|x|\leqslant B} e(\alpha_1 x^3+\alpha_2 x).$$

Notice that by a change of variables, one has

$$\overline{f(\alpha_1,\alpha_2)}=f(-\alpha_1,-\alpha_2)=f(\alpha_1,\alpha_2),$$

and hence $f(\alpha_1,\alpha_2)$ is real. In particular, one has

$$|f(\alpha_1,\alpha_2)|^2=f(\alpha_1,\alpha_2)f(-\alpha_1,-\alpha_2)=f(\alpha_1,\alpha_2)^2. \tag{3.1}$$

We shall feel free in what follows to insert or remove absolute values around even powers of such generating functions, implicitly making use of the relations (3.1), without further comment. Then, by orthogonality, one has

$$N(B)=\int_{[0,1)^3} f(\alpha,\beta)^6 f(\alpha,\gamma)^4\,\mathrm{d}\boldsymbol{\alpha}, \tag{3.2}$$

where we use $\boldsymbol{\alpha}$ to denote (α,β,γ).

We analyse $N(B)$ by means of the circle method, and this entails a certain Hardy-Littlewood dissection of non-standard type. Put $\delta=1/9$, and let $\mathfrak{M}$ denote the union of the intervals

$$\mathfrak{M}(q,a)=\{\alpha\in[0,1):|\alpha-a/q|\leqslant B^{\delta-3}\},$$

with $0\leqslant a\leqslant q\leqslant B^\delta$ and $(a,q)=1$. Complementary to this set of major arcs are the minor arcs $\mathfrak{m}=[0,1)\setminus\mathfrak{M}$. Thus, on writing

$$N(B;\mathfrak{B})=\int_{\mathfrak{B}}\int_0^1\int_0^1 f(\alpha,\beta)^6 f(\alpha,\gamma)^4\,\mathrm{d}\gamma\,\mathrm{d}\beta\,\mathrm{d}\alpha, \tag{3.3}$$

it follows that

$$N(B)=N(B;\mathfrak{M})+N(B;\mathfrak{m}). \tag{3.4}$$

Next we introduce the auxiliary integral

$$u(n;\mathfrak{B})=\int_{\mathfrak{B}}\int_0^1 f(\alpha,\beta)^6 e(-n\alpha)\,\mathrm{d}\beta\,\mathrm{d}\alpha. \tag{3.5}$$

Write also $v(n)$ for the number of representations of the integer n in the form

$$n=x_1^3+x_2^3+x_3^3+x_4^3,$$

with

$$-B\leqslant x_1,\dots,x_4\leqslant B\quad\text{and}\quad x_1+x_2+x_3+x_4=0. \tag{3.6}$$

Thus, we have

$$\sum_{|n|\leqslant 4B^3} u(n;\mathfrak{B})v(n)=\sum_{x_1,\dots,x_4}\int_{\mathfrak{B}}\int_0^1 f(\alpha,\beta)^6 e\left(-(x_1^3+x_2^3+x_3^3+x_4^3)\alpha\right)\,\mathrm{d}\beta\,\mathrm{d}\alpha,$$

in which the summation over $x_1,\dots,x_4$ is again subject to the conditions (3.6). Hence, by employing orthogonality and recalling the notation (3.3), one infers that

$$N(B;\mathfrak{B})=\sum_{|n|\leqslant 4B^3} u(n;\mathfrak{B})v(n). \tag{3.7}$$

We compute $N(B;\mathfrak{m})$ in the next section, and $N(B;\mathfrak{M})$ in §§5–9.

4. The minor arcs

The point of departure in our analysis of the minor arcs is the relation (3.7), and here we isolate the term with $n = 0$ for special attention. Note that $v(0)$ counts the number of integral solutions of the system

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$$
$$x_1 + x_2 + x_3 + x_4 = 0,$$

with $|x_i| \leqslant B$ $(1 \leqslant i \leqslant 4)$. There are $(2B)^2 + O(B)$ integral solutions with $x_1 + x_2 = x_3 + x_4 = 0$, and when $x_1 + x_2$ is non-zero, one sees that

$$x_1^2 - x_1x_2 + x_2^2 = x_3^2 - x_3x_4 + x_4^2,$$

whence $x_1x_2 = x_3x_4$. In this second class of solutions, one therefore has $\{x_1, x_2\} = \{-x_3, -x_4\}$, and there are $2(2B)^2 + O(B)$ such integral solutions. We may thus conclude that

$$v(0) = 3(2B)^2 + O(B). \tag{4.1}$$

Meanwhile, by orthogonality $u(0; [0, 1))$ counts the number of integral solutions of the system

$$\sum_{i=1}^{6} x_i^3 = \sum_{i=1}^{6} x_i = 0, \tag{4.2}$$

with $|x_i| \leqslant B$ $(1 \leqslant i \leqslant 6)$. By the methods of [5] and [15], one therefore has

$$u(0; [0, 1)) = 15(2B)^3 + O\left(B^2(\log B)^5\right). \tag{4.3}$$

A few words of explanation are required to justify this assertion. What these methods show is that there are $O\left(B^2(\log B)^5\right)$ solutions of (4.2) with $|x_i| \leqslant B$ not lying on the linear spaces defined by the equations

$$x_{j_1} + x_{j_2} = x_{j_3} + x_{j_4} = x_{j_5} + x_{j_6} = 0,$$

wherein $\{j_1, j_2, \ldots, j_6\} = \{1, 2, \ldots, 6\}$. The number of such linear spaces is

$$\frac{1}{3!}\binom{6}{2}\binom{4}{2} = 15,$$

and each contains $(2B)^3 + O(B^2)$ integral points. This confirms the asymptotic formula (4.3).

An asymptotic formula for $u(n; \mathfrak{m})$ is obtained from a crude upper bound for $u(n; \mathfrak{M})$, which we now derive. Observe that $\mathrm{mes}(\mathfrak{M}) = O(B^{3\delta-3})$, and by orthogonality one has

$$u(0; \mathfrak{M}) = \int_{\mathfrak{M}} \sum_{\substack{x_1+\ldots+x_6=0 \\ -B\leqslant x_1,\ldots,x_6\leqslant B}} e\left(\alpha(x_1^3 + \ldots + x_6^3)\right) \,\mathrm{d}\alpha$$
$$\ll B^5 \,\mathrm{mes}(\mathfrak{M}) \ll B^{2+3\delta}.$$

Consequently,

$$u(0; \mathfrak{m}) = u(0; [0, 1)) - u(0; \mathfrak{M}) = 15(2B)^3 + O(B^{2+3\delta}),$$

and it follows from (4.1) that

$$u(0;\mathfrak{m})v(0) = 45(2B)^5 + O(B^{4+3\delta}). \tag{4.4}$$

When n is non-zero, it follows from its definition that $v(n)$ is bounded above by the number of integral solutions of the equation

$$n = (x_1^3 + x_2^3 + x_3^3) - (x_1 + x_2 + x_3)^3,$$

and hence of

$$n = -3(x_1 + x_2)(x_2 + x_3)(x_3 + x_1).$$

By employing an elementary estimate for the divisor function, one sees that for a fixed non-zero integer n, there are $O(n^\varepsilon)$ possible choices for $x_1 + x_2$, $x_2 + x_3$ and $x_3 + x_1$, and hence also for x_1, x_2 and x_3. Thus $v(n) = O(n^\varepsilon)$. It therefore follows from (3.5) via the inequalities of Cauchy and Bessel that

$$\begin{aligned}\Bigg(\sum_{0<|n|\leqslant 4B^3} u(n;\mathfrak{m})v(n)\Bigg)^2 &\ll B^{3+\varepsilon} \sum_{|n|\leqslant 4B^3} |u(n;\mathfrak{m})|^2 \\ &\ll B^{3+\varepsilon} \int_{\mathfrak{m}} \bigg(\int_0^1 f(\alpha,\beta)^6 \,\mathrm{d}\beta\bigg)^2 \mathrm{d}\alpha \\ &\ll B^{3+\varepsilon} \int_{\mathfrak{m}} \int_0^1 \int_0^1 f(\alpha,\beta)^6 f(\alpha,\gamma)^6 \,\mathrm{d}\gamma \,\mathrm{d}\beta \,\mathrm{d}\alpha.\end{aligned}$$

As a consequence of Weyl's inequality (see [14, Lemma 2.5]), one sees that

$$\sup_{\alpha\in\mathfrak{m}} \sup_{\gamma\in\mathbb{R}} |f(\alpha,\gamma)| \ll B^{1-\delta/4+\varepsilon}.$$

Thus, by reference to (3.2), we deduce that

$$\begin{aligned}\Bigg(\sum_{0<|n|\leqslant 4B^3} u(n;\mathfrak{m})v(n)\Bigg)^2 &\ll B^{3+\varepsilon}(B^{1-\delta/4})^2 \int_{[0,1)^3} f(\alpha,\beta)^6 f(\alpha,\gamma)^4 \,\mathrm{d}\boldsymbol{\alpha} \\ &\ll B^{5-\delta/3} N(B),\end{aligned}$$

whence

$$\sum_{0<|n|\leqslant 4B^3} u(n;\mathfrak{m})v(n) \ll B^{5/2-\delta/6} N(B)^{1/2}.$$

On recalling (3.7) and (4.4), we conclude that

$$N(B;\mathfrak{m}) = 45(2B)^5 + O\left(B^{4+3\delta} + B^{5/2-\delta/6} N(B)^{1/2}\right). \tag{4.5}$$

5. The major arcs

We begin by extending our definition of one dimensional major arcs to corresponding three dimensional arcs. When $0 \leqslant a, b, c \leqslant q \leqslant B^\delta$ and $(a, q) = 1$, let $\mathfrak{N}(q, a, b, c)$ denote the set of triples $(\alpha, \beta, \gamma) \in [0, 1)^3$ such that $\alpha \in \mathfrak{M}(q, a)$,

$$-\frac{1}{2q} \leqslant \beta - \frac{b}{q} < \frac{1}{2q} \quad \text{and} \quad -\frac{1}{2q} \leqslant \gamma - \frac{c}{q} < \frac{1}{2q}.$$

We observe that these boxes $\mathfrak{N}(q,a,b,c)$ are disjoint, and hence their union $\mathfrak{N}$ is equal to $\mathfrak{M}\times[0,1)^2$.

We next introduce the standard major arc approximants to $f(\alpha_1,\alpha_2)$. When $a_1,a_2\in\mathbb{Z}$ and $q\in\mathbb{N}$, we put

$$S(q,a_1,a_2)=\sum_{r=1}^{q} e_q(a_1r^3+a_2r).$$

Likewise, when $\beta_1,\beta_2\in\mathbb{R}$, we write

$$v(\beta_1,\beta_2;B)=\int_{-B}^{B} e(\beta_1\gamma^3+\beta_2\gamma)\,\mathrm{d}\gamma.$$

It is convenient in what follows to abbreviate $v(\beta_1,\beta_2;B)$ to $v(\beta_1,\beta_2)$, and $v(\beta_1,\beta_2;\frac12)$ to $v_1(\beta_1,\beta_2)$. Finally, when $(\alpha,\beta,\gamma)\in\mathfrak{N}(q,a,b,c)\subseteq\mathfrak{N}$, we write

$$f^*(\alpha,\beta)=q^{-1}S(q,a,b)v(\alpha-a/q,\beta-b/q) \tag{5.1}$$

and

$$f^*(\alpha,\gamma)=q^{-1}S(q,a,c)v(\alpha-a/q,\gamma-c/q). \tag{5.2}$$

Notice that the cosmetic ambiguity in this definition would arise only when $\beta=\gamma$, in which case the two definitions coincide. Then, by appealing to the final conclusion (2.4) of [6, Theorem 3], we see that when $(\alpha,\beta,\gamma)\in\mathfrak{N}(q,a,b,c)\subseteq\mathfrak{N}$, one has

$$f(\alpha,\beta)-f^*(\alpha,\beta)\ll q^{2/3+\varepsilon}\leqslant B^\delta \tag{5.3}$$

and

$$f(\alpha,\gamma)-f^*(\alpha,\gamma)\ll q^{2/3+\varepsilon}\leqslant B^\delta. \tag{5.4}$$

Our first lemma relates $N(B;\mathfrak{M})$ to the mean value

$$N^*(B)=\int_{\mathfrak{N}} f^*(\alpha,\beta)^6 f^*(\alpha,\gamma)^4\,\mathrm{d}\boldsymbol{\alpha}. \tag{5.5}$$

Lemma 5.1. *One has* $N(B;\mathfrak{M})-N^*(B)\ll B^{5-\delta}$.

Proof. Suppose that $(\alpha,\beta,\gamma)\in\mathfrak{N}$. Then, by making use of the trivial estimate $f(\alpha_1,\alpha_2)=O(B)$ in combination with (5.3) and (5.4), we see that

$$f^*(\alpha,\beta)^6f^*(\alpha,\gamma)^4=\big(f(\alpha,\beta)+O(B^\delta)\big)^6\big(f(\alpha,\gamma)+O(B^\delta)\big)^4,$$

whence

$$f^*(\alpha,\beta)^6f^*(\alpha,\gamma)^4-f(\alpha,\beta)^6f(\alpha,\gamma)^4\ll B^\delta T_1(\boldsymbol{\alpha})+B^{2\delta}T_2(\boldsymbol{\alpha})+B^{7+3\delta}, \tag{5.6}$$

where

$$T_1(\boldsymbol{\alpha})=|f(\alpha,\beta)^5f(\alpha,\gamma)^4|+|f(\alpha,\beta)^6f(\alpha,\gamma)^3|$$

and

$$T_2(\boldsymbol{\alpha})=|f(\alpha,\beta)^4f(\alpha,\gamma)^4|+|f(\alpha,\beta)^5f(\alpha,\gamma)^3|+|f(\alpha,\beta)^6f(\alpha,\gamma)^2|.$$

Making use of the trivial estimate $|z_1\cdots z_n|\leqslant|z_1|^n+\ldots+|z_n|^n$, we find that

$$T_1(\boldsymbol{\alpha})\ll|f(\alpha,\beta)^7f(\alpha,\gamma)^2|+|f(\alpha,\beta)^2f(\alpha,\gamma)^7|$$

and

$$T_2(\boldsymbol{\alpha}) \ll f(\alpha,\beta)^6 f(\alpha,\gamma)^2 + f(\alpha,\beta)^2 f(\alpha,\gamma)^6.$$

Integrating $T_1(\boldsymbol{\alpha})$ over $\boldsymbol{\alpha} \in \mathfrak{N}$ and invoking symmetry, we obtain

$$\int_{\mathfrak{N}} T_1(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha} \ll \int_0^1 \int_0^1 |f(\alpha,\beta)|^7 \int_0^1 |f(\alpha,\gamma)|^2 \,\mathrm{d}\gamma\,\mathrm{d}\beta\,\mathrm{d}\alpha.$$

By orthogonality, therefore, together with an application of Hölder's inequality, we see that

$$\begin{aligned}\int_{\mathfrak{N}} T_1(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha} &\ll B \int_0^1 \int_0^1 |f(\alpha,\beta)|^7 \,\mathrm{d}\beta\,\mathrm{d}\alpha \\ &\ll B \Upsilon_6^{3/4} \Upsilon_{10}^{1/4},\end{aligned}$$

where for even exponents t we write

$$\Upsilon_t = \int_0^1 \int_0^1 f(\alpha,\beta)^t \,\mathrm{d}\beta\,\mathrm{d}\alpha.$$

The case $k = 3$ of [9, Lemma 5.2] delivers the bound

$$\Upsilon_6 \ll B^{3+\varepsilon}, \tag{5.7}$$

and by equation (2) in [9, §5 of Chapter V], meanwhile, one has $\Upsilon_{10} \ll B^{6+\varepsilon}$. Note that these sources in fact count solutions of the underlying Diophantine equations in which the variables are positive, whereas our bounds assert that the number of solutions, both positive and negative, be so bounded. The reader should have no difficulty, however, in either adapting the methods underlying these cited bounds, or indeed deriving the stated results through application of the triangle inequality. Improved bounds for the former mean value are the subject of [15], whilst the second is handled more precisely in [6] and [16]. Thus we obtain the estimate

$$\int_{\mathfrak{N}} T_1(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha} \ll B^{1+\varepsilon}(B^3)^{3/4}(B^6)^{1/4} \ll B^{5-2\delta}.$$

Similarly, in view of (5.7), one deduces that

$$\begin{aligned}\int_{\mathfrak{N}} T_2(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha} &\ll \int_0^1 \int_0^1 f(\alpha,\beta)^6 \int_0^1 f(\alpha,\gamma)^2 \,\mathrm{d}\gamma\,\mathrm{d}\beta\,\mathrm{d}\alpha \\ &\ll B \int_0^1 \int_0^1 f(\alpha,\beta)^6 \,\mathrm{d}\beta\,\mathrm{d}\alpha \ll B^{4+\varepsilon}.\end{aligned}$$

Since, in addition, one has $\mathrm{mes}(\mathfrak{M}) \ll B^{2\delta-3}$, we conclude from (5.5) and (5.6) that

$$\begin{aligned}N^*(B) - N(B;\mathfrak{M}) &\ll B^{\delta}(B^{5-2\delta}) + B^{2\delta}(B^{4+\varepsilon}) + B^{7+3\delta}(B^{3\delta-3}) \\ &\ll B^{5-\delta}.\end{aligned}$$

This completes the proof of the lemma. □

On combining (4.5) and the conclusion of Lemma 5.1 by means of (3.4), we may conclude thus far that

$$N(B) - N^*(B) - 45(2B)^5 \ll B^{5/2-\delta/6} N(B)^{1/2} + B^{5-\delta}. \tag{5.8}$$

The remainder of this paper will be consumed by the task of showing that

$$N^*(B) = \mathcal{C}(2B)^5 + O(B^{5-\delta/20}).$$

As is evident from (5.8), this asymptotic relation suffices to confirm that

$$N(B) - (45 + \mathcal{C})(2B)^5 \ll B^{5/2-\delta/6} N(B)^{1/2} + B^{5-\delta/20},$$

whence

$$N(B) = (45 + \mathcal{C})(2B)^5 + O(B^{5-\delta/20}). \tag{5.9}$$

This confirms (1.5) and, subject to verifying that $\mathcal{C} > 0$, completes the proof of Theorem 1.1.

Before proceeding further, we introduce the truncated singular integral

$$I(q) = \int_{-B^{\delta-3}}^{B^{\delta-3}} \int_{-1/(2q)}^{1/(2q)} \int_{-1/(2q)}^{1/(2q)} v(\xi,\eta)^6 v(\xi,\zeta)^4 \, d\zeta \, d\eta \, d\xi \tag{5.10}$$

and the auxiliary sum

$$A(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{b=1}^{q} \sum_{c=1}^{q} q^{-10} S(q,a,b)^6 S(q,a,c)^4. \tag{5.11}$$

Then in view of the definitions (5.1), (5.2) and (5.5), we may write

$$N^*(B) = \sum_{1 \leqslant q \leqslant B^\delta} I(q) A(q). \tag{5.12}$$

6. The completion of the singular integral

Our next step is to complete the truncated singular integral defined in (5.10) so as to obtain the completed singular integral

$$\mathfrak{J}(B) = \int_{\mathbb{R}^3} v(\xi,\eta)^6 v(\xi,\zeta)^4 \, d\zeta \, d\eta \, d\xi. \tag{6.1}$$

In pursuit of this goal, we recall from [14, Theorem 7.3] the bound

$$v(\alpha_1, \alpha_2) \ll B(1 + |\alpha_2| B + |\alpha_1| B^3)^{-1/3}.$$

This delivers the estimate

$$v(\xi,\eta)^6 v(\xi,\zeta)^4 \ll B^{10} \left((1 + |\xi| B^3)(1 + |\eta| B)(1 + |\zeta| B) \right)^{-10/9}.$$

In particular, it is apparent from (6.1) that the singular integral $\mathfrak{J}(B)$ exists, and that

$$\mathfrak{J}(B) \ll B^5. \tag{6.2}$$

Also, when $1 \leqslant q \leqslant B^\delta$, one sees that whenever

$$(\xi,\eta,\zeta) \in \mathbb{R}^3 \setminus \left([-B^{\delta-3}, B^{\delta-3}] \times [-1/(2q), 1/(2q)]^2 \right),$$

then

$$(1+|\xi|B^3)(1+|\eta|B)(1+|\zeta|B) > B^\delta.$$

Hence,

$$\begin{aligned}\mathfrak{J}(B) - I(q) &\ll B^{10-\delta/18}\int_{\mathbb{R}^3}\big((1+|\xi|B^3)(1+|\eta|B)(1+|\zeta|B)\big)^{-19/18}\,\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}\xi\\ &\ll B^{5-\delta/18}. \qquad (6.3)\end{aligned}$$

We summarise (6.2) and (6.3) in the form of a lemma.

Lemma 6.1. *When* $1 \leqslant q \leqslant B^\delta$, *one has*

$$I(q) \ll B^5 \quad \textit{and} \quad I(q) = \mathfrak{J}(B) + O(B^{5-\delta/18}).$$

In preparation for the next section of our argument, we introduce the truncated singular series

$$\mathfrak{S}(Q) = \sum_{1\leqslant q\leqslant Q} A(q), \qquad (6.4)$$

in which $A(q)$ is defined by (5.11). Then, on substituting the conclusion of Lemma 6.1 into (5.12), we infer that

$$N^*(B) = \big(\mathfrak{J}(B) + O(B^{5-\delta/18})\big)\,\mathfrak{S}(B^\delta). \qquad (6.5)$$

7. The completion of the singular series

The next phase of our discussion is focused on the completion of the truncated singular series (6.4), thereby delivering the completed singular series

$$\mathfrak{S} = \sum_{q=1}^{\infty} A(q). \qquad (7.1)$$

This step in our argument makes use of the following auxiliary lemma.

Lemma 7.1. *When* $(a,q)=1$, *one has*

$$\sum_{b=1}^{q} S(q,a,b)^4 \ll q^{3+\varepsilon}.$$

Proof. By orthogonality, one has

$$\begin{aligned} q^{-1}\sum_{b=1}^{q}\sum_{r_1,\ldots,r_4 \pmod q} & e_q\big(a(r_1^3+\ldots+r_4^3)+b(r_1+\ldots+r_4)\big)\\ &= \sum_{\substack{r_1,\ldots,r_4 \pmod q\\ r_1+\ldots+r_4\equiv 0 \pmod q}} e_q\big(a(r_1^3+\ldots+r_4^3)\big)\\ &= \sum_{r_1,r_2,r_3 \pmod q} e_q\left(-3a(r_1+r_2)(r_2+r_3)(r_3+r_1)\right). \qquad (7.2)\end{aligned}$$

On substituting $u = r_1 + r_2$ and $v = r_2 + r_3$, we see that the right hand side of (7.2) is equal to

$$\sum_{u,v,r_3 \pmod q} e_q\left(-3auv(2r_3 + u - v)\right).$$

The sum over r_3 here makes a non-zero contribution only when $q|6auv$. Thus, since $(a, q) = 1$, we find that this expression is bounded above in absolute value by

$$q \sum_{\substack{u,v \pmod q \\ q|6uv}} 1 \ll q^{2+\varepsilon}.$$

On recalling (7.2), the conclusion of the lemma now follows. □

We next recall an estimate due to Hua (see [14, Theorem 7.1]) asserting that whenever $(a_1, q) = 1$, one has

$$S(q, a_1, a_2) \ll q^{2/3+\varepsilon}.$$

On recalling (5.11), we deduce via Lemma 7.1 that

$$A(q) \ll q^{\varepsilon-2/3} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(q^{-4} \sum_{b=1}^{q} S(q, a, b)^4\right)^2 \ll q^{3\varepsilon-5/3}. \tag{7.3}$$

It therefore follows from (6.4) and (7.1) that the singular series $\mathfrak{S} = \lim_{Q\to\infty} \mathfrak{S}(Q)$ converges, and moreover that $\mathfrak{S} - \mathfrak{S}(B^\delta) \ll B^{-\delta/2}$. In particular, on recalling Lemma 6.1 and (6.5), we may conclude thus far that

$$N^*(B) = \mathfrak{S}\mathfrak{J}(B) + O(B^{5-\delta/18}). \tag{7.4}$$

8. The evaluation of the singular series

We have yet to interpret the singular series as a product of local densities, the first step towards this goal being that of establishing the multiplicative nature of $A(q)$. We put

$$\Psi(\mathbf{y}) = \sum_{i=1}^{8} y_i^3 - (y_1 + \ldots + y_5)^3 - (y_6 + y_7 + y_8)^3,$$

and then set

$$T(q, a) = \sum_{y_1,\ldots,y_8 \pmod q} e_q\left(a\Psi(\mathbf{y})\right).$$

Then, by orthogonality, we may eliminate two variables and infer that

$$q^{-2} \sum_{b=1}^{q} \sum_{c=1}^{q} S(q, a, b)^6 S(q, a, c)^4 = T(q, a).$$

Thus we have

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-8} T(q, a) = A(q).$$

The standard theory of singular series (see [14, Lemmata 2.10 and 2.11]) therefore shows that $A(q)$ is a multiplicative function of q.

Observe next that when p is prime and H is a non-negative integer, it follows from (7.3) that

$$\sum_{h=0}^{H} A(p^h) = 1 + O(p^{-3/2}).$$

Thus, writing

$$\widetilde{\chi}_p = \sum_{h=0}^{\infty} A(p^h),$$

we have $\widetilde{\chi}_p = 1 + O(p^{-3/2})$, and hence the product $\prod_p \widetilde{\chi}_p$ converges. But one has

$$p^{7H} \sum_{h=0}^{H} A(p^h) = p^{-H} \sum_{a=1}^{p^H} T(p^H, a),$$

and by orthogonality, this is equal to the number of solutions of the congruence $\Psi(\mathbf{y}) \equiv 0 \pmod{p^H}$, with $1 \leqslant y_i \leqslant p^H$ $(1 \leqslant i \leqslant 8)$. Next back substituting $y_9 = -(y_1 + \ldots + y_5)$ and $y_{10} = -(y_6 + y_7 + y_8)$, it is apparent that this, in turn, is equal to $M_p(H)$. Hence

$$\widetilde{\chi}_p = \lim_{H\to\infty} \sum_{h=0}^{H} A(p^h) = \lim_{H\to\infty} (p^H)^{-7} M_p(H) = \chi_p,$$

where χ_p is defined as in (1.2).

Since $A(q)$ is multiplicative and the series $\mathfrak{S} = \sum_{q=1}^{\infty} A(q)$ is absolutely convergent, we deduce that

$$\mathfrak{S} = \prod_p \sum_{h=0}^{\infty} A(p^h) = \prod_p \chi_p. \tag{8.1}$$

Here we note that, by the positivity evident in (5.11), one has $\chi_p \geqslant 1$ for each prime number p, whence $\mathfrak{S} \geqslant 1$. In particular, the singular series $\mathfrak{S}$ is positive.

9. The evaluation of the singular integral

The evaluation of the singular integral $\mathfrak{J}(B)$ begins with the observation that, by employing a change of variable in (6.1), one obtains

$$\mathfrak{J}(B) = B^5 \int_{\mathbb{R}^3} v_1(\xi,\eta)^6 v_1(\xi,\zeta)^4 \,\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}\xi = B^5 \mathfrak{J}(1). \tag{9.1}$$

We next pursue a strategy proposed by Schmidt [12, 13] in a somewhat refined form.

When $0 < \eta \leqslant 1$, we define the auxiliary function

$$w_\eta(\beta) = \eta \left(\frac{\sin(\pi\eta\beta)}{\pi\eta\beta} \right)^2, \tag{9.2}$$

having Fourier transform

$$\widehat{w}_\eta(\gamma)=\int_{-\infty}^{\infty} w_\eta(\beta)e(-\beta\gamma)\,\mathrm{d}\beta=\max\{0,1-|\gamma|/\eta\}. \tag{9.3}$$

Here, the integral converges absolutely. One can apply the formula (9.3) to construct a continuous approximation to the indicator function of a box. When $0<\delta<\eta$, we define

$$W_{\eta,\delta}(\gamma)=\begin{cases}1, & \text{when } |\gamma|\leqslant \eta,\\ 1-\dfrac{|\gamma|-\eta}{\delta}, & \text{when } \eta<|\gamma|<\eta+\delta,\\ 0, & \text{when } |\gamma|\geqslant \eta+\delta.\end{cases}$$

Next we put

$$W_\eta^+(\gamma)=W_{\eta,\eta^2}(\gamma)\quad\text{and}\quad W_\eta^-(\gamma)=W_{\eta-\eta^2,\eta^2}(\gamma).$$

We observe that $W_\eta^+(\gamma)$ and $W_\eta^-(\gamma)$ supply upper and lower bounds for the characteristic function of the interval $[-\eta,\eta]$.

Since it follows from (9.3) that

$$W_{\eta,\delta}(\gamma)=(1+\eta/\delta)\widehat{w}_{\eta+\delta}(\gamma)-(\eta/\delta)\widehat{w}_\eta(\gamma),$$

we find that

$$W_\eta^+(\gamma)=(1+\eta^{-1})\widehat{w}_{\eta+\eta^2}(\gamma)-\eta^{-1}\widehat{w}_\eta(\gamma)$$

and

$$W_\eta^-(\gamma)=\eta^{-1}\widehat{w}_\eta(\gamma)+(1-\eta^{-1})\widehat{w}_{\eta-\eta^2}(\gamma).$$

Next set

$$F_1(\boldsymbol{\xi})=\sum_{i=1}^{10}\xi_i^3,\quad F_2(\boldsymbol{\xi})=\sum_{i=1}^{6}\xi_i,\quad F_3(\boldsymbol{\xi})=\sum_{i=7}^{10}\xi_i,$$

and define

$$V_\eta^\pm(\boldsymbol{\xi})=\prod_{i=1}^{3}W_\eta^\pm(F_i(\boldsymbol{\xi})). \tag{9.4}$$

For convenience, we write $\mathfrak{U}$ for the unit box $\left[-\frac{1}{2},\frac{1}{2}\right]^{10}$. Then our discussion thus far demonstrates that the volume $M_\infty(\eta)$ defined via (1.4) satisfies

$$\int_{\mathfrak{U}} V_\eta^-(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\leqslant M_\infty(\eta)\leqslant \int_{\mathfrak{U}} V_\eta^+(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}. \tag{9.5}$$

We now proceed by turning our attention to the Fourier side. Define

$$U(\boldsymbol{\eta})=\int_{\mathfrak{U}}\prod_{i=1}^{3}\eta_i^{-1}\widehat{w}_{\eta_i}(F_i(\boldsymbol{\xi}))\,\mathrm{d}\boldsymbol{\xi}. \tag{9.6}$$

Lemma 9.1. *Let η be a real number with $0<\eta<1$. Suppose that η_i is a real number with $|\eta_i-\eta|\leqslant\eta^2$ for $i=1,2,3$. Then*

$$U(\boldsymbol{\eta})=\mathfrak{J}(1)+O(\eta^{1/36}).$$

Proof. Put

$$K(\boldsymbol{\beta}) = \prod_{i=1}^{3} \eta_i^{-1} w_{\eta_i}(\beta_i).$$

Then by interchanging orders of integration, it follows from (9.6) that

$$U(\boldsymbol{\eta}) = \int_{\mathbb{R}^3} v_1(\beta_1, \beta_2)^6 v_1(\beta_1, \beta_3)^4 K(\boldsymbol{\beta}) \, \mathrm{d}\boldsymbol{\beta}.$$

Hence, on recalling (6.1), we see that

$$U(\boldsymbol{\eta}) - \mathfrak{J}(1) = \int_{\mathbb{R}^3} v_1(\beta_1, \beta_2)^6 v_1(\beta_1, \beta_3)^4 \left(K(\boldsymbol{\beta}) - 1\right) \mathrm{d}\boldsymbol{\beta}. \tag{9.7}$$

Next, put $\mathfrak{D} = [-\eta^{-1/2}, \eta^{-1/2}]^3$ and $\mathfrak{E} = \mathbb{R}^3 \setminus \mathfrak{D}$. From the power series expansion of $w_\eta(\beta)$ underlying (9.2), we have

$$0 \leqslant 1 - K(\boldsymbol{\beta}) \ll \min\{1, \eta^2(\beta_1^2 + \beta_2^2 + \beta_3^2)\}.$$

Hence, the absolute convergence of the integral $\mathfrak{J}(1)$ ensures that the contribution from integrating over $\mathfrak{D}$ on the right hand side of (9.7) is at most

$$\sup_{\boldsymbol{\beta} \in \mathfrak{D}} |1 - K(\boldsymbol{\beta})| \int_{\mathbb{R}^3} v_1(\beta_1, \beta_2)^6 v_1(\beta_1, \beta_3)^4 \, \mathrm{d}\boldsymbol{\beta} \ll \eta.$$

Meanwhile, on making use of an argument akin to that delivering (6.3), one finds that the corresponding contribution from $\mathfrak{E}$ is bounded above by

$$\int_{\mathfrak{E}} ((1 + |\beta_1|)(1 + |\beta_2|)(1 + |\beta_3|))^{-10/9} \, \mathrm{d}\boldsymbol{\beta} \ll (\eta^{-1/2})^{-1/18} = \eta^{1/36}.$$

Thus we infer from (9.7) that

$$U(\boldsymbol{\eta}) - \mathfrak{J}(1) \ll \eta^{1/36},$$

completing the proof of the lemma. □

By using the definitions of $W_\eta^\pm(\gamma)$ to expand the products (9.4) defining $V_\eta^\pm(\boldsymbol{\xi})$ as a linear combination of terms of the shape

$$\widehat{w}_{\eta_1}(F_1(\boldsymbol{\xi}))\widehat{w}_{\eta_2}(F_2(\boldsymbol{\xi})\widehat{w}_{\eta_3}(F_3(\boldsymbol{\xi})),$$

it follows from Lemma 9.1 that

$$\begin{aligned}\int_{\mathfrak{U}} V_\eta^+(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi} &= \left((\eta + \eta^2)(1 + 1/\eta) - \eta(1/\eta)\right)^3 \left(\mathfrak{J}(1) + O(\eta^{1/36})\right) \\ &= \left(8\eta^3 + O(\eta^4)\right) \left(\mathfrak{J}(1) + O(\eta^{1/36})\right) \end{aligned} \tag{9.8}$$

and

$$\begin{aligned}\int_{\mathfrak{U}} V_\eta^-(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi} &= \left(\eta(1/\eta) + (\eta - \eta^2)(1 - 1/\eta)\right)^3 \left(\mathfrak{J}(1) + O(\eta^{1/36})\right) \\ &= \left(8\eta^3 + O(\eta^4)\right) \left(\mathfrak{J}(1) + O(\eta^{1/36})\right). \end{aligned} \tag{9.9}$$

We conclude from (9.5), (9.8) and (9.9) that

$$(2\eta)^{-3} M_\infty(\eta) = \mathfrak{J}(1) + O(\eta^{1/36}).$$

Consequently, the definition (1.3) shows that

$$\chi_\infty = \lim_{\eta\to 0+} \big(\mathfrak{J}(1) + O(\eta^{1/36})\big) = \mathfrak{J}(1).$$

On recalling (7.4), (8.1) and (9.1), we may conclude that

$$N^*(B) = B^5\mathfrak{S}\mathfrak{J}(1) + O(B^{5-\delta/18}) = \mathcal{C}B^5 + O(B^{5-\delta/18}),$$

where $\mathcal{C} = \chi_\infty \prod_p \chi_p$. The conclusion of Theorem 1.1 now follows on verifying that $\mathcal{C} > 0$, as described in the argument leading to (5.9) above.

It remains to confirm that the real density χ_∞ is positive. This is routine. Observe first that the point

$$\mathbf{x}_0 = (1/\sqrt{3}, 0, 0, -1/\sqrt{3}, 0, \ldots, 0)$$

is a non-singular solution of the system of equations

$$F_1(\mathbf{x}) = F_2(\mathbf{x}) = F_3(\mathbf{x}) = 0. \tag{9.10}$$

By the Implicit Function Theorem (see [1, Theorem 7-6]), there exists a positive number δ with the property that whenever

$$|z_4 + 1/\sqrt{3}| < \delta \quad \text{and} \quad |z_i| < \delta \qquad (5 \leqslant i \leqslant 10),$$

then the equations (9.10) possesses a solution (x_1, x_2, x_3) with coordinates $x_i = x_i(\mathbf{z})$ satisfying

$$|x_1(\mathbf{z}) - 1/\sqrt{3}| \ll \delta, \quad |x_2(\mathbf{z})| \ll \delta, \quad |x_3(\mathbf{z})| \ll \delta.$$

Here, the implicit constants are absolute. Notice that whenever δ is sufficiently small, then the partial derivatives of the polynomials $F_i(x_1, x_2, x_3, \mathbf{z})$ $(i = 1, 2, 3)$ remain close to their values at $\mathbf{x}_0$. It therefore follows that there is an absolute constant Λ having the property that whenever $|\eta_i| < \Lambda\eta$ $(i = 1, 2, 3)$, then

$$|F_i(x_1(\mathbf{z}) + \eta_1, x_2(\mathbf{z}) + \eta_2, x_3(\mathbf{z}) + \eta_3, z_4, \ldots, z_{10})| < \eta.$$

In particular,

$$\begin{aligned} M_\infty(\eta) &\geqslant (2\Lambda\eta)^3 \mathrm{mes}\{(z_4, \ldots, z_{10}) \in (-\delta, \delta)^7\} \\ &= (2\Lambda\eta)^3(2\delta)^7 \gg \eta^3. \end{aligned}$$

Hence

$$\chi_\infty = \lim_{\eta\to 0+} (2\eta)^{-3} M_\infty(\eta) > 0.$$

Our method to treat the singular integral works in broad generality. Write $\mathfrak{U}_s = \left[-\frac{1}{2}, \frac{1}{2}\right]^s$. Suppose that an affine variety is defined by polynomial equations

$$F_1(x_1, \ldots, x_s) = \ldots = F_r(x_1, \ldots, x_s) = 0.$$

Then, whenever the formal singular integral

$$I = \int_{\mathbb{R}^r} \int e\left(\beta_1 F_1(\mathbf{x}) + \ldots + \beta_r F_r(\mathbf{x})\right) \mathrm{d}\mathbf{x}\, \mathrm{d}\boldsymbol{\beta}$$

converges absolutely, the above method shows that

$$I = \lim_{\eta\to 0+} (2\eta)^{-r} M_\infty(\eta),$$

where now $M_\infty(\eta)$ is the Lebesgue measure of the set of all $\mathbf{x} \in \mathfrak{U}_s$ for which $|F_j(\mathbf{x})| < \eta$ $(1 \leqslant j \leqslant r)$. Further, if the variety contains a non-singular point $\mathbf{x}_0 \in \mathfrak{U}_s$, then our methods also show that $I > 0$.

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