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# Fujii's development on Chebyshev's conjecture

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## Abstract

Chebyshev presented a conjecture after observing the apparent bias towards primes congruent to 3 (mod 4). His conjecture is equivalent to a version of the Generalised Riemann Hypothesis. Fujii strengthened this conjecture; we strengthen it still further using detailed computations of zeroes of Dirichlet  $L$ -functions.

## 1 Introduction

Chebyshev observed that there appear to be more primes congruent to 3 (mod 4) than 1 (mod 4). This bias has spawned much research — see, e.g. the seminal work by Rubinstein and Sarnak [10], and also Ford and Konyagin [2]. In 1853 Chebyshev conjectured that

$$\sum_{p>2} (-1)^{(p-1)/2} e^{-xp} \rightarrow -\infty, \quad (1)$$

as  $x \rightarrow 0$ . Hardy and Littlewood [4] and Landau [6] showed that (1) is equivalent to all of the non-trivial zeroes of  $L(s, \chi_4)$  having real part  $\sigma = \frac{1}{2}$ , where we use  $\chi_4$  to denote the

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non-principal Dirichlet character modulo 4. We shall refer to this specialised version of the Generalised Riemann Hypothesis for  $\chi_4$  as ‘GRH for  $\chi_4$ ’.

In this article we examine the attenuation factor  $e^{-xp}$  in (1). Fujii [3, Thm 1] showed that for all  $0 < \alpha < 4.19$  the statement

$$\sum_{p>2} (-1)^{(p-1)/2} e^{-(xp)^\alpha} \rightarrow -\infty \text{ as } x \rightarrow 0 \quad (2)$$

is equivalent to GRH for  $\chi_4$ . Note that the larger one can take  $\alpha$  the quicker the summands in (2) attenuate, and hence there must be an even greater bias towards primes congruent to 3 (mod 4).

Fujii’s argument is elegant; his result of  $\alpha < 4.19$  is a result of some numerical calculations involving the first few zeroes  $\beta + i\gamma$  of  $L(s, \chi_4)$ . In fact, Fujii only uses the fact that  $\gamma_1 > 6$  and that  $\sum_{\gamma>0} \gamma^{-2} < 1/5$ .

We use some more extensive calculations on the zeroes of  $L(s, \chi_4)$  and some optimisation to improve Fujii’s work. The result is the following theorem.

**Theorem 1.** *Suppose that  $0 < \alpha < 20.40442$ . Then the statement*

$$\sum_{p>2} (-1)^{(p-1)/2} e^{-(xp)^\alpha} \rightarrow -\infty \text{ as } x \rightarrow 0$$

*is equivalent to all of the non-trivial zeroes of  $L(s, \chi)$  having real part  $\sigma = \frac{1}{2}$ , where  $\chi$  is the non-principal Dirichlet character modulo 4.*

We introduce Fujii’s work in §2, and prove Theorem 1 in §3. We remark at the end of §3 that it appears impossible to improve Theorem 2 further using Fujii’s method.

## 2 Fujii’s method and some lemmas

Proceeding as in Fujii [3, §3], we have, under the assumption of GRH for  $\chi_4$ ,

$$S = \sum_{p>2} (-1)^{(p-1)/2} e^{-(xp)^\alpha} = S_1 + S_2 + S_3 + S_4,$$

where

$$\begin{aligned} S_1 &= -\frac{1}{2} \Gamma\left(\frac{1}{2\alpha}\right) x^{-1/2} + o(x^{-1/2}) \\ S_2 + S_4 &= o(x^{-1/2}) \\ |S_3| &\leq x^{-1/2} \sum_{\rho} \left| \Gamma\left(\frac{1}{2\alpha} + \frac{i\gamma}{\alpha}\right) \right|. \end{aligned}$$

Therefore, to show that  $S \rightarrow -\infty$  as  $x \rightarrow 0$  it is sufficient to show that

$$\sum_{\rho} \left| \Gamma\left(\frac{1}{2\alpha} + \frac{i\gamma}{\alpha}\right) \right| < \frac{1}{2} \Gamma\left(\frac{1}{2\alpha}\right). \quad (3)$$

We require an explicit version of Stirling's formula to bound the summands in (3). Many versions abound in the literature: we shall use the one given by Olver [8, p. 294], namely

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log 2\pi + \frac{\vartheta}{6|z|}, \quad (|\arg z| \leq \frac{\pi}{2}). \quad (4)$$

Using (4) and (3) and that fact that  $\tan^{-1} x < x$  for all  $x$ , we see that we have

$$\left| \Gamma\left(\frac{1}{2\alpha} + \frac{i\gamma}{\alpha}\right) \right| \leq (2\pi)^{1/2} \left( \frac{\sqrt{\gamma^2 + \frac{1}{4}}}{\alpha} \right)^{\frac{1}{2\alpha} - \frac{1}{2}} \exp\left(-\frac{\pi\gamma}{2\alpha} + \frac{\alpha}{6\sqrt{\gamma^2 + \frac{1}{4}}}\right).$$

We aim at writing the sum in (3) as  $\Sigma = \Sigma_1 + \Sigma_2$  where  $0 < \gamma \leq T_1$  in  $\Sigma_1$  and  $\gamma > T_1$  in  $\Sigma_2$ . We shall choose  $T_1$  such that we have detailed information on the location of zeroes with  $\gamma \leq T_1$ . We shall sum the contribution from these zeroes explicitly. We shall then estimate  $\Sigma_2$  using (2) and bounds on  $N(T, \chi_4)$ , the number of zeroes of  $L(s, \chi_4)$  with  $|\gamma| \leq T$ . We have such an estimate in [11], namely, that

$$\left| N(T, \chi_4) - \frac{T}{\pi} \log \frac{2T}{\pi e} \right| \leq C_1 \log 4T + C_2, \quad (T \geq 1), \quad (5)$$

where  $C_1$  and  $C_2$  are explicitly given constants. Note that the definition of  $N(T, \chi)$  counts zeroes with  $|\gamma| \leq T$ . We actually wish to count the zeroes with  $\gamma \geq 0$ . Therefore, we divide (5) by 2, giving us

$$N(T, \chi) = \frac{T}{2\pi} \log \frac{2T}{\pi} - \frac{T}{2\pi} + Q(T),$$

where

$$|Q(T)| \leq \frac{C_1}{2} \log 4T + \frac{C_2}{2} \leq \theta_1 \log T, \quad (T \geq T_1)$$

say. Henceforth we consider everything in terms of  $T_1$ , which will be the truncation point in the sum. We need the following, which is a trivial adaptation of a result by Lehman.

**Lemma 1.** *Let  $\phi(t)$  be a decreasing function with continuous derivative on  $[T_1, T_2]$ . For  $L(s, \chi)$  the non-principal  $L$ -function with  $\chi$  to the modulus 4 we have, for any  $T_1 \geq 1$  that*

$$\sum_{T_1 < \gamma \leq T_2} \phi(\gamma) = \frac{1}{2\pi} \int_{T_1}^{T_2} \phi(t) \log \frac{2t}{\pi} dt + \theta_1 \left\{ 2\phi(T_1) \log T_1 + \int_{T_1}^{T_2} \frac{\phi(t)}{t} dt \right\},$$

where  $C_1$  and  $C_2$  are in (5) and  $\theta_1$  is such that

$$\theta_1 \geq \frac{\frac{C_1}{2} \log 4T_1 + \frac{C_2}{2}}{\log T_1}. \quad (6)$$

*Proof.* The proof follows the proof given in Lehman [7, p. 400]. □

We now apply Lemma 1 with

$$\phi(t) = \exp \left\{ \frac{-\pi t}{2\alpha} \right\}$$

and send  $T_2 \rightarrow \infty$ . We obtain

$$\sum_{\gamma > T_1} \phi(\gamma) \leq \exp \left( -\frac{\pi T_1}{2\alpha} \right) \left[ \frac{\alpha}{\pi^2} \log \frac{2T_1 e}{\pi} + \theta_1 \log T_1^2 e \right]. \quad (7)$$

Putting this together with (2) we find that  $\alpha$  is admissible in Theorem 1 if

$$\begin{aligned} & 2(2\pi)^{1/2} \left( \frac{\sqrt{T_1^2 + \frac{1}{4}}}{\alpha} \right)^{\frac{1}{2\alpha} - \frac{1}{2}} \exp \left( \frac{\alpha}{6\sqrt{T_1^2 + \frac{1}{4}}} \right) \exp \left( -\frac{\pi T_1}{2\alpha} \right) \left[ \frac{\alpha}{\pi^2} \log \frac{2T_1 e}{\pi} + \theta_1 \log T_1^2 e \right] \\ & + 2 \sum_{0 < \gamma < T_1} \left| \Gamma \left( \frac{1}{2\alpha} + \frac{i\gamma}{\alpha} \right) \right| \leq \frac{1}{2} \Gamma \left( \frac{1}{2\alpha} \right), \end{aligned} \quad (8)$$

subject to (6). We replicate (6) here for convenience, and choose  $C_1 = 0.315$  and  $C_2 = 6.445$  as in [11]. Therefore, we require that (8) be satisfied along with

$$\theta_1 = \frac{0.1575 \log 4T_1 + 3.2225}{\log T_1}.$$

### 3 Computations and proof of Theorem 1

We used “lcalc” [9] to produce a list of the lowest 1 000 zeroes of  $L(s, \chi_4)$ . The output gives 11 decimal places reducing to 10 for the highest zeroes. We checked each  $t$  actually did represent a zero by using ARB [5] to rigorously compute

$$\frac{4^{s/2}}{\pi} \Gamma \left( \frac{s+1}{2} \right) 4^{-s} \left[ \zeta \left( s, \frac{1}{4} \right) - \zeta \left( s, \frac{3}{4} \right) \right]$$

with  $s = 1/2 + i(t - \delta)$  and  $s = 1/2 + i(t + \delta)$  with  $\delta = 10^{-10}$  and checking that we saw a sign change in every case. We then use a rigorous version of Turing’s method [1] to confirm that “lcalc” had (as expected) found all the zeroes with  $\Im \rho \in [0, 1127]$ .

Taking these lowest 1 000 zeroes we can set  $T_1 = 1127$  and we see that  $\alpha = 20.40442$  gives us

$$\sum_{|\Im \rho| < T_1} \left| \Gamma \left( \frac{1}{2\alpha} + \frac{i\gamma}{\alpha} \right) \right| < 20.1276643$$

whereas

$$\frac{1}{2} \Gamma \left( \frac{1}{2\alpha} \right) > 20.1276649.$$

We also find that the contribution from zeroes with  $|\Im\rho| > T_1$  is strictly less in absolute terms than  $4 \times 10^{-38}$  so  $\alpha = 20.40442$  is admissible.

Further, if we take  $\alpha = 20.40443$  we find that the sum over the zeroes with imaginary part  $< 1\,127$  exceeds  $\Gamma(1/(2\alpha))/2$  even if we ignore the contribution from the rest of the zeroes. Thus regardless of how many more zeroes we consider, how precisely we know their imaginary parts or how small we can make the constant  $\theta_1$ , we will never be able to show that  $\alpha = 20.40443$  is admissible by this method.

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