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ON A GENERALIZATION OF SPIKES*

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Abstract. We consider matroids with the property that every subset of the ground set of size t is contained in both an ℓ-element circuit and an ℓ-element cocircuit; we say that such a matroid has the (t, ℓ)-*property*. We show that for any positive integer t, there is a finite number of matroids with the (t, ℓ)-property for $\ell < 2t$; however, matroids with the $(t, 2t)$-property form an infinite family. We say a matroid is a t-*spike* if there is a partition of the ground set into pairs such that the union of any t pairs is a circuit and a cocircuit. Our main result is that if a sufficiently large matroid has the $(t, 2t)$-property, then it is a t-spike. Finally, we present some properties of t-spikes.

Key words. matroid, spike, circuit, cocircuit

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1. Introduction.

For all $r \geq 3$, a rank-r *spike* is a matroid on $2r$ elements with a partition $(X_1, X_2, \ldots, X_r)$ into pairs such that $X_i \cup X_j$ is a circuit and a cocircuit for all distinct $i, j \in \{1, 2, \ldots, r\}$. Spikes frequently arise in the matroid theory literature (see, for example, [2, 4, 8, 10]) as a seemingly benign, yet wild, class of matroids. Miller [5] proved that if M is a sufficiently large matroid having the property that every two elements share both a 4-element circuit and a 4-element cocircuit, then M is a spike.

We consider generalizations of this result. We say that a matroid M has the (t, ℓ)-*property* if every t-element subset of $E(M)$ is contained in both an ℓ-element circuit and an ℓ-element cocircuit. It is well known that the only matroids with the $(1, 3)$-property are wheels and whirls, and Miller's result shows that if M is a sufficiently large matroid with the $(2, 4)$-property, then M is a spike.

We first show that when $\ell < 2t$, there are only finitely many matroids with the (t, ℓ)-property. However, for any positive integer t, the matroids with the $(t, 2t)$-property form an infinite class: when $t = 1$, this is the class of matroids obtained by taking direct sums of copies of $U_{1,2}$; when $t = 2$, the class contains the infinite family of spikes. Our main result is the following theorem.

THEOREM 1.1. *There exists a function f such that if M is a matroid with the*

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$(t, 2t)$-property, and $|E(M)| \geq f(t)$, then $E(M)$ has a partition into pairs such that the union of any t pairs is both a circuit and a cocircuit.

We call a matroid with such a partition a *t-spike*. (A traditional spike is a 2-spike. Note also that what we call a spike is sometimes referred to as a *tipless spike*.)

We also prove some properties of t-spikes, which demonstrate that t-spikes are highly structured matroids. In particular, a t-spike has $2r$ elements for some positive integer r, it has rank r (and corank r), any circuit that is not a union of t pairs avoids at most $t-2$ of the pairs, and any sufficiently large t-spike is $(2t-1)$-connected. We show that a t-spike's partition into pairs describes crossing $(2t-1)$-separations in the matroid; that is, an appropriate concatenation of this partition is a $(2t-1)$-flower (more specifically, a $(2t-1)$-anemone), following the terminology of [1]. We also describe a construction of a $(t+1)$-spike from a t-spike, and show that every $(t+1)$-spike can be obtained from some t-spike in this way.

Our methods in this paper are extremal, so the lower bounds on $|E(M)|$ that we obtain, given by the function f, are extremely large, and we make no attempts to optimize these. For $t = 2$, Miller [5] showed that $f(2) = 13$ is best possible, and he described the other matroids with the $(2, 4)$-property when $|E(M)| \leq 12$. We see no reason why a similar analysis could not be undertaken for, say, $t = 3$.

There are a number of interesting variants of the (t, ℓ)-property. In particular, we say that a matroid has the *$(t_1, \ell_1, t_2, \ell_2)$-property* if every t_1-element set is contained in an ℓ_1-element circuit, and every t_2-element set is contained in an ℓ_2-element cocircuit. Although we focus here on the case where $t_1 = t_2$ and $\ell_1 = \ell_2$, we show, in section 3, that there are only finitely many matroids with the $(t_1, \ell_1, t_2, \ell_2)$-property when $\ell_1 < 2t_1$ or $\ell_2 < 2t_2$. Oxley et al. [7] recently considered the case where $(t_1, \ell_1, t_2, \ell_2) = (2, 4, 1, k)$ and $k \in \{3, 4\}$. In particular, they proved, for $k \in \{3, 4\}$, that a k-connected matroid M with $|E(M)| \geq k^2$ has the $(2, 4, 1, k)$-property if and only if $M \cong M(K_{k,n})$ for some $n \geq k$. This gives credence to the idea that sufficiently large matroids with the $(t_1, \ell_1, t_2, \ell_2)$-property, for appropriate values of t_1, ℓ_1, t_2, ℓ_2, may form structured classes. In particular, we conjecture the following generalization of Theorem 1.1.

Conjecture 1.2. *There exists a function $f(t_1, t_2)$ such that if M is a matroid with the $(t_1, 2t_1, t_2, 2t_2)$-property, for positive integers t_1 and t_2, and $|E(M)| \geq f(t_1, t_2)$, then $E(M)$ has a partition into pairs such that the union of any t_1 pairs is a circuit, and the union of any t_2 pairs is a cocircuit.*

The study of matroids with the $(t, 2t)$-property was motivated by problems in matroid connectivity. Tutte proved that wheels and whirls (that is, matroids with the $(1, 3)$-property) are the only 3-connected matroids with no element whose deletion or contraction preserves 3-connectivity [11]. Moreover, spikes (matroids with the $(2, 4)$-property) are the only 3-connected matroids with $|E(M)| \geq 13$ having no triangles or triads, and no pair of elements whose deletion or contraction preserves 3-connectivity [12]. We envision that t-spikes could also play a role in a connectivity "chain theorem": they are $(2t-1)$-connected matroids, having no circuits or cocircuits of size $(2t-1)$, with the property that for every t-element subset $X \subseteq E(M)$, neither M/X nor $M \backslash X$ is $(t+1)$-connected. We conjecture the following.

Conjecture 1.3. *There exists a function $f(t)$ such that if M is a $(2t-1)$-connected matroid with no circuits or cocircuits of size $2t-1$, and $|E(M)| \geq f(t)$, then either*

(i) *there exists a t-element set $X \subseteq E(M)$ such that either M/X or $M \backslash X$ is $(t+1)$-connected, or*

(ii) *M is a t-spike.*

This paper is structured as follows. In section 3, we prove that there are only finitely many matroids with the (t,ℓ)-property, for $\ell < 2t$. In section 4, we define t-echidnas and t-spikes, and show that a matroid with the $(t,2t)$-property and having a sufficiently large t-echidna is a t-spike. We prove Theorem 1.1 in section 5. Finally, we present some properties of t-spikes in section 6.

2. Preliminaries.

Our notation and terminology follow Oxley [6]. We refer to the fact that a circuit and a cocircuit cannot intersect in exactly one element as "orthogonality." We say that a k-element set is a *k-set*. A set S_1 *meets* a set S_2 if $S_1 \cap S_2 \neq \emptyset$. We denote $\{1,2,\dots,n\}$ by $[n]$, and, for positive integers $i<j$, we denote $\{i,i+1,\dots,j\}$ by $[i,j]$. We denote the set of positive integers by $\mathbb{N}$.

LEMMA 2.1. *There exists a function $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that, if $\mathcal{S}$ is a collection of distinct s-sets and $|\mathcal{S}| \geq f(s,n)$, then there is some $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = n$, and a set J with $0 \leq |J| < s$, such that $S_1 \cap S_2 = J$ for all distinct $S_1, S_2 \in \mathcal{S}'$.*

Proof. We define $f(1,n)=n$ and $f(s,n)=s(n-1)f(s-1,n)$ for $s>1$. Note that f is increasing. We claim that this function satisfies the lemma. We proceed by induction on s. If $s=1$, then the claim holds with $J=\emptyset$.

Let $\mathcal{S}$ be a collection of s-sets with $|\mathcal{S}| \geq f(s,n)$. Suppose there are n pairwise disjoint sets in $\mathcal{S}$. Then the desired conditions are satisfied if we take $J=\emptyset$. Thus, we may assume that there is some maximal $\mathcal{D} \subseteq \mathcal{S}$ consisting of pairwise disjoint sets, with $|\mathcal{D}| \leq n-1$. Each $S \in \mathcal{S}-\mathcal{D}$ meets some $D \in \mathcal{D}$. Each such D has s elements. Therefore, each $S \in \mathcal{S}$ contains at least one of $(n-1)s$ elements $e \in \cup\mathcal{D}$. By the pigeonhole principle, there is some $e \in \cup\mathcal{D}$ such that

$$|\{S \in \mathcal{S} : e \in S\}| \geq \frac{f(s,n)}{(n-1)s} = f(s-1,n).$$

Let $\mathcal{T} = \{S-\{e\} : e \in S \in \mathcal{S}\}$. Then, for every $T \in \mathcal{T}$, we have $|T| = s-1$. Moreover, $|\mathcal{T}| = |\{S \in \mathcal{S} : e \in S\}| \geq f(s-1,n)$. By the induction assumption, there is a subset $\mathcal{T}' \subseteq \mathcal{T}$, with $|\mathcal{T}'| = n$, and a set J', with $|J'| < s-1$, such that $T_1 \cap T_2 = J'$ for all distinct $T_1, T_2 \in \mathcal{T}'$. Let $\mathcal{S}' = \{T \cup \{e\} : T \in \mathcal{T}'\}$. Then, $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = n$ such that $S_1 \cap S_2 = J' \cup \{e\}$ for all distinct $S_1, S_2 \in \mathcal{S}'$ and $|J \cup \{e\}| < s$. ◻

3. Matroids with the (t,ℓ)-property for $\ell < 2t$.

Recall that a matroid has the (t_1,ℓ_1,t_2,ℓ_2)-property if every t_1-element set is contained in an ℓ_1-element circuit, and every t_2-element set is contained in an ℓ_2-element cocircuit. In this section, we prove that there are only finitely many matroids with the (t_1,ℓ_1,t_2,ℓ_2)-property if $\ell_2 < 2t_2$. By duality, the same is true if $\ell_1 < 2t_1$. As a special case, we have that there are only finitely many matroids with the (t,ℓ)-property for $\ell < 2t$.

LEMMA 3.1. *Let $\mathcal{C}$ be a collection of circuits of a matroid M such that, for some $J \subseteq E(M)$ with $|J| \leq k$, we have $C \cap C' = J$ for all distinct $C, C' \in \mathcal{C}$. Then, for every subcollection $\{C_1,\dots,C_{2^k}\} \subseteq \mathcal{C}$ of size 2^k, there is a circuit contained in $\bigcup_{i=1}^{2^k} C_i - J$.*

Proof. We may assume $|\mathcal{C}| \geq 2^k$; otherwise, the result holds vacuously. Also, we may assume $k>0$ as the result holds for any singleton subcollection of $\mathcal{C}$ with $J=\emptyset$. Therefore, $\mathcal{C}$ has at least one subcollection $\mathcal{C}' = \{C_1,\dots C_{2^k}\}$, with $|\mathcal{C}'| = 2^k \geq 2$.

Let $x_1,x_2,\dots,x_{|J|}$ be the elements of J. Define $Z_{i,0} = C_i$, for $i \in [2^k]$, and recursively define $Z_{i,j} = Z_{2i-1,j-1} \cup Z_{2i,j-1}$ for $j \in [k]$ and $i \in [2^{k-j}]$. Note that each $Z_{i,j}$ is the union of 2^j members of $\mathcal{C}$. We will show, by induction on j, that $Z_{i,j} - \{x_1, x_2, \ldots, x_j\}$ contains a circuit. This is clear when $j = 0$. Now let $j \geq 1$. By the induction hypothesis, $Z_{2i-1,j-1}$ and $Z_{2i,j-1}$ each contain a circuit, C_1' and C_2', respectively, disjoint from $\{x_1, x_2, \ldots, x_{j-1}\}$, for each $i \in [2^{k-j}]$. (Moreover, $C_1' \neq C_2'$ since $C_1' \cap C_2' \subseteq Z_{2i-1,j-1} \cap Z_{2i,j-1} \subseteq J$, which is independent since J is the intersection of at least two circuits.) We may assume that neither $Z_{2i-1,j-1}$ nor $Z_{2i,j-1}$ contains a circuit disjoint from $\{x_1, x_2, \ldots, x_j\}$; otherwise, so does $Z_{i,j}$. Thus, C_1' and C_2' both contain x_j. By circuit elimination, there is a circuit C_3' contained in $(C_1' \cup C_2') - \{x_j\} \subseteq Z_{i,j} - \{x_1, x_2, \ldots, x_j\}$. This completes the induction argument. In particular, there is a circuit contained in $Z_{1,k} - \{x_1, x_2, \ldots, x_{|J|}\} = \bigcup_{i=1}^{2^k} C_i - J$, as required. □

LEMMA 3.2. *There exists a function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that if M is a matroid having at least $g(\ell, d)$-many ℓ-element circuits, then M has a collection of d pairwise disjoint circuits.*

Proof. Let $\mathcal{C}$ be the collection of ℓ-element circuits of M, let f be the function of Lemma 2.1, and let $g(\ell, d) = f(\ell, 2^{\ell-1}d)$. Then, by Lemma 2.1, there is a subset $\mathcal{C}' \subseteq \mathcal{C}$, with $|\mathcal{C}'| = 2^{\ell-1}d$, and a set J, with $0 \leq |J| \leq \ell - 1$, such that $C \cap C' = J$ for every pair $C, C' \in \mathcal{C}'$. Say $\mathcal{C}' = \{C_1, C_2, \ldots, C_{2^{\ell-1}d}\}$.

If $J = \emptyset$, then M has $2^{\ell-1}d \geq d$ pairwise disjoint circuits, as required. Thus, we may assume that $J \neq \emptyset$. For each $C_i \in \mathcal{C}'$, let $D_i = C_i - J$, and observe that the D_i's are pairwise disjoint. For $j \in [d]$, let

$$D_j' = \bigcup_{i=1}^{2^{\ell-1}} D_{(j-1)(2^{\ell-1})+i}.$$

By Lemma 3.1, each D_j' contains a circuit C_j', and the C_j''s are pairwise disjoint. □

THEOREM 3.3. *Let t_1, ℓ_1, t_2, and ℓ_2 be positive integers. If $\ell_1 < 2t_1$ or $\ell_2 < 2t_2$, then there is a finite number of matroids with the $(t_1, \ell_1, t_2, \ell_2)$-property.*

Proof. By duality, it suffices to prove the result when $\ell_2 < 2t_2$. So let $\ell_2 < 2t_2$, and let g be the function given in Lemma 3.2.

Suppose M has at least $g(\ell_1, t_2)$-many ℓ_1-element circuits. By Lemma 3.2, M has a collection of t_2 pairwise disjoint circuits. Call this collection $\mathcal{C} = \{C_1, \ldots, C_{t_2}\}$. Let b_i be an element of C_i, for each $i \in [t_2]$. By the $(t_1, \ell_1, t_2, \ell_2)$-property, there is an ℓ_2-element cocircuit C^* containing $\{b_1, \ldots, b_{t_2}\}$. By orthogonality, for each $i \in [t_2]$ there is an element $b_i' \neq b_i$ such that $b_i' \in C_i \cap C^*$. This implies that $\ell_2 = |C^*| \geq 2t_2$; a contradiction. Thus, M has fewer than $g(\ell_1, t_2)$-many ℓ_1-element circuits.

Suppose $|E(M)| \geq \ell_1 \cdot g(\ell_1, t_2)$. Partition a subset of $E(M)$ into $\lfloor \ell_1/t_1 \rfloor \cdot g(\ell_1, t_2)$ pairwise disjoint t_1-sets. By the $(t_1, \ell_1, t_2, \ell_2)$-property, each of these t_1-sets is contained in an ℓ_1-element circuit. The collection consisting of these ℓ_1-element circuits contains at least $g(\ell_1, t_2)$ distinct circuits. This contradicts the fact that M has fewer than $g(\ell_1, t_2)$-many ℓ_1-element circuits. Therefore, $|E(M)| < \ell_1 \cdot g(\ell_1, t_2)$. The result follows. □

Note that there may still be infinitely many matroids where every t_1-element set is in an ℓ_1-element circuit for fixed $\ell_1 < 2t_1$; it is necessary that the matroids in Theorem 3.3 have the property that every t_2-element set is in an ℓ_2-element cocircuit, for fixed t_2 and ℓ_2. To see this, observe that projective geometries on at least three elements form an infinite family of matroids with the property that every pair of elements is in a 3-element circuit.

COROLLARY 3.4. *Let t and ℓ be positive integers. When $\ell < 2t$, there is a finite number of matroids with the (t, ℓ)-property.*

4. Echidnas and t-spikes.

We now focus on matroids with the $(t, 2t)$-property. In section 5, we will show that every sufficiently large matroid with the $(t, 2t)$-property has a partition into pairs such that the union of any t of these pairs is both a circuit and a cocircuit. We call such a matroid a t-spike. We first define a related structure: a t-echidna.

DEFINITION 4.1. *Let M be a matroid. A t-*echidna *of order n is a partition $(S_1, \ldots, S_n)$ of a subset of $E(M)$ such that*
(i) *$|S_i| = 2$ for all $i \in [n]$ and*
(ii) *$\bigcup_{i \in I} S_i$ is a circuit for all $I \subseteq [n]$ with $|I| = t$.*
For $i \in [n]$, we say S_i is a spine. *We say $(S_1, \ldots, S_n)$ is a t-*coechidna *of M if $(S_1, \ldots, S_n)$ is a t-echidna of M^*.*

DEFINITION 4.2. *A matroid M is a t-*spike *of order r if there exists a partition $\pi = (A_1, \ldots, A_r)$ of $E(M)$ such that π is a t-echidna and a t-coechidna, for some $r \geq t$. We say π is the* associated partition *of the t-spike M, and A_i is an* arm *of the t-spike for each $i \in [r]$.*

Note that if M is a t-spike, then M^* is a t-spike.

In this section, we prove, as Lemma 4.5, that if M is a matroid with the $(t, 2t)$-property, and M has a t-echidna of order $4t - 3$, then M is a t-spike.

LEMMA 4.3. *Let M be a matroid with the $(t, 2t)$-property. If M has a t-echidna $(S_1, \ldots, S_n)$, where $n \geq 3t - 1$, then $(S_1, \ldots, S_n)$ is also a t-coechidna of M.*

Proof. Let $S_i = \{x_i, y_i\}$ for each $i \in [n]$. By definition, if J is a t-element subset of $[n]$, then $\bigcup_{j \in J} S_j$ is a circuit. Consider such a circuit C; without loss of generality, we let $C = \{x_1, y_1, \ldots, x_t, y_t\}$. By the $(t, 2t)$-property, there is a $2t$-element cocircuit C^* that contains $\{x_1, \ldots, x_t\}$.

Suppose that $C^* \neq C$. Then there is some $i \in [t]$ such that $y_i \notin C^*$. Without loss of generality, say $y_1 \notin C^*$. Let I be a $(t-1)$-element subset of $[t+1, n]$. For any such I, the set $S_1 \cup (\bigcup_{i \in I} S_i)$ is a circuit that meets C^*. By orthogonality, $\bigcup_{i \in I} S_i$ meets C^* for every $(t-1)$-element subset I of $[t+1, n]$. Thus, C^* avoids at most $t-2$ of the S_i's for $i \in [t+1, n]$. In fact, as C^* meets each S_i with $i \in [t]$, the cocircuit C^* avoids at most $t-2$ of the S_i's with $i \in [n]$. Thus $|C^*| \geq n - (t-2) \geq (3t-1) - (t-2) = 2t+1 > 2t$; a contradiction. Therefore, we conclude that $C^* = C$, and the result follows. ∎

LEMMA 4.4. *Let M be a matroid with the $(t, 2t)$-property, and let $(S_1, \ldots, S_n)$ be a t-echidna of M with $n \geq 3t - 1$. Let I be a $(t-1)$-element subset of $[n]$. For $z \in E(M) - \bigcup_{i \in I} S_i$, there is a $2t$-element circuit and a $2t$-element cocircuit each containing $\{z\} \cup (\bigcup_{i \in I} S_i)$.*

Proof. By duality, it suffices to show that there is a $2t$-element circuit containing $\{z\} \cup (\bigcup_{i \in I} S_i)$. For $i \in [n]$, let $S_i = \{x_i, y_i\}$. By the $(t, 2t)$-property, there is a $2t$-element circuit C containing $\{z\} \cup \{x_i : i \in I\}$. Let J be a $(t-1)$-element subset of $[n]$ such that C and $\bigcup_{j \in J} S_j$ are disjoint (such a set exists since $|C| = 2t$ and $n \geq 3t-1$). For $i \in I$, let $C_i^* = S_i \cup (\bigcup_{j \in J} S_j)$, and observe that $x_i \in C_i^* \cap C$, and $C_i^* \cap C \subseteq S_i$. By Lemma 4.3, $(S_1, \ldots, S_n)$ is a t-coechidna as well as a t-echidna; therefore, C_i^* is a cocircuit. Now, for each $i \in I$, orthogonality implies that $|C_i^* \cap C| \geq 2$, and hence $y_i \in C$. So C contains $\{z\} \cup (\bigcup_{i \in I} S_i)$, as required. ∎

Let $(S_1, \ldots, S_n)$ be a t-echidna of a matroid M. If $(S_1, \ldots, S_m)$ is a t-echidna of

M, for some $m \geq n$, we say that $(S_1, \ldots, S_n)$ *extends* to $(S_1, \ldots, S_m)$. We say that $\pi = (S_1, \ldots, S_n)$ is *maximal* if there is no echidna other than π to which π extends.

LEMMA 4.5. *Let M be a matroid with the $(t, 2t)$-property, with $t \geq 2$. If M has a t-echidna $(S_1, \ldots, S_n)$, where $n \geq 4t - 3$, then $(S_1, \ldots, S_n)$ extends to a partition of $E(M)$ that is both a t-echidna and a t-coechidna.*

Proof. Suppose that $(S_1, \ldots, S_n)$ extends to $\pi = (S_1, \ldots, S_m)$, where π is maximal. Let $X = \bigcup_{i=1}^{m} S_i$. By Lemma 4.3, π is a t-coechidna as well as a t-echidna. The result holds if $X = E(M)$. Therefore, towards a contradiction, we suppose that $E(M) - X \neq \emptyset$. Let $z \in E(M) - X$. By Lemma 4.4, there is a $2t$-element circuit $C = \{z, z'\} \cup (\bigcup_{i \in [t-1]} S_i)$, for some $z' \in E(M) - \big(\{z\} \cup (\bigcup_{i \in [t-1]} S_i)\big)$.

We claim that $z' \notin X$. Towards a contradiction, suppose that $z' \in S_k$ for some $k \in [t, m]$. Let J be a t-element subset of $[t, m]$ containing k. Then, since $(S_1, \ldots, S_m)$ is a t-coechidna, $\bigcup_{j \in J} S_j$ is a cocircuit that contains z'. Now, by orthogonality, $z \in X$; a contradiction. Thus, $z' \notin X$, as claimed.

We next show that $(\{z, z'\}, S_t, S_{t+1}, \ldots, S_m)$ is a t-coechidna. It suffices to show that $\{z, z'\} \cup (\bigcup_{i \in I} S_i)$ is a cocircuit for each $(t-1)$-element subset I of $[t, m]$. Let I be such a set. Lemma 4.4 implies that there is a $2t$-element cocircuit C^* of M containing $\{z\} \cup (\bigcup_{i \in I} S_i)$. By orthogonality, $|C \cap C^*| > 1$. Therefore, $z' \in C^*$. Thus, $(\{z, z'\}, S_t, S_{t+1}, \ldots, S_m)$ is a t-coechidna. Since this t-coechidna has order $1 + m - (t-1) \geq 3t - 1$, the dual of Lemma 4.3 implies that $(\{z, z'\}, S_t, S_{t+1}, \ldots, S_m)$ is also a t-echidna.

Now, we claim that $(\{z, z'\}, S_1, S_2, \ldots, S_m)$ is a t-coechidna. It suffices to show that $\{z, z'\} \cup (\bigcup_{i \in I} S_i)$ is a cocircuit for any $(t-1)$-element subset I of $[m]$. Let I be such a set, and let J be a $(t-1)$-element subset of $[t, m] - I$. By Lemma 4.4, there is a $2t$-element cocircuit C^* containing $\{z\} \cup (\bigcup_{i \in I} S_i)$. Moreover, $C = \{z, z'\} \cup (\bigcup_{j \in J} S_j)$ is a circuit since $(\{z, z'\}, S_t, S_{t+1}, \ldots, S_m)$ is a t-echidna. By orthogonality, $z' \in C^*$. Therefore, $(\{z, z'\}, S_1, S_2, \ldots, S_m)$ is a t-coechidna. By the dual of Lemma 4.3, it is also a t-echidna, contradicting the maximality of $(S_1, \ldots, S_m)$. ∎

5. Matroids with the $(t, 2t)$-property. In this section, we prove that every sufficiently large matroid with the $(t, 2t)$-property is a t-spike. Our primary goal is to show that a sufficiently large matroid with the $(t, 2t)$-property has a large t-echidna or t-coechidna; it then follows, by Lemma 4.5, that the matroid is a t-spike.

LEMMA 5.1. *Let M be a matroid with the $(t, 2t)$-property, and let $X \subseteq E(M)$.*
(i) *If $r(X) < t$, then X is independent.*
(ii) *If $r(X) = t$, then $M|X \cong U_{t,|X|}$ and $|X| < 3t$.*

Proof. Clearly, as M has the $(t, 2t)$-property, M has no circuits of size at most t. Thus, if $r(X) < t$, then X contains no circuits and is therefore independent. If $r(X) = t$, then a subset of X is a circuit if and only if it has size $t + 1$. Therefore, $M|X \cong U_{t,|X|}$.

Suppose towards a contradiction that $M|X \cong U_{t,3t}$. Let $x \in X$, and let C^* be a cocircuit of M containing x. Then $E(M) - C^*$ is closed, so $\mathrm{cl}(X - C^*) \subseteq \mathrm{cl}(E(M) - C^*) = E(M) - C^*$. Therefore, $r(X - C^*) < r(X) = t$, implying that $|C^*| > 2t$. But then every cocircuit containing x has size greater than $2t$, contradicting the $(t, 2t)$-property. ∎

LEMMA 5.2. *Let M be a matroid with the $(t, 2t)$-property. Let $C_1^*, C_2^*, \ldots, C_{t-1}^*$ be a collection of $t-1$ pairwise disjoint cocircuits of M, and let $Y = E(M) - \bigcup_{i \in [t-1]} C_i^*$. For all $y \in Y$, there is a $2t$-element circuit C_y containing y such that either*

(i) $|C_y \cap C_i^*| = 2$ *for all* $i \in [t-1]$ *or*

(ii) $|C_y \cap C_j^*| = 3$ *for some* $j \in [t-1]$*, and* $|C_y \cap C_i^*| = 2$ *for all* $i \in [t-1] - \{j\}$*.*

Moreover, if $C_y = S \cup \{y\}$ *satisfies* (ii)*, then there are at most* $3t-1$ *elements* $w \in Y$ *such that* $S \cup \{w\}$ *is a circuit.*

Proof. Choose an element $c_i \in C_i^*$ for each $i \in [t-1]$. By the $(t, 2t)$-property, there is a $2t$-element circuit C_y containing $\{c_1, c_2, \ldots, c_{t-1}, y\}$, for each $y \in Y$. By orthogonality, C_y satisfies (i) or (ii).

Suppose C_y satisfies (ii), and let $S = C_y - Y = C_y - \{y\}$. Let $W = \{w \in Y : S \cup \{w\}$ is a circuit$\}$. It remains to prove that $|W| < 3t$. Observe that $W \subseteq \mathrm{cl}(S) \cap Y$, and, since S contains $t-1$ elements in pairwise disjoint cocircuits that avoid Y, we have $r(\mathrm{cl}(S) \cup Y) \geq r(Y) + (t-1)$. Thus,

$$\begin{aligned} r(W) &\leq r(\mathrm{cl}(S) \cap Y) \\ &\leq r(\mathrm{cl}(S)) + r(Y) - r(\mathrm{cl}(S) \cup Y) \\ &\leq (2t-1) + r(Y) - (r(Y) + (t-1)) \\ &= t, \end{aligned}$$

using submodularity of the rank function at the second line.

Now, by Lemma 5.1(i), if $r(W) < t$, then W is independent, so $|W| = r(W) < t$. On the other hand, by Lemma 5.1(ii), if $r(W) = t$, then $M|W \cong U_{t,|W|}$ and $|W| < 3t$, as required. ∎

LEMMA 5.3. *There exists a function* h *such that if* M *is a matroid with the* $(t, 2t)$*-property and having at least* $h(\ell, d, t)$ ℓ*-element circuits, then* M *has a collection of* d *pairwise disjoint* $2t$*-element cocircuits.*

Proof. By Lemma 3.2, there is a function g such that if M has at least $g(\ell, d)$ ℓ-element circuits, then M has a collection of d pairwise disjoint circuits. We define $h(\ell, d, t) = g(\ell, td)$, and claim that a matroid with the $(t, 2t)$-property and having at least $h(\ell, d, t)$ ℓ-element circuits has a collection of d pairwise disjoint $2t$-element cocircuits.

Let M be such a matroid. By Lemma 3.2, M has a collection of td pairwise disjoint circuits. We partition these into d groups of size t: call this partition $(\mathcal{C}_1, \ldots, \mathcal{C}_d)$. Since the t circuits in any cell of this partition are pairwise disjoint, it now suffices to show that, for each $i \in [d]$, there is a $2t$-element cocircuit contained in the union of the members of $\mathcal{C}_i$. Let $\mathcal{C}_i = \{C_1, \ldots, C_t\}$ for some $i \in [d]$. Pick some $c_j \in C_j$ for each $j \in [t]$. Then, by the $(t, 2t)$-property, $\{c_1, c_2, \ldots, c_t\}$ is contained in a $2t$-element cocircuit, which, by orthogonality, is contained in $\bigcup_{j \in [t]} C_j$. ∎

LEMMA 5.4. *There exists a function* g *such that if* M *is a matroid with the* $(t, 2t)$*-property and* $|E(M)| \geq g(t, q)$*, then, for some* $M' \in \{M, M^*\}$*, the matroid* M' *has* $t-1$ *pairwise disjoint cocircuits* $C_1^*, C_2^*, \ldots, C_{t-1}^*$*, and there is some* $Z \subseteq E(M') - \bigcup_{i \in [t-1]} C_i^*$ *such that*

(i) $r_{M'}(Z) \geq q$ *and*

(ii) *for each* $z \in Z$*, there exists an element* $z' \in Z - \{z\}$ *such that* $\{z, z'\}$ *is contained in a* $2t$*-element circuit* C *of* M' *with* $|C \cap C_i^*| = 2$ *for each* $i \in [t-1]$*.*

Proof. By Lemma 5.3, there is a function h such that if M' has at least $h(\ell, d, t)$ ℓ-element circuits, for $M' \in \{M, M^*\}$, then M' has a collection of d pairwise disjoint $2t$-element cocircuits.

Suppose $|E(M)| \geq 2t \cdot h(2t, t-1, t)$. Then, by the $(t, 2t)$-property, M' has at least $h(2t, t-1, t)$ distinct $2t$-element circuits. Hence, by Lemma 5.3, M' has a collection of $t-1$ pairwise disjoint $2t$-element cocircuits $C_1^*, C_2^*, \dots, C_{t-1}^*$.

Let $X = \bigcup_{i\in[t-1]} C_i^*$ and $Y = E(M) - X$. By Lemma 5.2, for each $y \in Y$ there is a $2t$-element circuit C_y containing y such that $|C_y \cap C_j^*| = 3$ for at most one $j \in [t-1]$ and $|C_y \cap C_i^*| = 2$ otherwise. Let W be the set of all $w \in Y$ such that w is in a $2t$-element circuit C with $|C \cap C_j^*| = 3$ for some $j \in [t-1]$, and $|C \cap C_i^*| = 2$ for all $i \in [t-1] - \{j\}$. Now, letting $Z = Y - W$, we see that (ii) is satisfied for both $M' = M$ and $M' = M^*$.

Since the C_i^*'s have size $2t$, there are $(t-1)\binom{2t}{3}\binom{2t}{2}^{t-2}$ sets $X' \subseteq X$ with $|X' \cap C_j^*| = 3$ for some $j \in [t-1]$ and $|X' \cap C_i^*| = 2$ for all $i \in [t-1] - \{j\}$. It follows, by Lemma 5.2, that $|W| \le s(t)$ where

$$s(t) = (3t-1)\left[(t-1)\binom{2t}{3}\binom{2t}{2}^{t-2}\right].$$

We define

$$g(t,q) = \max\left\{2t \cdot h(2t, t-1, t), 2\big(q + s(t) + 2t(t-1)\big)\right\}.$$

Suppose that $|E(M)| \ge g(t,q)$. Recall that (ii) holds for both $M' = M$ and $M' = M^*$. Moreover, we can choose $M' \in \{M, M^*\}$ such that $r(M') \ge q + s(t) + 2t(t-1)$. Then,

$$\begin{aligned} r_{M'}(Z) &\ge r_{M'}(Y) - |W| \\ &\ge \big(r(M') - 2t(t-1)\big) - s(t) \\ &\ge q, \end{aligned}$$

so (i) holds as well, as required. □

LEMMA 5.5. *Let M be a matroid with the $(t,2t)$-property. Suppose M has $t-1$ pairwise disjoint cocircuits $C_1^*, C_2^*, \dots, C_{t-1}^*$, and, for some positive integer p, there is some $Z \subseteq E(M) - \bigcup_{i\in[t-1]} C_i^*$ such that*

(a) *$r_M(Z) \ge \binom{2t}{2}^{t-1}(p + 2(t-1))$ and*
(b) *for each $z \in Z$, there exists an element $z' \in Z - \{z\}$ such that $\{z, z'\}$ is contained in a $2t$-element circuit C of M with $|C \cap C_i^*| = 2$ for each $i \in [t-1]$.*

Then there exist a subset $Z' \subseteq Z$ and a partition $\mathcal{Z}' = (Z_1', \dots, Z_p')$ of Z' into pairs such that

(i) *each circuit of $M|Z'$ is a union of pairs in $\mathcal{Z}'$ and*
(ii) *the union of any t pairs of $\mathcal{Z}'$ contains a circuit.*

Proof. We first prove the following claim.

Claim 5.5.1. There exist a $(2t-2)$-element set X, with $|X \cap C_i^*| = 2$ for each $i \in [t-1]$, and a set $Z' \subseteq Z$, with a partition $\mathcal{Z}' = (Z_1', \dots, Z_p')$ into p pairs, such that

(I) $X \cup Z_i'$ is a circuit for each $i \in [p]$ and
(II) $\mathcal{Z}'$ partitions the ground set of $(M/X)|Z'$ into parallel classes, and we have that $r_{M/X}\big(\bigcup_{i\in[p]} Z_i'\big) = p$.

Proof. For each $z \in Z$, there exist an element $z' \in Z - \{z\}$ and a set X' such that $\{z, z'\} \cup X'$ is a circuit of M, and X' is the union of pairs Y_i for $i \in [t-1]$, with $Y_i \subseteq C_i^*$. There are $\binom{2t}{2}^{t-1}$ choices of such pairs $Y_i \subseteq C_i^*$ for $i \in [t-1]$. Thus, for some $m \le \binom{2t}{2}^{t-1}$, there are $(2t-2)$-element sets $X_1, \dots, X_m$, each of which intersects C_i^* in two elements for each $i \in [t-1]$, and sets $Z_1, \dots, Z_m$ whose union is Z, such that for each $j \in [m]$ and each $z_j \in Z_j$, there is an element $z'_j \in Z_j$ such that $X_j \cup \{z_j, z'_j\}$ is a circuit. Moreover, $r(Z_1) + \cdots + r(Z_m) \geq r(Z)$. Thus, by the pigeonhole principle, there exists some $j \in [m]$ with

$$r(Z_j) \geq \frac{r(Z)}{\binom{2t}{2}^{t-1}} \geq p + 2(t-1).$$

Let $Z' = Z_j$ and $X = X_j$. Now, observe that $X \cup \{z, z'\}$ is a circuit, for some pair $\{z, z'\} \subseteq Z'$, if and only if $\{z, z'\}$ is a parallel pair in M/X. So the ground set of $(M/X)|Z'$ has a partition into parallel classes, where each parallel class has size at least two. Let $\mathcal{Z}' = \{\{z_1, z'_1\}, \ldots, \{z_n, z'_n\}\}$ be a collection of pairs from each parallel class such that $\{z_1, z_2, \ldots, z_n\}$ is independent in $(M/X)|Z'$. Since $r_{M/X}(Z') = r(Z' \cup X) - r(X) \geq r(Z') - 2(t-1) \geq p$, there exists such a collection $\mathcal{Z}'$ of size p, and this collection satisfies Claim 5.5.1. ☐

Let X and $\mathcal{Z}' = \{Z'_1, \ldots, Z'_p\}$ be as described in Claim 5.5.1, let $Z' = \bigcup_{i \in [p]} Z'_i$, and let $\mathcal{X} = \{X_1, \ldots, X_{t-1}\}$, where $X_i = \{x_i, x'_i\} = X \cap C^*_i$.

Claim 5.5.2. Each circuit of $M|(X \cup Z')$ is a union of pairs in $\mathcal{X} \cup \mathcal{Z}'$.

Proof. Let C be a circuit of $M|(X \cup Z')$. If $x_i \in C$, for some $\{x_i, x'_i\} \in \mathcal{X}$, then, by orthogonality with C^*_i, we have $x'_i \in C$. Towards a contradiction, say $\{z, z'\} \in \mathcal{Z}'$ and $C \cap \{z, z'\} = \{z\}$. Choose W to be the union of the pairs of $\mathcal{Z}'$ that contain elements of $(C - \{z\}) \cap Z'$. Then $z \in \mathrm{cl}(X \cup W)$. Hence $z \in \mathrm{cl}_{M/X}(W)$, contradicting Claim 5.5.1(II). ☐

Claim 5.5.3. The union of any t pairs of $\mathcal{X} \cup \mathcal{Z}'$ contains a circuit.

Proof. Let $\mathcal{W}$ be a subcollection of $\mathcal{X} \cup \mathcal{Z}'$ of size t. We proceed by induction on the number of pairs in $\mathcal{W} \cap \mathcal{Z}'$. If there is only one pair in $\mathcal{W} \cap \mathcal{Z}'$, then the union of the pairs in $\mathcal{W}$ contains a circuit (indeed, is a circuit) by Claim 5.5.1(I). Suppose the result holds for any subcollection containing k pairs in $\mathcal{Z}'$, and let $\mathcal{W}$ be a subcollection containing $k+1$ pairs in $\mathcal{Z}'$. Let $\{x, x'\}$ be a pair in $\mathcal{X} - \mathcal{W}$, and let $W = \bigcup_{W' \in \mathcal{W}} W'$. By the induction hypothesis, $W \cup \{x, x'\}$ contains a circuit C_1. If $\{x, x'\} \subseteq E(M) - C_1$, then $C_1 \subseteq W$, in which case the union of the pairs in $\mathcal{W}$ contains a circuit, as desired. Therefore, we may assume, by Claim 5.5.2, that $\{x, x'\} \subseteq C_1$. Since X is independent, there is a pair $\{z, z'\} \subseteq Z' \cap C_1$. By the induction hypothesis, there is a circuit C_2 contained in $(W - \{z, z'\}) \cup \{x, x'\}$. Observe that C_1 and C_2 are distinct, and $\{x, x'\} \subseteq C_1 \cap C_2$. By circuit elimination on C_1 and C_2, and Claim 5.5.2, there is a circuit $C_3 \subseteq (C_1 \cup C_2) - \{x, x'\} \subseteq W$, as desired. The result now follows by induction. ☐

Now, Claim 5.5.3 implies that the union of any t pairs of $\mathcal{Z}'$ contains a circuit, and the result follows. ☐

In order to prove Theorem 1.1, we use some hypergraph Ramsey theory [9].

THEOREM 5.6 (Ramsey's theorem for k-uniform hypergraphs). *For positive integers k and n, there exists an integer $r_k(n)$ such that if H is a k-uniform hypergraph on $r_k(n)$ vertices, then H has either a clique on n vertices, or a stable set on n vertices.*

We now prove Theorem 1.1, restated below as Theorem 5.7.

THEOREM 5.7. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if M is a matroid with the $(t, 2t)$-property, and $|E(M)| \geq f(t)$, then M is a t-spike.*

Proof. We first consider the case where $t = 1$. Let M be a nonempty matroid with the $(1, 2)$-property. Then, for every $e \in E(M)$, the element e is in a parallel pair P and a series pair S. By orthogonality, $P = S$, and P is a connected component of M. Then $M \cong U_{1,2} \oplus M \backslash P$, and the result easily follows.

We may now assume that $t \geq 2$. We define the function $h_k : \mathbb{N} \to \mathbb{N}$, for each $k \in [t]$, as follows:

$$h_k(t) = \begin{cases} 4t - 3 & \text{if } k = t, \\ r_k(h_{k+1}(t)) & \text{if } k \in [t-1], \end{cases}$$

where $r_k(n)$ is the Ramsey number described in Theorem 5.6. Note that $h_k(t) \geq h_{k+1}(t) \geq 4t - 3$, for each $k \in [t-1]$. Let $p(t) = h_1(t)$, and let $q(t) = \binom{2t}{2}^{t-1}(p(t) + 2(t-1))$.

By Lemma 5.4, there exists a function g such that if $|E(M)| \geq g(t, q(t))$, then, for some $M' \in \{M, M^*\}$, the matroid M' has $t - 1$ pairwise disjoint cocircuits $C_1^*, C_2^*, \ldots, C_{t-1}^*$, and there is some $Z' \subseteq E(M') - \bigcup_{i \in [t-1]} C_i^*$ such that $r_{M'}(Z') \geq q(t)$, and, for each $z \in Z'$, there exists an element $z' \in Z' - \{z\}$ such that $\{z, z'\} \cup (\bigcup_{i \in [t-1]} \{x_i, x_i'\})$ is a circuit of M', where $\{x_i, x_i'\} \subseteq C_i^*$.

Let $f(t) = g(t, q(t))$, and suppose that $|E(M)| \geq f(t)$. For ease of notation, we assume that $M' = M$. Then, by Lemma 5.5, there exist a subset $Z \subseteq Z'$ and a partition $\mathcal{Z} = (Z_1, \ldots, Z_{p(t)})$ of Z into $p(t)$ pairs such that

(I) each circuit of $M|Z$ is a union of pairs in $\mathcal{Z}$ and
(II) the union of any t pairs of $\mathcal{Z}$ contains a circuit.

By Lemma 4.5, and since $t \geq 2$, it suffices to show that M has a t-echidna or a t-coechidna of order $4t - 3$. If the smallest circuit in $M|Z$ has size $2t$, then, by (II), $\mathcal{Z}$ is a t-echidna of order $p(t) \geq 4t - 3$. So we may assume that the smallest circuit in $M|Z$ has size $2j$ for some $j \in [t-1]$.

Claim 5.7.1. If the smallest circuit in $M|Z$ has size $2j$, for $j \in [t-1]$, and $|\mathcal{Z}| \geq h_j(t)$, then either

(i) M has a t-coechidna of order $4t - 3$ or
(ii) there exists some $Z' \subseteq Z$ that is the union of $h_{j+1}(t)$ pairs of $\mathcal{Z}$ for which the smallest circuit in $M|Z'$ has size at least $2(j + 1)$.

Proof. Let $2j$ be the size of the smallest circuit in $M|Z$. We define H to be the j-uniform hypergraph with vertex set $\mathcal{Z}$ whose hyperedges are the j-subsets of $\mathcal{Z}$ that are partitions of circuits in $M|Z$. By Theorem 5.6 and the definition of h_k, as H has at least $h_j(t)$ vertices, it has either a clique or a stable set, on $h_{j+1}(t)$ vertices. If H has a stable set $\mathcal{Z}'$ on $h_{j+1}(t)$ vertices, then clearly (ii) holds, with $Z' = \bigcup_{P \in \mathcal{Z}'} P$.

So we may assume that there are $h_{j+1}(t)$ pairs in $\mathcal{Z}$ such that the union of any j of these pairs is a circuit. Let Z'' be the union of these $h_{j+1}(t)$ pairs. We claim that the union of any set of t pairs contained in Z'' is a cocircuit. Let T be a transversal of t pairs of $\mathcal{Z}$ contained in Z'', and let C^* be the $2t$-element cocircuit containing T. Towards a contradiction, suppose that there exists some pair $P \in \mathcal{Z}$ with $P \subseteq Z''$ such that $|C^* \cap P| = 1$. Select $j-1$ pairs $Z_1'', \ldots, Z_{j-1}''$ of $\mathcal{Z}$ that are each contained in $Z'' - C^*$ (these exist since $h_{j+1}(t) \geq 3t - 1 \geq 2t + j - 1$). Then $P \cup (\bigcup_{i \in [j-1]} Z_i'')$ is a circuit that intersects the cocircuit C^* in a single element, contradicting orthogonality. We deduce that the union of any t pairs of $\mathcal{Z}$ that are contained in Z'' is a cocircuit. So M has a t-coechidna of order $h_{j+1}(t) \geq 4t - 3$, satisfying (i). ◻

We now apply Claim 5.7.1 iteratively, for a maximum of $t - j$ iterations. If (i) holds, at any iteration, then M has a t-coechidna of order $4t - 3$, as required.

Otherwise, we let $\mathcal{Z}'$ be the partition of Z' induced by $\mathcal{Z}$; then, at the next iteration, we relabel $Z = Z'$ and $\mathcal{Z} = \mathcal{Z}'$. If (ii) holds for each of $t-j$ iterations, then we obtain a subset Z' of Z such that the smallest circuit in $M|Z'$ has size $2t$. Then, by (II), M has a t-echidna of order $h_t(t) = 4t-3$. This completes the proof. ∎

6. Properties of *t*-spikes. In this section, we prove some properties of t-spikes, which demonstrate that t-spikes form a class of highly structured matroids. In particular, we show that a t-spike has order at least $2t-1$; a t-spike of order r has $2r$ elements and rank r; the circuits of a t-spike that are not a union of t arms meet all but at most $t-2$ of the arms; and a t-spike of order at least $4t-4$ is $(2t-1)$-connected. We also show that an appropriate concatenation of the associated partition of a t-spike is a $(2t-1)$-anemone, following the terminology of [1].

It is straightforward to see that the family of 1-spikes consists of matroids obtained by taking direct sums of copies of $U_{1,2}$. We also describe a construction that can be used to obtain a $(t+1)$-spike from a t-spike, and show that every $(t+1)$-spike can be constructed from some t-spike in this way.

Basic properties.

Lemma 6.1. *Let M be a t-spike of order r. Then $r \geq 2t-1$.*

Proof. Let $(A_1, \dots, A_r)$ be the associated partition of M. By definition, $r \geq t$. Let J be a t-element subset of $[r]$, and let $Y = \bigcup_{j \in J} A_j$. Pick some $y \in Y$. Since Y is a cocircuit and a circuit, $Z = (E(M) - Y) \cup \{y\}$ spans and cospans M. Since $|Z| = 2(r-t)+1$,

$$2r = |E(M)| = r(M) + r^*(M) \leq \big(2(r-t)+1\big) + \big(2(r-t)+1\big).$$

It follows that $r \geq 2t-1$. ∎

Lemma 6.2. *Let M be a t-spike of order r. Then $r(M) = r^*(M) = r$.*

Proof. Let $(A_1, \dots, A_r)$ be the associated partition of M, and label $A_i = \{x_i, y_i\}$ for each $i \in [r]$. Pick $I \subseteq J \subseteq [r]$ such that $|I| = t-1$ and $|J| = r-t$. Let $X = \big(\bigcup_{i \in I} A_i\big) \cup \{x_j : j \in J\}$, and observe that $|X| = |I| + |J| = r-1$. Now, since $(A_1, \dots, A_r)$ is a t-echidna, $\bigcup_{j \in J} A_j \subseteq \mathrm{cl}(X)$. As $E(M) - \bigcup_{j \in J} A_j$ is a cocircuit, we deduce that $r(M) - 1 \leq r(X) \leq |X| = r-1$, so $r(M) \leq r$. Similarly, as $(A_1, \dots, A_r)$ is a t-coechidna, we deduce that $r^*(M) \leq r$. Since $r(M) + r^*(M) = |E(M)| = 2r$, the lemma follows. ∎

The next lemma shows that a circuit C of a t-spike is either a union of t arms, or else C meets all but at most $t-2$ of the arms.

Lemma 6.3. *Let M be a t-spike of order r with associated partition $(A_1, \dots, A_r)$, and let C be a circuit of M. Then either*

(i) *$C = \bigcup_{j \in J} A_j$ for some t-element set $J \subseteq [r]$ or*

(ii) *$|\{i \in [r] : A_i \cap C \neq \emptyset\}| \geq r-(t-2)$ and $|\{i \in [r] : A_i \subseteq C\}| < t$.*

Proof. Let $S = \{i \in [r] : A_i \cap C \neq \emptyset\}$, so S is the minimal subset of $[r]$ such that $C \subseteq \bigcup_{i \in S} A_i$. If C is properly contained in $\bigcup_{j \in J} A_j$ for some t-element set $J \subseteq [r]$, then C is independent; a contradiction. So $|S| \geq t$. If $|S| = t$, then $C = \bigcup_{i \in S} A_i$, implying C is a circuit, which satisfies (i). So we may assume that $|S| > t$. Now $|\{i \in [r] : A_i \subseteq C\}| < t$; otherwise C properly contains a circuit. Thus, there exists some $j \in S$ such that $A_j - C \neq \emptyset$. If $|S| \geq r-(t-2)$, then (ii) holds; thus we assume that $|S| \leq r-(t-1)$. Let $T = ([r]-S) \cup \{j\}$. Then $|T| \geq t$, so $\bigcup_{i \in T} A_i$ contains a cocircuit that intersects C in one element, contradicting orthogonality. ∎

Connectivity. Let M be a matroid with ground set E. Recall that the *connectivity function* of M, denoted by λ, is defined as

$$\lambda(X) = r(X) + r(E - X) - r(M)$$

for all subsets X of E. It is easily verified that

$$\lambda(X) = r(X) + r^*(X) - |X|. \tag{6.1}$$

A subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda(X) < k$. A k-separating partition $(X, E - X)$ is a *k-separation* if $|X| \geq k$ and $|E - X| \geq k$. The matroid M is *n-connected* if, for all $k < n$, it has no k-separations.

LEMMA 6.4. *Suppose M is a t-spike with associated partition $(A_1, \ldots, A_r)$. Then, for all partitions (J, K) of $[r]$ with $|J| \leq |K|$,*

$$\lambda\left(\bigcup_{j\in J} A_j\right) = \begin{cases} 2|J| & \text{if } |J| < t, \\ 2t - 2 & \text{if } |J| \geq t. \end{cases}$$

Proof. Let (J, K) be a partition of $[r]$ with $|J| \leq |K|$.

Claim 6.4.1. The lemma holds when $|J| \leq t$.

Proof. Suppose $|J| < t$. Since $(A_1, \ldots, A_r)$ is a t-echidna (respectively, t-coechidna), $\bigcup_{j\in J} A_j$ is independent (respectively, coindependent). So, by (6.1), $\lambda\big(\bigcup_{j\in J} A_j\big) = 2|J| + 2|J| - 2|J| = 2|J|$.

Now suppose $|J| = t$. Then, by definition, $\bigcup_{j\in J} A_j$ is a circuit and a cocircuit. So $\lambda\big(\bigcup_{j\in J} A_j\big) = (2t - 1) + (2t - 1) - 2t = 2t - 2$, by (6.1). □

Claim 6.4.2. Let $X \subseteq Y \subseteq [r]$ such that $|X| \geq t - 1$. Then

$$\lambda\left(\bigcup_{x\in X} A_x\right) \geq \lambda\left(\bigcup_{y\in Y} A_y\right).$$

Proof. Let X' be a $(t - 1)$-element subset of X, and let $y \in Y - X$. Then $\lambda\big(\bigcup_{x\in X'} A_x\big) = 2(t - 1)$, and $\lambda\big(A_y \cup (\bigcup_{x\in X'} A_x)\big) = 2t - 2$, by Claim 6.4.1. By submodularity of the connectivity function,

$$\begin{aligned}\lambda\left(A_y \cup \bigcup_{x\in X} A_x\right) &\leq \lambda\left(A_y \cup \bigcup_{x\in X'} A_x\right) + \lambda\left(\bigcup_{x\in X} A_x\right) - \lambda\left(\bigcup_{x\in X'} A_x\right)\\ &= (2t - 2) + \lambda\left(\bigcup_{x\in X} A_x\right) - (2t - 2)\\ &= \lambda\left(\bigcup_{x\in X} A_x\right).\end{aligned}$$

Claim 6.4.2 now follows by induction. □

Now suppose $|J| > t$. By Claims 6.4.1 and 6.4.2, $\lambda\big(\bigcup_{j\in J} A_j\big) \leq 2t - 2$. Recall that $|K| \geq |J| > t$. Let K' be a t-element subset of K. Let $J' = [r] - K'$, and note that $J \subseteq J'$. So, by Claim 6.4.2,

$$\lambda\left(\bigcup_{j\in J} A_j\right) \geq \lambda\left(\bigcup_{j\in J'} A_j\right) = \lambda\left(\bigcup_{k\in K'} A_k\right) = 2t - 2.$$

We deduce that $\lambda\big(\bigcup_{j\in J} A_j\big) = 2t-2$, as required. ∎

Given a t-spike M with associated partition $(A_1,\dots,A_r)$, suppose that $(P_1,\dots,P_m)$ is a partition of $E(M)$ such that, for each $i \in [m]$, $P_i = \bigcup_{i\in I} A_i$ for some subset I of $[r]$, with $|P_i| \geq 2t-2$. Using the terminology of [1], it follows immediately from Lemma 6.4 that $(P_1,\dots,P_m)$ is a $(2t-1)$-anemone. (Note that a partition whose concatenations give rise to a flower in this way has previously appeared in the literature [3] under the name of "quasi-flowers.")

LEMMA 6.5. *Let M be a t-spike of order at least $4t-4$, for $t \geq 2$. Then M is $(2t-1)$-connected.*

Proof. Let r be the order of the t-spike M, and let $(A_1,\dots,A_r)$ be the associated partition of M. Towards a contradiction, suppose M is not $(2t-1)$-connected, and let (P,Q) be a k-separation for some $k < 2t-1$. Without loss of generality, we may assume that $|P| \geq |Q|$. Note, in particular, that $\lambda(P) < k \leq |Q|$ and $\lambda(P) < 2t-2$.

Suppose $|P\cap A_j| \neq 1$ for all $j \in [r]$. Then, by Lemma 6.4, $\lambda(P) = |Q|$ if $|Q| < 2t$, otherwise $\lambda(P) = 2t-2$; either case is contradictory. So $|P \cap A_j| = 1$ for some $j \in [r]$.

Suppose $|Q| \leq 2t-2$. Then, by Lemma 6.3 and its dual, Q is independent and coindependent, so $\lambda(P) = |Q|$ by (6.1); a contradiction.

Now we may assume that $|Q| > 2t-2$. Suppose $\bigcup_{i\in I} A_i \subseteq P$, for some $(t-1)$-element set $I \subseteq [r]$. Then $A_j \subseteq \mathrm{cl}(P)$ for each $j \in [r]$ such that $|P \cap A_j| = 1$. For such a j, it follows, by the definition of λ, that $\lambda(P\cup A_j) \leq \lambda(P)$; we use this repeatedly in what follows. Let $U = \{u \in [r] : |P\cap A_u| = 1\}$. For any subset $U' \subseteq U$, we have $\lambda\left(P \cup (\bigcup_{u\in U'} A_u)\right) \leq \lambda(P) < 2t-2$. Let $P' = P \cup (\bigcup_{u\in U} A_u)$, and let $Q' = E(M) - P'$. If $|Q'| > 2t-2$, then $\lambda(P') = 2t-2$ by Lemma 6.4, contradicting that $\lambda(P') \leq \lambda(P) < 2t-2$. So $|Q'| \leq 2t-2$. Now, let $d = |Q|-(2t-2)$, and let U' be a d-element subset of U. Then $\lambda(P) \geq \lambda\left(P\cup(\bigcup_{u\in U'} A_u)\right) = \lambda\left(Q - \bigcup_{u\in U'} A_u\right)$. Since $\left|Q - \bigcup_{u\in U'} A_u\right| = 2t-2$, we have that $\lambda\left(Q - \bigcup_{u\in U'} A_u\right) = 2t-2$, so $\lambda(P) \geq 2t-2$; a contradiction. We deduce that $|\{i \in [r] : A_i \subseteq P\}| < t-1$. Since $|Q| \leq |P|$, it follows that $|\{i\in[r] : A_i \subseteq Q\}| \leq |\{i \in [r] : A_i \subseteq P\}| < t-1$.

Now $|\{i \in [r] : A_i \cap Q \neq \emptyset\}| \geq r-(t-2)$, so $r(Q) \geq r-(t-1)$ by Lemma 6.3. Similarly, $r(P) \geq r-(t-1)$. So

$$\begin{aligned}\lambda(P) &= r(P) + r(Q) - r(M)\\ &\geq (r-(t-1)) + (r-(t-1)) - r\\ &\geq (4t-4) - 2(t-1) = 2t-2;\end{aligned}$$

a contradiction. This completes the proof. ∎

Constructions. We first describe a construction that can be used to obtain a $(t+1)$-spike of order r from a t-spike of order r, when $r \geq 2t+1$. We then show that every $(t+1)$-spike can be constructed from some t-spike in this way.

Recall that M_1 is an *elementary quotient* of M_0 if there is a single-element extension M_0^+ of M_0 by an element e such that $M_1 = M_0^+/e$. A matroid M_1 is an *elementary lift* of M_0 if M_1^* is an elementary quotient of M_0^*. Note also that if M_1 is an elementary quotient of M_0, then M_0 is an elementary lift of M_1.

Let M_0 be a t-spike of order $r \geq 2t+1$ with associated partition π. Let M_0' be an elementary quotient of M_0 such that none of the $2t$-element cocircuits are preserved (that is, extend M_0 by an element e that blocks all of the $2t$-element cocircuits, and then contract e). Now, in M_0', the union of any t cells of π is still a $2t$-element circuit, but, as $r(M_0') = r(M_0)-1$, the union of any $t+1$ cells of π is a $2(t+1)$-element cocircuit. We then repeat this in the dual; that is, let M_1 be an elementary lift of M_0' such that none of the $2t$-element circuits are preserved. Then M_1 is a $(t+1)$-spike. Note that M_1 is not unique; more than one $(t+1)$-spike can be constructed from a given t-spike M_0 in this way.

Given a $(t+1)$-spike M_1, for some positive integer t, we now describe how to obtain a t-spike M_0 from M_1 by a specific elementary quotient, followed by a specific elementary lift. This process reverses the construction from the previous paragraph. The next lemma describes the single-element extension (or coextension, in the dual) that gives rise to the elementary quotient (or lift) we desire. Intuitively, the extension adds a "tip" to a t-echidna. In the proof of this lemma, we assume knowledge of the theory of modular cuts (see [6, section 7.2]).

LEMMA 6.6. *Let M be a matroid with a t-echidna $\pi = (S_1, \ldots, S_n)$. Then there is a single-element extension M^+ of M by an element e such that $e \in \mathrm{cl}_{M^+}(X)$ if and only if X contains at least $t-1$ spines of π for all $X \subseteq E(M)$.*

Proof. Let

$$\mathcal{F} = \left\{ \bigcup_{i \in I} S_i : I \subseteq [n] \text{ and } |I| = t-1 \right\}.$$

By the definition of a t-echidna, $\mathcal{F}$ is a collection of flats of M. Let $\mathcal{M}$ be the set of all flats of M containing some flat $F \in \mathcal{F}$. We claim that $\mathcal{M}$ is a modular cut. Recall that, for distinct $F_1, F_2 \in \mathcal{M}$, the pair (F_1, F_2) is *modular* if $r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2)$. It suffices to prove that for any $F_1, F_2 \in \mathcal{M}$ such that (F_1, F_2) is a modular pair, $F_1 \cap F_2 \in \mathcal{M}$.

For any $F \in \mathcal{M}$, since F contains at least $t-1$ spines of π, and the union of any t spines is a circuit (by the definition of a t-echidna), it follows that F is a union of spines of π. So let $F_1, F_2 \in \mathcal{M}$ such that $F_1 = \bigcup_{i \in I_1} S_i$ and $F_2 = \bigcup_{i \in I_2} S_i$, where I_1 and I_2 are distinct subsets of $[n]$ with $u_1 = |I_1| \geq t-1$ and $u_2 = |I_2| \geq t-1$. Then

$$\begin{aligned} r(F_1) + r(F_2) &= (t-1+u_1) + (t-1+u_2) \\ &= 2(t-1) + u_1 + u_2. \end{aligned}$$

Suppose that $|I_1 \cap I_2| < t-1$. Let $s = |I_1 \cap I_2|$. Then $F_1 \cup F_2$ is the union of $u_1 + u_2 - s \geq t-1$ spines of π. So

$$\begin{aligned} r(F_1 \cup F_2) + r(F_1 \cap F_2) &= \big(t-1+(u_1+u_2-s)\big) + 2s \\ &= (t-1) + s + u_1 + u_2. \end{aligned}$$

Since $s < t-1$, it follows that $r(F_1 \cup F_2) + r(F_1 \cap F_2) < r(F_1) + r(F_2)$. So, for every modular pair (F_1, F_2) with $F_1, F_2 \in \mathcal{M}$, we have $|I_1 \cap I_2| \geq t-1$, in which case $F_1 \cap F_2$ is a flat containing the union of $t-1$ spines of π, and hence $F_1 \cap F_2 \in \mathcal{M}$ as required.

Now, there is a single-element extension corresponding to the modular cut $\mathcal{M}$, and this extension satisfies the requirements of the lemma (see, for example, [6, Theorem 7.2.3]). ∎

Let M be a t-spike with associated partition $\pi = (A_1, \ldots, A_r)$, for some integer $t \geq 2$, where $r \geq 2t-1$ by Lemma 6.1. Let M^+ be the single-element extension of M by an element e described in Lemma 6.6.

Consider M^+/e. We claim that π is a $(t-1)$-echidna and a t-coechidna of M^+/e. Let X be the union of any $t-1$ spines of π. Then X is independent in M, and $X \cup \{e\}$ is a circuit in M^+, so X is a circuit in M^+/e. So π is a $(t-1)$-echidna of M^+/e.

Now let C^* be the union of any t spines of π, and let $H = E(M) - C^*$. Then H is the union of at least $t-1$ spines, so $e \in \mathrm{cl}_{M^+}(H)$. Now $H \cup \{e\}$ is a hyperplane in M^+, so C^* is a cocircuit in M^+. Hence π is a t-coechidna of M^+/e.

We now repeat this process on $N = (M^+/e)^*$. In N, the partition π is a t-echidna and $(t-1)$-coechidna. By Lemma 6.6, there is a single-element extension N^+ of N (a single-element coextension of M^+/e) by an element e'. By the same argument as in the previous paragraph, π is a $(t-1)$-echidna and $(t-1)$-coechidna of N^+/e, so N^+/e is a $(t-1)$-spike. Let $M' = (N^+/e)^*$.

Note that M^+/e is an elementary quotient of M, so M is an elementary lift of M^+/e where none of the $2(t-1)$-element circuits of M^+/e are preserved in M. Similarly, M^+/e is an elementary quotient of M' where none of the $2(t-1)$-element cocircuits are preserved. So the t-spike M can be obtained from the $(t-1)$-spike M' using the earlier construction.

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